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BOUR'S THEOREM IN 4-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. In this paper we generalize 3-dimensional Bour's Theorem to the case of 4-dimension. We proved that a helicoidal surface in $\mathbb{R}^4$ is isometric to a family of surfaces of revolution in $\mathbb{R}^4$ in such a way that helices on the helicoidal surface correspond to parallel circles on the surfaces of revolution. Moreover, if the surfaces are required further to have the same Gauss map, then they are hyperplanar and minimal. Parametrizations for such minimal surfaces are given explicitly.

1. Introduction

Consider the deformation determined by a family of parametric surfaces given by

$$\begin{aligned} X_\theta(u,v) &= (x_\theta(u,v), y_\theta(u,v), z_\theta(u,v)),\\ x_\theta(u,v) &= \cos\theta\sinh v\sin u + \sin\theta\cosh v\cos u,\\ y_\theta(u,v) &= -\cos\theta\sinh v\cos u + \sin\theta\cosh v\sin u,\\ z_\theta(u,v) &= u\cos\theta + v\sin\theta; \end{aligned}$$

where $-\pi < u \le \pi$, $-\infty < v < \infty$ and the deformation parameter $-\pi < \theta \le \pi$. A direct computation shows that all surfaces X_θ are minimal, have the same first fundamental form and normal vector field. We can see that X_0 is the helicoid and $X_{\pi/2}$ is the catenoid. Thus, locally the helicoid and catenoid are isometric and have the same Gauss map. Moreover, helices on the helicoid correspond to parallel circles on the catenoid. The classical Bour's theorem (see [1]) can be seen as a weak generalization of this fact.

Bour's theorem. *A helicoidal surface is isometric to a surface of revolution so that helices on the helicoidal surface correspond to parallel circles on the surface of revolution.*

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Ikawa [7] studied a pair of surfaces by Bour's theorem in $\mathbb{R}^3$ with an additional condition that they have the same Gauss map and proved that they are minimal, i.e they are the helicoid and catenoid, respectively.

Bour's theorem is studied not only in $\mathbb{R}^3$, but also in Lorentz-Minkowski 3-space (see [2], [3], [5], [4], [6], [7], [8], [9], [10]).

A helicoidal surface in $\mathbb{R}^3$ is the orbit of a plane curve under a screw motion, that is a rotation around a line followed by a translation along the line. In $\mathbb{R}^4$, we can define the rotation around a plane, therefore we can define surfaces of revolution as well as helicoidal surfaces similarly as those in $\mathbb{R}^3$.

So it is natural to generalize Bour's theorem to 4-dimensional case, i.e., the ambient space is $\mathbb{R}^4$, and consider the case when the surfaces are required further to have the same Gauss map. The purpose of this paper is to settle these problems. The first main result of the paper is Theorem 3.1, a generalization of the classical Bour's theorem. The second one is Theorem 3.2 asserting that a pair of surfaces related by Bour's theorem having the same Gauss map must be hyperplanar, i.e. belong to hyperplanes, and minimal. The parametrizations of such minimal surfaces are determined explicitly in Corollary 3.3.

2. Preliminaries

2.1. Basic concepts

Let $X : D \to \mathbb{R}^4,\ D \subset \mathbb{R}^2$ be a smooth parametric surface in $\mathbb{R}^4$. The tangent space to M at an arbitrary point $p = X(u, v)$ is the span$\{X_u, X_v\}$. The coefficients of the first fundamental form of M are given by

$$g_{11} = \langle X_u, X_u \rangle\,, \quad g_{12} = g_{21} = \langle X_u, X_v \rangle\,, \quad g_{22} = \langle X_v, X_v \rangle\,,$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^4$. We assume that $W^2 = \det(g_{ij}) = g_{11}g_{22} - g_{12}^2 > 0$, i.e., the surface M is regular. Let $\{e_1, e_2, N_1, N_2\}$ be a local orthonormal frame field on M such that e_1, e_2 are tangent to M and N_1, N_2 are normal to M. The coefficients of the second fundamental form of M with respect to the unit normal vector field $N_i, i = 1, 2$ are given by

$$b_{11}^i = \langle X_{uu}, N_i \rangle\,, \quad b_{12}^i = b_{21}^i = \langle X_{uv}, N_i \rangle\,, \quad b_{22}^i = \langle X_{vv}, N_i \rangle\,.$$

Denote by

(1) $H_i = \frac{b_{11}^i g_{22} - 2b_{12}^i g_{12} + b_{22}^i g_{11}}{2W^2}$ the mean curvature of M with respect to $N_i,\ i = 1, 2$;
(2) $\overrightarrow{H} = H_1 n_1 + H_2 n_2$ the mean curvature vector of M;
(3) $K = \frac{b_{11}^1 b_{22}^1 - (b_{12}^1)^2 + b_{11}^2 b_{22}^2 - (b_{12}^2)^2}{W^2} = \frac{\det(b_{ij}^1) + \det(b_{ij})^2}{W^2}$ the Gauss curvature of M.

A surface M is said to be *minimal* if its mean curvature vector vanishes identically.

Denote $G(2, 4)$ the Grassmann manifold consisting of all oriented 2-planes passing through the origin in $\mathbb{R}^4$. The map $G : M \to G(2, 4)$ that assigns each point of M to its oriented tangent space in $\mathbb{R}^4$ is called the Gauss map of the surface M. We can identify naturally an oriented 2-planes passing through the origin in $\mathbb{R}^4$ with a unit simple 2-vector in $\mathbb{R}^4$, i.e., an element of $\bigwedge^2 \mathbb{R}^4$. The space $\bigwedge^2 \mathbb{R}^4$ is a 6-dimensional Euclidean space with the canonical orthonormal basic $\{\epsilon_i \wedge \epsilon_j : i < j; \quad i, j = 1, 2, 3, 4\}$, where $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \}$ is the canonical basis of $\mathbb{R}^4$. For more details about the space of k-vectors, we refer the reader to Chapter 4 of [11].

We can choose an orthonormal tangent frame field $\{e_1, e_2\}$ on M as follows

$$e_1 = \frac{1}{\sqrt{g_{11}}} X_u, \quad e_2 = \frac{1}{W\sqrt{g_{11}}}(g_{11}X_v - g_{12}X_u).$$

and Gauss map of M can be written as

$$G = \frac{1}{W} X_u \wedge X_v.$$

In the rest of this paper, sometime a vector $(a, b, c, d) \in \mathbb{R}^4$ is identified with its transpose, i.e., a column matrix.

2.2. Helicoidal surfaces in $\mathbb{R}^4$

Let I be an interval in $\mathbb{R}$, Π be a hyperplane in $\mathbb{R}^4$, $P \subset \Pi$ be a 2-plane and $c : I \to \Pi$ be a smooth parametric curve in Π, such that $c(I) \cap P = \emptyset$.

The orbit of the curve c under the rotation around the plane P is a *surface of revolution* in $\mathbb{R}^4$. When c rotates around the plane P, it simultaneously translates along a line l parallel to P in such a way that the speed of the translation is proportional to the speed of the rotation, then the resulting surface is a *helicoidal surface* in $\mathbb{R}^4$.

Let x, y, z, w be the coordinates in $\mathbb{R}^4$. By a suitable change of coordinates, we may assume that Π is the xzw-hyperplane, P is zw-plane, l is parallel to the z-axis. Now suppose that $c(u) = (u,\, 0,\, f(u),\, g(u))$, then a parametrization of the helicoidal surface $\mathcal{H}$ generated by c is

$$\mathcal{H}(u, v) = (u \cos v,\, u \sin v,\, f(u) + av,\, g(u)), \tag{1}$$

where $u > 0$, $0 \leq v < 2\pi$ and $a > 0$. When g is a constant function, then the surface is just a helicoidal surface in $\mathbb{E}^3$.

A direct computation yields

$$\mathcal{H}_u = (\cos v,\, \sin v,\, f',\, g'), \quad \mathcal{H}_v = (-u \sin v,\, u \cos v,\, a,\, 0),$$

and

$$g_{11} = 1 + f'^2 + g'^2, \quad g_{12} = g_{21} = af', \quad g_{22} = u^2 + a^2.$$

We consider the following orthonormal frame field $\{e_1, e_2, N_1, N_2\}$ on $\mathcal{H}$ such that e_1, e_2 are tangent to $\mathcal{H}$ and N_1, N_2 are normal to $\mathcal{H}$

$$e_1 = \frac{1}{\sqrt{g_{11}}} \mathcal{H}_u, \quad e_2 = \frac{1}{W\sqrt{g_{11}}}(g_{11}\mathcal{H}_v - g_{12}\mathcal{H}_u),$$

$$N_1 = \frac{1}{\sqrt{1+g'^2}}(g'\cos v,\ g'\sin v,\ 0,\ -1),$$

$$N_2 = \frac{1}{W\sqrt{1+g'^2}}\begin{bmatrix} a(1+g'^2)\sin v - uf'\cos v \\ -a(1+g'^2)\cos v - uf'\sin v \\ u(1+g'^2) \\ -uf'g' \end{bmatrix},$$

where $W = \det(g_{ij}) = \sqrt{g_{11}g_{22} - g_{12}^2} = \sqrt{(u^2+a^2)(1+g'^2)+u^2f'^2} \neq 0$. The coefficients of the second fundamental form, the Gauss curvature K and the mean curvatures $H_i\ (i=1,2)$ are

$$b_{11}^1 = \frac{-g''}{\sqrt{1+g'^2}}, \qquad b_{12}^1 = 0, \qquad b_{22}^1 = \frac{-ug'}{\sqrt{1+g'^2}},$$

$$b_{11}^2 = \frac{uf''(1+g'^2) - uf'g'g''}{W\sqrt{1+g'^2}}, \quad b_{12}^2 = \frac{-a(1+g'^2)}{W\sqrt{1+g'^2}}, \quad b_{22}^2 = \frac{u^2f'}{W\sqrt{1+g'^2}},$$

(2)
$$K = \frac{(u^2+a^2)ug'g'' + u^3f'f'' - a^2(1+g'^2)}{W^4},$$

$$H_1 = \frac{-(u^2+a^2)g'' - ug'(1+f'^2+g'^2)}{2W^2\sqrt{1+g'^2}},$$

(3)
$$H_2 = \frac{u(u^2+a^2)\left[f''(1+g'^2) - f'g'g''\right] + 2a^2f'(1+g'^2) + u^2f'(1+f'^2+g'^2)}{2W^3\sqrt{1+g'^2}};$$

respectively.

3. Generalized Bour's theorem

The following theorem is a generalization of the classical Bour's theorem.

Theorem 3.1. *In $\mathbb{R}^4$, let $\mathcal{H}$ be a helicoidal surface given by* (1)*. Suppose that $k(u), l(u), u > 0$, are differentiable functions satisfying the following ODE*

(4)
$$k^2 + l^2 = \frac{a^2 + u^2{f'}^2 + (u^2+a^2){g'}^2}{u^2}.$$

Then the helicoidal surface $\mathcal{H}$ is isometric to a family of surfaces of revolution determined by

(5)
$$\mathcal{R}(u,v) = \begin{bmatrix} \sqrt{u^2+a^2}\ \cos\left(v + \int \frac{af'}{u^2+a^2}du\right) \\ \sqrt{u^2+a^2}\ \sin\left(v + \int \frac{af'}{u^2+a^2}du\right) \\ \int \frac{u\,k(u)}{\sqrt{u^2+a^2}}du \\ \int \frac{u\,l(u)}{\sqrt{u^2+a^2}}du \end{bmatrix},$$

in such a way that helices on the helicoidal surface correspond to parallel circles on the surfaces of revolution.

Proof. The first fundamental form of the helicoidal surface (1) is given by

$$ds^2 = (1 + f'^2 + g'^2)\,du^2 + 2af'\,dudv + (u^2 + a^2)\,dv^2.$$

Set $\overline{u} = u, \overline{v} = v + \int \frac{af'}{u^2+a^2}\,du$. Since the Jacobian $\partial(\overline{u},\overline{v})/\partial(u,v)$ is non-zero, it follows that $(\overline{u},\overline{v})$ are new parameters for $\mathcal{H}$. In terms of these new parameters, the first fundamental form becomes

$$ds^2 = \left(1 + \frac{u^2 f'^2}{u^2+a^2} + g'^2\right) d\bar{u}^2 + \left(u^2 + a^2\right) d\bar{v}^2.$$

On the other hand, the first fundamental form of a surface of revolution in $\mathbb{R}^4$

$$\mathcal{R}(w,t) = (w\cos t,\ w\sin t,\ \varphi(w),\ \psi(w)) \tag{6}$$

is

$$ds^2 = (1 + \varphi'^2 + \psi'^2)\,dw^2 + w^2\,dt^2.$$

Set $w = \sqrt{u^2+a^2}$ and $k(u) = \varphi', l(u) = \psi'$, we obtain

$$\varphi = \int \frac{u\,k(u)}{\sqrt{u^2+a^2}}du, \qquad \psi = \int \frac{u\,l(u)}{\sqrt{u^2+a^2}}du,$$

and therefore the required ODE (4) implies that the helicoidal surface (1) is isometric to the surface of revolution (5).

It is easy to see that, a helix on $\mathcal{H}$ is defined by $u = u_0$, where u_0 is a constant, and it corresponds to the curves on $\mathcal{R}$ defined by $w = \sqrt{u_0^2 + a^2}$, i.e., circles on the plane $\{x_3 = \varphi(w),\ x_4 = \psi(w)\}$. □

Next we consider the isometric surfaces in Theorem 3.1 with an additional condition that they have the same Gauss map.

Theorem 3.2. *Let $\mathcal{H}$ and $\mathcal{R}$ be a helicoidal surface and a surface of revolution that are isometrically related by Theorem 3.1. If the surfaces have the same Gauss map, then they are hyperplanar and minimal.*

Proof. Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal basis in $\mathbb{R}^4$ and denote $e_{ij} := e_i \wedge e_j,\ i,j = 1,2,3,4,\ i<j.$

Then, the Gauss map of the helicoidal surface (1) is

$$\begin{aligned} G_{\mathcal{H}} &= \frac{1}{W}(\mathcal{H}_u \wedge \mathcal{H}_v) \\ &= \frac{1}{W}\Big(u\,e_{12} + (a\cos v + uf'\sin v)\,e_{13} + ug'\sin v\,e_{14} \\ &\qquad + (a\sin v - uf'\cos v)\,e_{23} - ug'\cos v\,e_{24} - ag'\,e_{34}\Big), \end{aligned} \tag{7}$$

while the Gauss map of the surface of revolution (5) is

$$G_{\mathcal{R}} = \frac{1}{W}\bigg(u\,e_{12} + uk\sin\left(v + \int \tfrac{af'}{u^2+a^2}du\right) e_{13} + ul\sin\left(v + \int \tfrac{af'}{u^2+a^2}du\right) e_{14}$$

$$- uk\cos\left(v+\int\tfrac{af'}{u^2+a^2}du\right)e_{23} - ul\cos\left(v+\int\tfrac{af'}{u^2+a^2}du\right)e_{24}\bigg). \tag{8}$$

If $G_{\mathcal{H}}$ is identically equal to $G_{\mathcal{R}}$, then (7) and (8) yield

$$a\cos v + uf'\sin v = uk\sin\left(v+\int\frac{af'}{u^2+a^2}du\right), \tag{9}$$

$$a\sin v - uf'\cos v = -uk\cos\left(v+\int\frac{af'}{u^2+a^2}\right), \tag{10}$$

$$ug'\sin v = ul\sin\left(v+\int\frac{af'}{u^2+a^2}du\right), \tag{11}$$

$$-ug'\cos v = -ul\cos\left(v+\int\frac{af'}{u^2+a^2}du\right), \tag{12}$$

$$-ag' = 0. \tag{13}$$

From (11), (12) and (13) we obtain

$$g' = 0, \qquad l = 0;$$

i.e., $\mathcal{H}$ and $\mathcal{R}$ are hyperplanar. We will prove that they are minimal (see [7] for the proof of 3-dimensional case).

Because $l = 0$, $k \neq 0$ by (4). Multiplying (9) by $\cos v$, (10) by $\sin v$ and then summing them up, we obtain

$$a = uk\sin\left(\int\frac{af'}{u^2+a^2}du\right). \tag{14}$$

Multiplying (9) by $\sin v$, (10) by $\cos v$ and then subtracting the first from the later one, we obtain

$$uf' = uk\cos\left(\int\frac{af'}{u^2+a^2}du\right). \tag{15}$$

Thus,

$$\left(\frac{uf'}{a}\right) = \cot\left(\int\frac{af'}{u^2+a^2}du\right),$$

or

$$\cot^{-1}\left(\frac{uf'}{a}\right) = \int\frac{af'}{u^2+a^2}du. \tag{16}$$

Taking the derivative both sides of (16), we obtain

$$(u^2+a^2)uf'' + u^2f'(1+f'^2) + 2a^2f' = 0. \tag{17}$$

The mean curvatures of the helicoidal surface with respect to

$$N_1^{\mathcal{H}} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \quad \text{and} \quad N_2^{\mathcal{H}} = \begin{bmatrix}a\sin v - uf'\cos v\\ -a\cos v - uf'\sin v\\ u\\ 0\end{bmatrix}$$

are

$$H_1^{\mathcal{H}} = 0, \quad \text{and} \quad H_2^{\mathcal{H}} = \frac{(u^2+a^2)uf'' + u^2 f'(1+f'^2) + 2a^2 f'}{2\left(a^2+u^2+u^2f'^2\right)^{3/2}}.$$

The mean curvatures of the surface of revolution with respect to

$$N_1^{\mathcal{R}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and}$$

$$N_2^{\mathcal{R}} = \frac{1}{\sqrt{a^2+u^2+u^2f'^2}} \begin{bmatrix} -\sqrt{a^2+u^2f'^2}\ \cos\left(v + \int \frac{af'}{u^2+a^2}du\right) \\ -\sqrt{a^2+u^2f'^2}\ \sin\left(v + \int \frac{af'}{u^2+a^2}du\right) \\ u \\ 0 \end{bmatrix}$$

are

$$H_1^{\mathcal{R}} = 0 \quad \text{and} \quad H_2^{\mathcal{R}} = \frac{u^2 f'\left[(u^2+a^2)uf'' + u^2f'(1+f'^2) + 2a^2f'\right]}{2\sqrt{u^2+a^2}\sqrt{a^2+u^2f'^2}\left(a^2+u^2+u^2f'^2\right)^{3/2}}.$$

Therefore, both $\mathcal{H}$ and $\mathcal{R}$ are minimal by (17). □

Corollary 3.3. *Let $\mathcal{H}$ and $\mathcal{R}$ be a helicoidal surface and a surface of revolution having the same Gauss map that are isometrically related by Theorem 3.1. Then their parametrizations can be determined explicitly as follows.*

$$\mathcal{H}(u,v) = (u\cos v,\ u\sin v,\ f(u) + av,\ c_1),$$

$$\mathcal{R}(u,v) = \begin{bmatrix} \sqrt{u^2+a^2}\ \cos\left(v + \int \frac{af'}{u^2+a^2}du\right) \\ \sqrt{u^2+a^2}\ \sin\left(v + \int \frac{af'}{u^2+a^2}du\right) \\ b\cosh^{-1}\left(\frac{\sqrt{u^2+a^2}}{b}\right) \\ c_2 \end{bmatrix},$$

where

$$f(u) = \sqrt{b^2-a^2}\,\ln\sqrt{\frac{\sqrt{u^2+a^2}+\sqrt{u^2+a^2-b^2}}{\sqrt{u^2+a^2}-\sqrt{u^2+a^2-b^2}}} - a\arctan\left(\frac{\sqrt{b^2-a^2}}{a}\sqrt{\frac{u^2+a^2}{u^2+a^2-b^2}}\right),$$

and c_1, c_2, b are constants, $b \geq a$ and $u > \sqrt{b^2-a^2}$.

Proof. By the above proof, $\mathcal{H}$ and $\mathcal{R}$ are hyperplanar. Suppose that $\mathcal{H}$ belongs to the hyperplane $w = c_1$ and $\mathcal{R}$ belongs to the hyperplane $w = c_2$. Because $\mathcal{R}$ is minimal, it must be the catenoid and therefore $\varphi(w) = b\cosh^{-1}\left(\frac{w}{b}\right)$, where b is a nonzero constant. We can assume b is positive. Thus,

$$b\cosh^{-1}\left(\frac{\sqrt{u^2+a^2}}{b}\right) = \int\sqrt{\frac{a^2+u^2f'^2}{u^2+a^2}}du.$$

It follows that

$$f' = \frac{\sqrt{b^2 - a^2}\sqrt{u^2 + a^2}}{u\sqrt{u^2 + a^2 - b^2}}. \tag{18}$$

Set $t = \sqrt{\frac{u^2+a^2}{u^2+a^2-b^2}} > 0$, we have

$$\begin{aligned} f &= \sqrt{b^2 - a^2} \int \frac{-b^2 t^2}{(t^2 - 1)((b^2 - a^2)t^2 + a^2)}\, dt \\ &= \sqrt{b^2 - a^2} \ln \sqrt{\frac{t+1}{t-1}} - a \arctan \frac{\sqrt{b^2 - a^2}}{a}\, t. \end{aligned}$$

□

Remark 3.4. If $f' = 0$, then the helicoidal surface reduces to the helicoid and the mean curvature vector of rotation surface is identically zero, i.e., it is the catenoid. In this case, by (18), it follows that $a = b$.

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