Bull. Korean Math. Soc. **54** (2017), No. 6, pp. 2065–2079
https://doi.org/10.4134/BKMS.b160740
pISSN: 1015-8634 / eISSN: 2234-3016

MINIMAL AND MAXIMAL BOUNDED SOLUTIONS FOR QUADRATIC BSDES WITH STOCHASTIC CONDITIONS

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Abstract. This paper is devoted to the minimal and maximal bounded solutions for general time interval quadratic backward stochastic differential equations with stochastic conditions. A general existence result is established by the method of convolution, the exponential transform, Girsanov's transform and a priori estimates, where the terminal time is allowed to be finite or infinite, and the generator g is allowed to have a stochastic semi-linear growth and a general growth in y, and a quadratic growth in z. This improves some existing results at some extent. Some new ideas and techniques are also applied to prove it.

1. Introduction and preliminaries

Throughout this paper, let d be a positive integer and $(B_t)_{t\geq 0}$ a standard d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and $(\mathcal{F}_t)_{t\geq 0}$ the completed natural σ-algebra generated by $(B_t)_{t\geq 0}$ and $\mathcal{F}_T := \mathcal{F}$, where the time terminal $0 < T \leq +\infty$. We consider the following one-dimensional backward stochastic differential equation (BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad t \in [0, T], \tag{1}$$

where the terminal condition ξ is an $\mathcal{F}_T$-measurable random variable, and the generator $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \mapsto \mathbf{R}$ is an $(\mathcal{F}_t)$-progressively measurable random function for each (y, z) and almost everywhere continuous in (y, z). The triple (ξ, T, g) is called the parameters of BSDE (1) and a pair of $(\mathcal{F}_t)$-progressively measurable processes $(y_t, z_t)_{t\in[0,T]}$ satisfying (1) is called a solution of BSDE (1).

Received September 13, 2016; Revised December 5, 2016; Accepted December 26, 2016.
2010 *Mathematics Subject Classification.* 60H10.
Key words and phrases. backward stochastic differential equations, minimal and maximal bounded solutions, stochastic conditions, quadratic growth.
Supported by the National Natural Science Foundation of China (No. 11371362) and the fundamental Research Funds for the Central Universities (No. 2012QNA36).

Since the nonlinear version of finite time interval multidimensional BSDEs with square-integrable parameters was introduced by [18], which established an existence and uniqueness result for square-integrable solutions under the Lipschitz assumption of the generator g, the BSDEs have been extensively investigated with great interest and they have become important tools in many fields such as stochastic games, the optimal control, financial mathematics and PDEs and so on, see [3], [11], [14] etc.

Many works are interested in improving the work of [18] by weakening the assumptions with respect to the parameters (ξ, T, g), see [1-10, 12-13, 15-17, 19] etc. In particular, the general time interval BSDEs are first introduced and investigated in [6], and further developed in [1], [3], [7], [10], and [19] etc. [14] first introduced a quadratic growth condition of the generator g in z and proved the existence, uniqueness and stability of the bounded solutions of BSDEs, and the quadratic BSDEs were further investigated in [2], [4], [5], [8], [9], [10], [12], [16], and so on. Especially, in [4-5] and [8-9] the authors studied the quadratic BSDEs with unbounded terminal value. Very recently, [17] put forward some stochastic Lipschitz conditions of the generator g in (y, z) and established several existence, uniqueness and comparison results of the L^p $(p > 1)$ solutions for general time interval BSDEs.

In light of these works, this paper is devoted to the minimal and maximal bounded solutions of general time interval quadratic BSDEs with some certain stochastic conditions. A general existence result is established, which improves some known results mentioned in the previous paragraph at some extent. The main reason lies in that our conditions for the generator g is allowed to be more general and our terminal time is allowed to be finite or infinite. It should be mentioned that under our weaker assumptions, the usual ODE-based or PDE-based method employed in existing works such as [10], [16] and [13] is not valid any longer, and some new ideas and techniques have been applied to prove our result.

Let us close this section by introducing some notations and definitions. Denote $\mathbf{R}_+ := [0, +\infty)$ and for every positive integer n, denote by $|\cdot|$ the norm of Euclidean space $\mathbf{R}^n$. Let $L^2(\Omega, \mathcal{F}_T, P)$ represent the set of $\mathcal{F}_T$-measurable random variables ξ such that $\mathbf{E}[|\xi|^2] < +\infty$, and $L^\infty(\Omega, \mathcal{F}_T, P)$ the set of $\mathcal{F}_T$-measurable bounded random variables endowed with the infinity norm. And, let $\mathcal{S}^2(0, T; \mathbf{R})$ ($\mathcal{S}^2$ for short) represent the set of real-valued, $(\mathcal{F}_t)$-adapted and continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{\mathcal{S}^2} := \left(\mathbf{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right]\right)^{\frac{1}{2}} < +\infty,$$

and $\mathcal{S}^\infty$ the set of $(\mathcal{F}_t)$-adapted, continuous and bounded processes. Furthermore, let $\mathrm{M}^2(0, T; \mathbf{R}^d)$ (M^2 for short) represent the set of $\mathbf{R}^d$-valued and

$(\mathcal{F}_t)$-progressively measurable processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{\mathrm{M}^2} := \left(\mathbf{E}\left[\int_0^T |Z_t|^2 \mathrm{d}t\right]\right)^{\frac{1}{2}} < +\infty.$$

Clearly, $\mathcal{S}^2$ and M^2 are Banach spaces. Finally, let $L^\infty(\Omega; L^1([0,T]; \mathbf{R}_+))$ and $L^\infty(\Omega; L^2([0,T]; \mathbf{R}_+))$ represent respectively the set of $(\mathcal{F}_t)$-progressively measurable processes $u_t(\omega) : \Omega \times [0,T] \mapsto \mathbf{R}_+$ satisfying

$$\left\| \int_0^T u_s(\omega)\mathrm{d}s \right\|_\infty < +\infty \quad \text{and} \quad \left\| \int_0^T u_s^2(\omega)\mathrm{d}s \right\|_\infty < +\infty.$$

Definition 1. A pair of $(\mathcal{F}_t)$-progressively measurable processes $(y_\cdot, z_\cdot)$ is called a bounded (L^2) solution of BSDE (1) if $(y_\cdot, z_\cdot) \in \mathcal{S}^\infty \times \mathrm{M}^2$ ($\mathcal{S}^2 \times \mathrm{M}^2$) and satisfies BSDE (1).

Definition 2. We call $(y_t, z_t)_{t\in[0,T]}$ the minimal (resp. maximal) bounded solution of BSDE (1) if it is a bounded solution of BSDE (1) and $y_\cdot \le y'_\cdot$ (resp. $y_\cdot \ge y'_\cdot$) for any bounded solution $(y'_\cdot, z'_\cdot)$ of BSDE (1). Similarly, we can define the minimal (resp. maximal) L^2 solution of BSDE (1).

2. Main results and its proof

The following Theorem 1 is the main result of this paper which generalizes, at some extent, some existing results of bounded solutions obtained in [16] and [10] respectively. In stating it, we need the following assumptions on the generator g, where $0 < T \le +\infty$.

(H1) There exist two stochastic processes $f_\cdot, u_\cdot \in L^\infty(\Omega; L^1([0,T]; \mathbf{R}_+))$ and a continuous function $h(\cdot) : \mathbf{R} \to \mathbf{R}_+$ such that $\mathrm{d}P \times \mathrm{d}t - a.e.$, for each $y \in \mathbf{R}$ and $z \in \mathbf{R}^d$,

$$\mathrm{sgn}(y) g(\omega, t, y, z) \le f_t(\omega) + u_t(\omega)|y| + h(y)|z|^2.$$

(H2) There exist a stochastic process $v_\cdot \in L^\infty(\Omega; L^1([0,T]; \mathbf{R}_+))$ and two continuous functions $\psi(\cdot), \varphi(\cdot) : \mathbf{R} \mapsto \mathbf{R}_+$ such that $\mathrm{d}P \times \mathrm{d}t - a.e.$, for each $y \in \mathbf{R}$ and $z \in \mathbf{R}^d$,

$$|g(\omega, t, y, z)| \le v_t(\omega)\psi(y) + \varphi(y)|z|^2.$$

Remark 1. By Example 3.1 in [17] it is clear that (H1) and (H2) are respectively weaker than the corresponding assumptions (3A1) with $l(y) := 1+|y|$ and (3A2) in [10].

Theorem 1. *Let $0 < T \le +\infty$, $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$ and the generator g satisfy* (H1) *and* (H2). *Then BSDE (ξ, T, g) admits both a minimal and a maximal solution among all bounded solutions $(y_\cdot, z_\cdot)$. Moreover, for each $t \in [0,T]$, we have*

$$\mathrm{d}P - a.s., \quad |y_t| \le \left(\|\xi\|_\infty + \left\|\int_0^T f_t \,\mathrm{d}t\right\|_\infty\right) e^{\|\int_0^T u_t \,\mathrm{d}t\|_\infty}.$$

As said in the introduction, under our weaker assumptions (H1) and (H2), the usual PDE-based or ODE-based method applied in [16] and [10] is not valid any longer (see Remark 3 of this paper for details), we need to develop and use some new ideas and techniques to prove Theorem 1.

We shall establish the following four lemmas. Firstly, in the same way as in Lemma 2.4 of [19], we can establish the following a priori estimate of L^2 solutions, which will play an important role in the proof of the existence of bounded solutions. The proof is standard, we omit it here.

Lemma 1. *Let $0 < T \le +\infty$, $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, g is a generator of BSDE, and $(y_t, z_t)_{t\in[0,T]}$ is an L^2 solution of BSDE (ξ, T, g). If g satisfies the following assumption*:

(A) *There exist three $(\mathcal{F}_t)$-progressively measurable non-negative processes $\mu_\cdot \in L^\infty(\Omega; L^1([0,T]; \mathbf{R}_+))$, $\lambda_\cdot \in L^\infty(\Omega; L^2([0,T]; \mathbf{R}_+))$ and $(f_t)_{t\in[0,T]}$ with $\mathbf{E}\Big[(\int_0^T f_t\,\mathrm{d}t)^2\Big] < +\infty$ such that $\mathrm{d}P \times \mathrm{d}t - a.e.$, for each $y \in \mathbf{R}$ and $z \in \mathbf{R}^d$,*

$$\mathrm{sgn}(y) g(\omega, t, y, z) \le f_t(\omega) + \mu_t(\omega)|y| + \lambda_t(\omega)|z|.$$

Then there exists a constant $C > 0$ depending only on

$$\Big\{\Big\| \int_0^T \mu_t\,\mathrm{d}t\Big\|_\infty + \Big\| \int_0^T \lambda_t^2\,\mathrm{d}t\Big\|_\infty\Big\}$$

such that for each $0 \le r \le t \le T$, we have, $\mathrm{d}P - a.s.$,

$$\mathbf{E}\left[\sup_{s\in[t,T]} |y_s|^2 + \int_t^T |z_s|^2\,\mathrm{d}s \middle| \mathcal{F}_r\right] \le C\mathbf{E}\left[|\xi|^2 + \left(\int_t^T f_s\,\mathrm{d}s\right)^2 \middle| \mathcal{F}_r\right].$$

By virtue of Itô's formula we can prove the following Lemma 2, which is a sharp a priori estimate of bounded solutions and will be used several times later.

Lemma 2. *Let $0 < T \le +\infty$, $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$, the generator g satisfy* (H1), *and $(y_\cdot, z_\cdot)$ be a bounded solution of BSDE (ξ, T, g). Then for each $t \in [0, T]$, $\mathrm{d}P - a.s.$,*

$$|y_t| \le \left(\|\xi\|_\infty + \left\| \int_0^T f_t\,\mathrm{d}t \right\|_\infty\right) e^{\| \int_0^T u_t\,\mathrm{d}t \|_\infty}. \tag{2}$$

Proof. The proof is inspired by Proposition 1 in [5]. Suppose that g satisfies (H1) with $f_\cdot$, $u_\cdot$ and $h(\cdot)$, $(y_\cdot, z_\cdot)$ is a bounded solution of BSDE (ξ, T, g) and $\mathrm{d}P \times \mathrm{d}t - a.e.$, $|y_\cdot| \le C$ with $C \in \mathbf{R}_+$. First of all, we show that $\int_0^\cdot z_s \cdot \mathrm{d}B_s$ is a BMO-martingale, which is equivalent to

$$\sup_{\tau\in\Sigma_T} \left\| \mathbf{E}\left[\int_\tau^T |z_s|^2 \mathrm{d}s \middle| \mathcal{F}_\tau\right]\right\|_\infty < +\infty, \tag{3}$$

where Σ_T represents the set of all $(\mathcal{F}_t)$-stopping time τ such that $\tau \le T$. In fact, for each $\tau \in \Sigma_T$, applying Itô's formula to $e^{3K|y_\cdot|}$ with $K := \max_{|y|\le C} h(y)$

on $[\tau, T]$ and then taking the conditional expectation with respect to $\mathcal{F}_\tau$ yield that $\mathrm{d}P - a.s.$,

$$\frac{1}{2}\mathbf{E}\left[\int_\tau^T 9K^2 e^{3K|y_s|}|z_s|^2\mathrm{d}s\middle|\mathcal{F}_\tau\right]$$
$$\leq \mathbf{E}\left[e^{3K|\xi|}\middle|\mathcal{F}_\tau\right] + \mathbf{E}\left[\int_\tau^T 3Ke^{3K|y_s|}\mathrm{sgn}(y_s)g(s, y_s, z_s)\mathrm{d}s\middle|\mathcal{F}_\tau\right].$$

Then, from the assumptions of $y.$ and g we can deduce that, for each $\tau \in \Sigma_T$, $\mathrm{d}P - a.s.$,

$$\frac{3}{2}K^2\mathbf{E}\left[\int_\tau^T e^{3K|y_s|}|z_s|^2\mathrm{d}s\middle|\mathcal{F}_\tau\right]$$
$$\leq e^{3KC} + 3Ke^{3KC}\mathbf{E}\left[\int_\tau^T (f_s + Cu_s)\mathrm{d}s\middle|\mathcal{F}_\tau\right]$$
$$\leq e^{3KC} + 3Ke^{3KC}\left\{\left\|\int_0^T f_t\mathrm{d}t\right\|_\infty + C\left\|\int_0^T u_t\mathrm{d}t\right\|_\infty\right\},$$

from which the desired conclusion (3) follows immediately.

In the sequel, let us fix a $t \in [0, T]$ arbitrarily and for each $\bar{t} \in [t, T]$ and $x \in \mathbf{R}$, denote

$$\phi(\bar{t}, x) := xe^{\int_t^{\bar{t}} u_s\,\mathrm{d}s} + \int_t^{\bar{t}} f_s e^{\int_t^s u_r\,\mathrm{d}r}\,\mathrm{d}s.$$

Applying Itô's formula and Tanaka's formula to $e^{\phi(\cdot,|y_\cdot|)}$ on $[t, T]$ yields that

$$e^{\phi(t,|y_t|)} + \frac{1}{2}\int_t^T e^{\phi(s,|y_s|)}e^{2\int_t^s u_r\,\mathrm{d}r}|z_s|^2\mathrm{d}s$$
$$\leq e^{\phi(T,|\xi|)} + \int_t^T e^{\phi(s,|y_s|)}e^{\int_t^s u_r\,\mathrm{d}r}[\mathrm{sgn}(y_s)g(s, y_s, z_s) - f_s - u_s|y_s|]\mathrm{d}s$$
$$- \int_t^T e^{\phi(s,|y_s|)}e^{\int_t^s u_r\mathrm{d}r}\mathrm{sgn}(y_s)z_s \cdot \mathrm{d}B_s,$$

which means, in view of (H1),

$$e^{\phi(t,|y_t|)} \leq e^{\phi(T,|\xi|)} + \int_t^T Ke^{\phi(s,|y_s|)}e^{\int_t^s u_r\,\mathrm{d}r}|z_s|^2\mathrm{d}s$$
$$- \int_t^T e^{\phi(s,|y_s|)}e^{\int_t^s u_r\mathrm{d}r}\mathrm{sgn}(y_s)z_s \cdot \mathrm{d}B_s$$
$$= e^{\phi(T,|\xi|)} - \int_t^T e^{\phi(s,|y_s|)}e^{\int_t^s u_r\mathrm{d}r}\mathbf{1}_{s>t}\mathrm{sgn}(y_s)z_s \cdot \left[\mathrm{d}B_s - K\mathrm{sgn}(y_s)z_s\mathrm{d}s\right]. \tag{4}$$

Furthermore, it follows from (3) that the process

$$M_{\bar{t}} := \int_0^{\bar{t}} K\mathrm{sgn}(y_s)z_s \cdot \mathrm{d}B_s, \quad \bar{t} \in [0, T]$$

is a BMO-martingale under P. Then in view of Theorem 2.3 in [13], the stochastic exponential $\mathcal{E}(M)$ of M is a uniformly integrable martingale, where the stochastic exponential is denoted by

$$\mathcal{E}(M)(\bar{t}) = \exp\left(M(\bar{t}) - \frac{1}{2}\langle M\rangle_{\bar{t}}\right), \quad \bar{t} \in [0,T]$$

and the quadratic variation is denoted by $\langle M\rangle$. Now, let us define by Q the probability measure under $(\Omega, \mathcal{F}_T)$ given by $\frac{\mathrm{d}Q}{\mathrm{d}P} := \mathcal{E}(M)(T)$. Then, noticing by (3) that

$$\overline{M}_{\bar{t}} := \int_0^{\bar{t}} e^{\phi(s,|y_s|)} e^{\int_t^s u_r \mathrm{d}r} \mathbf{1}_{s>t} \mathrm{sgn}(y_s) z_s \cdot \mathrm{d}B_s, \quad \bar{t} \in [0,T]$$

is also a BMO-martingale under P, by Theorem 3.6 in [13] we know that the process

$$\int_0^{\bar{t}} e^{\phi(s,|y_s|)} e^{\int_t^s u_r \mathrm{d}r} \mathbf{1}_{s>t} \mathrm{sgn}(y_s) z_s \cdot \left[\mathrm{d}B_s - K\mathrm{sgn}(y_s) z_s \mathrm{d}s\right], \quad \bar{t} \in [0,T],$$

the Girsanov's transform of $\overline{M}$, is a BMO-martingale under Q. Thus, taking the conditional expectation with respect to $\mathcal{F}_t$ under Q in (4) yields that for each $t \in [0,T]$, $\mathrm{d}P - a.s.$,

$$\begin{aligned} \exp\{|y_t|\} &= \mathbf{E}_Q\left[e^{\phi(t,|y_t|)}\middle|\mathcal{F}_t\right] \leq \mathbf{E}_Q\left[e^{\phi(T,|\xi|)}\middle|\mathcal{F}_t\right] \\ &\leq \exp\left\{\left(\|\xi\|_\infty + \left\|\int_0^T f_s\,\mathrm{d}s\right\|_\infty\right) e^{\|\int_0^T u_s\,\mathrm{d}s\|_\infty}\right\}, \end{aligned} \tag{5}$$

from which (2) follows. Then the proof is complete. □

Remark 2. By Lemma 2, an important observation is that the first part $y.$ of the bounded solution admits a bound which is independent of the function $h(\cdot)$ in (H1). This fact will be utilized later.

By virtue of convolution, Itô's formula, Girsanov's transform and Lemmas 1-2, we can prove the following Lemma 3.

Lemma 3. *Let* $0 < T \leq +\infty$ *and* $\eta \in L^\infty(\Omega, \mathcal{F}_T, P)$*, and* $\mathrm{d}P - a.s.$*,* $0 < \alpha \leq \eta \leq \beta$ *with* $\alpha, \beta \in \mathbf{R}_+$*. Assume that the generator* $\bar{g}$ *satisfies for a stochastic process* $u. \in L^\infty(\Omega; L^1([0,T]; \mathbf{R}_+))$ *and a constant* $k > 0$ *the following restriction*:

(6)

$$\mathrm{d}P \times \mathrm{d}t - a.e., \forall y \in \mathbf{R} \text{ and } z \in \mathbf{R}^d,\ -u_t(\omega)|y| - k|z|^2 \leq \bar{g}(\omega, t, y, z) \leq u_t(\omega)|y|.$$

Then BSDE $(\eta, T, \bar{g})$ *admits a maximal bounded solution. Moreover, for any bounded solution* $(y., z.)$ *of BSDE* $(\eta, T, \bar{g})$ *we have for each* $t \in [0,T]$*,*

$$\mathrm{d}P - a.s., \quad S_0 \leq y_t \leq Q_0, \tag{7}$$

where

$$S_0 := \alpha e^{-\|\int_0^T u_t \mathrm{d}t\|_\infty} \quad \text{and} \quad Q_0 := \beta e^{\|\int_0^T u_t \mathrm{d}t\|_\infty}.$$

Proof. Suppose that $\eta \in L^\infty(\Omega, \mathcal{F}_T, P)$, $\mathrm{d}P - a.s.$, $0 < \alpha \le \eta \le \beta$ and (6) holds true. In view of (6), for each $n \ge 1$ the following $(\mathcal{F}_t)$-progressively measurable stochastic process is well defined:

$$\bar{g}_n(\omega, t, y, z) := \sup_{(u,v)\in\mathbf{R}^{1+d}} \{\bar{g}(\omega, t, u, v) - nu_t(\omega)|y-u| - ne^{-t}|z-v|\},$$
$$(\omega, t, y, z) \in \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d.$$

By a similar argument to that in [15], we can conclude that for each $n \ge 1$, $\mathrm{d}P \times \mathrm{d}t - a.e.$, the function $\bar{g}_n$ satisfies

(A1) $\forall\ y \in \mathbf{R}$ and $z \in \mathbf{R}^d, -u_t(\omega)|y| - k|z|^2 \le \bar{g}_n(\omega, t, y, z) \le u_t(\omega)|y|$;
(A2) $\forall\ y \in \mathbf{R}$ and $z \in \mathbf{R}^d$, $\bar{g}_n(\omega, t, y, z)$ is decreasing in n;
(A3) $\forall\ y_1, y_2 \in \mathbf{R}$ and $z_1, z_2 \in \mathbf{R}^d$, $|\bar{g}_n(\omega, t, y_1, z_1) - \bar{g}_n(\omega, t, y_2, z_2)| \le nu_t(\omega)|y_1 - y_2| + ne^{-t}|z_1 - z_2|$;
(A4) If $(y_n, z_n) \to (y, z)$, then $\bar{g}_n(\omega, t, y_n, z_n) \to \bar{g}(\omega, t, y, z)$, as $n \to +\infty$.

Now, in view of (A1) and (A3), it follows from Theorem 3.1 in [17] that for each $n \ge 1$, the following BSDE

$$\theta_t^n = \eta + \int_t^T \bar{g}_n(s, \theta_s^n, \Gamma_s^n)\mathrm{d}s - \int_t^T \Gamma_s^n \cdot \mathrm{d}B_s, \quad t \in [0,T]$$

admits a unique L^2 solution $(\theta_t^n, \Gamma_t^n)_{t\in[0,T]}$ and, in view of (A2), $\theta_\cdot^n$ is decreasing by Theorem 4.1 in [17]. Furthermore, it follows from (A1) and (A3) that for each $n \ge 1$, $\mathrm{d}P \times \mathrm{d}t - a.e.$, for each $y \in \mathbf{R}$ and $z \in \mathbf{R}^d$,

$$|\bar{g}_n(\omega, t, y, z)| \le nu_t(\omega)|y| + ne^{-t}|z|.$$

It then follows from Lemma 1 that there exists a constant $C_n > 0$ depending only on n and $\| \int_0^T u_t \,\mathrm{d}t\|_\infty$ such that for each $n \ge 1$ and $t \in [0,T]$, $\mathrm{d}P - a.s.$,

$$|\theta_t^n|^2 \le \mathbf{E}\left[\sup_{s\in[t,T]} |\theta_s^n|^2 \middle| \mathcal{F}_t\right] \le C_n \|\eta\|_\infty^2 < +\infty,$$

from which it follows that $(\theta_\cdot^n, \Gamma_\cdot^n)$ is a bounded solution of BSDE $(\eta, T, \bar{g}_n)$.

In the sequel, it follows from (A1) that $\bar{g}_n$ satisfies (H1) with $f_\cdot = 0$, $u_\cdot$ and $h(\cdot) = k$ for each $n \ge 1$. Note that $(\theta_\cdot^n, \Gamma_\cdot^n)$ is a bounded solution of BSDE $(\eta, T, \bar{g}_n)$ for each $n \ge 1$. By Lemma 2 (see also Remark 2 for details) we can conclude that for each $n \ge 1$ and $t \in [0,T]$,

$$\mathrm{d}P - a.s., \quad |\theta_t^n| \le \|\eta\|_\infty e^{\| \int_0^T u_t \,\mathrm{d}t\|_\infty} \le Q_0. \tag{8}$$

On the other hand, in view of the facts that for each $t \in [0,T]$, $\mathrm{d}P - a.s.$, $\alpha e^{-\int_0^T u_s \,\mathrm{d}s} \in L^\infty(\Omega, \mathcal{F}_T, P)$ and $\left| -ke^{\int_0^t u_s \,\mathrm{d}s}\right| \le ke^{\| \int_0^T u_s \,\mathrm{d}s\|_\infty}$, by [14], the following BSDE

$$\widetilde{y}_t = \alpha e^{-\int_0^T u_s \,\mathrm{d}s} - \int_t^T ke^{\int_0^s u_r \,\mathrm{d}r}|\widetilde{z}_s|^2 \,\mathrm{d}s - \int_t^T \widetilde{z}_s \cdot \mathrm{d}B_s, \quad t \in [0,T] \tag{9}$$

admits a bounded solution $(\widetilde{y}_\cdot, \widetilde{z}_\cdot)$. We define $y'_\cdot := e^{\int_0^\cdot u_s\,\mathrm{d}s}\widetilde{y}_\cdot$ and $z'_\cdot := e^{\int_0^\cdot u_s\,\mathrm{d}s}\widetilde{z}_\cdot$. From Itô's formula we can check that $(y'_\cdot, z'_\cdot)$ solves the following BSDE

$$y'_t = \alpha - \int_t^T (u_s y'_s + k|z'_s|^2)\,\mathrm{d}s - \int_t^T z'_s \cdot \mathrm{d}B_s, \quad t \in [0,T].$$

And, in view of (9), it follows from the proof of Lemma 2 that $\int_0^\cdot z'_s \cdot \mathrm{d}B_s$ is a BMO-martingale. Thus, by a similar argument to that from (4) to (5) in Lemma 2, taking the conditional expectation with respect to $\mathcal{F}_t$ in (9) under a new probability measure $\widetilde{Q}$ on $(\Omega, \mathcal{F}_T)$, which is equivalent to P, yields that for each $t \in [0,T]$, $\mathrm{d}P - a.s.$,

$$\widetilde{y}_t = \mathbf{E}_{\widetilde{Q}}\left[\alpha e^{-\int_0^T u_s\,\mathrm{d}s}\middle|\mathcal{F}_t\right] \geq \alpha e^{-\|\int_0^T u_s\,\mathrm{d}s\|_\infty} > 0,$$

which means that $(y'_\cdot, z'_\cdot)$ also solves BSDE

$$y'_t = \alpha - \int_t^T (u_s|y'_s| + k|z'_s|^2)\,\mathrm{d}s - \int_t^T z'_s \cdot \mathrm{d}B_s, \quad t \in [0,T]. \tag{10}$$

Moreover, in view of (A1), (A3), $\alpha < \eta$ and the fact that $(\theta^n_\cdot, \Gamma^n_\cdot)$ and $(y'_\cdot, z'_\cdot)$ are, respectively, an L^2 solution of BSDE $(\eta, T, \bar{g}_n)$ and BSDE (10), by Theorem 4.1 in [17], we know that for each $n \geq 1$ and $t \in [0,T]$, $\mathrm{d}P - a.s.$,

$$\theta^n_t \geq y'_t = e^{\int_0^t u_s\,\mathrm{d}s}\widetilde{y}_t = \mathbf{E}_{\widetilde{Q}}\left[\alpha e^{-\int_t^T u_s\,\mathrm{d}s}\middle|\mathcal{F}_t\right] \geq S_0. \tag{11}$$

Thus, combining (8)-(11) yields that for each $n \geq 1$ and $t \in [0,T]$,

$$\mathrm{d}P - a.s., \quad S_0 \leq \theta^n_t \leq Q_0. \tag{12}$$

Now, in view of (A1) and (12), we know that $\mathrm{d}P \times \mathrm{d}t - a.e.$,

$$\forall y \in [S_0, Q_0] \text{ and } z \in \mathbf{R}^d, \quad |g_n(\omega, t, y, z)| \leq u_t(\omega)Q_0 + k|z|^2.$$

Thus, in view of the facts that $\theta^n_\cdot$ is decreasing and (A4), following closely the proof procedure of Theorem 2 in [16], we can deduce that $(\theta^n_\cdot, \Gamma^n_\cdot)$ converges to a bounded solution $(\theta_\cdot, \Gamma_\cdot)$ of BSDE $(\eta, T, \bar{g})$, and $S_0 \leq \theta_\cdot \leq Q_0$.

Next, we show that $(\theta_\cdot, \Gamma_\cdot)$ is just the maximal bounded solution of BSDE $(\eta, T, \bar{g})$. In fact, for any bounded solution $(\bar{\theta}_\cdot, \bar{\Gamma}_\cdot)$ of BSDE $(\eta, T, \bar{g})$ which is also an L^2 solution, in view of (A3) and the fact that $(\theta^n_\cdot, \Gamma^n_\cdot)$ is the unique L^2 solution of BSDE $(\eta, T, \bar{g}_n)$ and $\bar{g} \leq \bar{g}_n$, by Theorem 4.1 in [17] again we have that for each $t \in [0,T]$ and $n \geq 1$, $\bar{\theta}_t \leq \theta^n_t$, $\mathrm{d}P - a.s.$, and then $\bar{\theta}_\cdot \leq \theta_\cdot$.

Finally, we show that (7) holds true for any bounded solution $(y_\cdot, z_\cdot)$ of BSDE $(\eta, T, \bar{g})$. In fact, set $\hat{y}_\cdot = y'_\cdot - y_\cdot$, $\hat{z}_\cdot = z'_\cdot - z_\cdot$. In view of (10) and the fact that $\mathrm{d}P - a.s.$, $(\alpha - \eta)^+ = 0$, Tanaka's formula yields that

(13)

$$\hat{y}^+_t e^{\int_0^t u_s\,\mathrm{d}s} \leq \int_t^T e^{\int_0^s u_r\,\mathrm{d}r}[\mathbf{1}_{\hat{y}_s > 0}(-u_s|y'_s| - k|z'_s|^2 - \bar{g}(s, y_s, z_s)) - u_s\hat{y}^+_s]\,\mathrm{d}s$$

$$-\int_t^T \mathbf{1}_{\hat{y}_s>0} e^{\int_0^s u_r\,\mathrm{d}r} \hat{z}_s \cdot \mathrm{d}B_s, \quad t\in[0,T].$$

In view of (6), we have, $\mathrm{d}P\times\mathrm{d}s - a.e.$,

$$\begin{aligned}
&\mathbf{1}_{\hat{y}_s>0}(-u_s|y'_s| - k|z'_s|^2 - \bar{g}(s,y_s,z_s)) \\
&\le \mathbf{1}_{\hat{y}_s>0}[-u_s|y'_s| - k|z'_s|^2 - (-u_s|y_s| - k|z_s|^2) \\
&\quad + (-u_s|y_s| - k|z_s|^2 - \bar{g}(s,y_s,z_s))] \\
&\le \mathbf{1}_{\hat{y}_s>0}(u_s(|y_s| - |y'_s|) + k(|z_s|^2 - |z'_s|^2)) \\
&\le u_s\hat{y}_s^+ + k\mathbf{1}_{\hat{y}_s>0}(|z_s| + |z'_s|)|\hat{z}_s|.
\end{aligned} \tag{14}$$

Thus, combining (13) with (14) yields that for each $t\in[0,T]$,

$$\begin{aligned}
\hat{y}_t^+ e^{\int_0^t u_s\,\mathrm{d}s} &\le \int_t^T k\mathbf{1}_{\hat{y}_s>0} e^{\int_0^s u_r\,\mathrm{d}r}(|z_s| + |z'_s|)|\hat{z}_s|\,\mathrm{d}s - \int_t^T \mathbf{1}_{\hat{y}_s>0} e^{\int_0^s u_r\,\mathrm{d}r}\hat{z}_s\cdot\mathrm{d}B_s \\
&= \int_t^T \mathbf{1}_{\hat{y}_s>0} e^{\int_0^s u_r\,\mathrm{d}r}\hat{z}_s\cdot\left[\mathrm{d}B_s - \frac{k(|z_s| + |z'_s|)\hat{z}_s}{|\hat{z}_s|}\mathbf{1}_{|\hat{z}_s|\neq 0}\,\mathrm{d}s\right].
\end{aligned} \tag{15}$$

Note by Lemma 2 that $\int_0^\cdot z_s\cdot\mathrm{d}B_s$ and $\int_0^\cdot z'_s\cdot\mathrm{d}B_s$ are both BMO-martingales. We have

$$\begin{aligned}
&\sup_{\tau\in\Sigma_T}\left\|\mathbf{E}\left[\int_\tau^T k^2(|z_s| + |z'_s|)^2\mathrm{d}s\middle|\mathcal{F}_\tau\right]\right\|_\infty \\
&\le 2k^2\sup_{\tau\in\Sigma_T}\left\|\mathbf{E}\left[\int_\tau^T |z_s|^2\mathrm{d}s\middle|\mathcal{F}_\tau\right]\right\|_\infty + 2k^2\sup_{\tau\in\Sigma_T}\left\|\mathbf{E}\left[\int_\tau^T |z'_s|^2\mathrm{d}s\middle|\mathcal{F}_\tau\right]\right\|_\infty < +\infty,
\end{aligned}$$

which means that for each $n\ge 1$, the process

$$\int_0^t \frac{k(|z_s| + |z'_s|)\hat{z}_s}{|\hat{z}_s|}\mathbf{1}_{|\hat{z}_s|\neq 0}\cdot\mathrm{d}B_s, \quad t\in[0,T]$$

is a BMO-martingale under P. Then by a similar argument again to that from (4) to (5) in Lemma 2 and taking the conditional expectation with respect to $\mathcal{F}_t$ in (15) under a new probability measure $\bar{Q}$ on $(\Omega,\mathcal{F}_T)$ which is equivalent to P, we can conclude that for each $t\in[0,T]$, $\mathrm{d}P - a.s.$, $\hat{y}_t^+\le 0$. Consequently, in view of (11), (12) and the fact that $(\theta_\cdot,\Gamma_\cdot)$ is the maximal bounded solution of BSDE $(\eta,T,\bar{g})$, we have for each $t\in[0,T]$, $\mathrm{d}P - a.s.$,

$$S_0 \le y'_t \le y_t \le \theta_t \le Q_0.$$

The proof of Lemma 3 is finally completed. □

Remark 3. An essential contribution of Lemma 3 lies in that S_0 and Q_0 in (7) do not depend on the constant k defined in (6). This fact will play a key role in the proof of the following Lemma 4, which is an important step to prove our main result — Theorem 1. On the other hand, we also especially point out that these two bounds S_0 and Q_0 in (7) cannot be obtained by the usual

ODE-based or PDE-based method employed in [16] and [10], because Lemma 3 in [10] or Lemma 3.2 in [16] is not valid any longer when the $u_t(\omega)$ is not a deterministic process. This is the main difficulty overcome in this paper.

By virtue of Lemma 3 we can prove the following Lemma 4, which is an existence result of the minimal and maximal bounded solutions. It improves the corresponding conclusion of Lemma 3.4 in [10], where $\bar{u}_t(\omega)$ in the following (16) is a deterministic process.

Lemma 4. *Let $0 < T \le +\infty$, $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$ and g be a generator. Assume that there exists a stochastic process $\bar{u}_\cdot \in L^\infty(\Omega; L^1([0,T]; \mathbf{R}_+))$ and a constant $\gamma > 0$ such that $\mathrm{d}P \times \mathrm{d}t - a.e.$, for each $y \in \mathbf{R}$ and $z \in \mathbf{R}^d$,*

$$|g(\omega,t,y,z)| \le \bar{u}_t(\omega) + \frac{\gamma}{2}|z|^2. \tag{16}$$

Then BSDE (ξ, T, g) admits both a minimal and a maximal solution among all bounded solutions $(y_\cdot, z_\cdot)$. Furthermore, for each $t \in [0,T]$, we have

$$\mathrm{d}P - a.s., \quad |y_t| \le \|\xi\|_\infty + \left\| \int_0^T \bar{u}_t \,\mathrm{d}t \right\|_\infty. \tag{17}$$

Proof. First of all, let us show the existence of the maximal bounded solution. Let

$$\begin{aligned} &\eta := e^{\gamma\xi} \in L^\infty(\Omega, \mathcal{F}_T, P), \\ &S_0^\gamma := e^{-\gamma(\|\xi\|_\infty + \|\int_0^T \bar{u}_t \mathrm{d}t\|_\infty)}, \\ &Q_0^\gamma := e^{\gamma(\|\xi\|_\infty + \|\int_0^T \bar{u}_t \mathrm{d}t\|_\infty)}. \end{aligned}$$

For each $(\omega,t,y,z) \in \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d$, define

$$G(\omega,t,y,z) := \left(\gamma y g\left(\omega, t, \frac{\ln y}{\gamma}, \frac{z}{\gamma y}\right) - \frac{|z|^2}{2y} \right) \cdot \mathbf{1}_{y>0}.$$

It then follows from (16) that $\mathrm{d}P \times \mathrm{d}t - a.e.$,

$$\forall\, y \in \mathbf{R} \text{ and } z \in \mathbf{R}^d,\ -\gamma \bar{u}_t(\omega)|y| - \frac{|z|^2}{|y|}\mathbf{1}_{|y|\neq 0} \le G(\omega,t,y,z) \le \gamma \bar{u}_t(\omega)|y|. \tag{18}$$

Furthermore, for each $(\omega,t,y,z) \in \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d$, define

$$G_\Psi(\omega,t,y,z) := \Psi(y) G(\omega,t,y,z),$$

where $\Psi : \mathbf{R} \mapsto [0,1]$ is a smooth function satisfying that

1) $0 \le \Psi \le 1$; 2) $\Psi(x) = 1$ if $x \in [S_0^\gamma, Q_0^\gamma]$; 3) $\Psi(x) = 0$ if $x \notin [S_0^\gamma/2, 2Q_0^\gamma]$.

Then in view of (18), $\mathrm{d}P \times \mathrm{d}t - a.e.$, we have

$$\forall\, y \in \mathbf{R} \text{ and } z \in \mathbf{R}^d, \quad -\gamma \bar{u}_t(\omega)|y| - \frac{2}{S_0^\gamma}|z|^2 \le G_\Psi(\omega,t,y,z) \le \gamma \bar{u}_t(\omega)|y|.$$

Thus, in view of $0 < e^{-\gamma\|\xi\|_\infty} \leq \eta \leq e^{\gamma\|\xi\|_\infty}$, it follows from Lemma 3 (see also Remark 3 for details) that BSDE (η, T, G_Ψ) admits a maximal bounded solution $(Y^\Psi_\cdot, Z^\Psi_\cdot)$ such that for each $t \in [0,T]$,

$$S_0^\gamma \leq Y_t^\Psi \leq Q_0^\gamma, \quad \mathrm{d}P - a.s.,$$

which means, in view of the definitions of functions Ψ and G_Ψ, that $(Y^\Psi_\cdot, Z^\Psi_\cdot)$ is a bounded solution of BSDE (η, T, G). Furthermore, it follows from Itô's formula that $(y^\Psi_\cdot, z^\Psi_\cdot)$ with

$$y^\Psi_\cdot := \frac{\ln Y^\Psi_\cdot}{\gamma} \quad \text{and} \quad z^\Psi_\cdot := \frac{Z^\Psi_\cdot}{\gamma Y^\Psi_\cdot}$$

is a bounded solution of BSDE (ξ, T, g) and $y^\Psi_\cdot$ satisfies (17).

In the sequel, we show that $(y^\Psi_\cdot, z^\Psi_\cdot)$ is also the maximal bounded solution of BSDE (ξ, T, g). In fact, let $(\bar{y}_\cdot, \bar{z}_\cdot)$ be any bounded solution of BSDE (ξ, T, g), with $|\bar{y}_\cdot| \leq B$ for some constant $B > 0$. Define $G_{\bar{\Psi}}(\omega, t, y, z) := \bar{\Psi}(y) G(\omega, t, y, z)$, where $\bar{\Psi} : \mathbf{R} \mapsto [0,1]$ is a smooth function satisfying that

1) $0 \leq \bar{\Psi} \leq 1$; 2) $\bar{\Psi}(x) = 1$ if $x \in [e^{-\gamma B}, e^{\gamma B}]$; 3) $\bar{\Psi}(x) = 0$ if $x \notin [e^{-\gamma B}/2, 2e^{\gamma B}]$.

It follows from Itô's formula that $(\bar{Y}_\cdot := e^{\gamma \bar{y}_\cdot}, \bar{Z}_\cdot := \gamma \bar{Y}_\cdot \bar{z}_\cdot)$ is a bounded solution of BSDE (η, T, G) and then, in view of $0 < e^{-\gamma B} \leq \bar{Y}_\cdot \leq e^{\gamma B}$, BSDE $(\eta, T, G_{\bar{\Psi}})$. Note by (18) that $\mathrm{d}P \times \mathrm{d}t - a.e.$,

$$\forall\ y \in \mathbf{R} \text{ and } z \in \mathbf{R}^d, \quad -\gamma \bar{u}_t(\omega)|y| - \frac{2}{e^{-\gamma B}}|z|^2 \leq G_{\bar{\Psi}}(\omega, t, y, z) \leq \gamma \bar{u}_t(\omega)|y|.$$

In view of $0 < e^{-\gamma\|\xi\|_\infty} \leq \eta \leq e^{\gamma\|\xi\|_\infty}$, it follows from Lemma 3 (see also Remark 3 for details) again that for each $t \in [0,T]$, $\mathrm{d}P - a.s.$, $0 < S_0^\gamma \leq \bar{Y}_t \leq Q_0^\gamma$, which means, in view of the definitions of functions Ψ and G_Ψ, that $(\bar{Y}_\cdot, \bar{Z}_\cdot)$ is also a bounded solution of BSDE (η, T, G_Ψ). Thus, in view of the fact that $(Y^\Psi_\cdot, Z^\Psi_\cdot)$ is the maximal bounded solution of BSDE (η, T, G_Ψ), we have $\mathrm{d}P - a.s.$, $e^{\gamma \bar{y}_\cdot} = \bar{Y}_\cdot \leq Y^\Psi_\cdot = e^{\gamma y^\Psi_\cdot}$, and then for each $t \in [0,T]$, $\mathrm{d}P - a.s.$, $\bar{y}_t \leq y_t^\Psi$.

Finally, a similar argument to that in Lemma 3.4 of [10] yields that BSDE (ξ, T, g) admits also a minimal bounded solution satisfying (17). The proof of Lemma 4 is then complete. □

Remark 4. Once again, an interesting observation is that the first part $y_\cdot$ of the bounded solution in Lemma 4 admits also a bound which is independent of the constant γ in (16).

We are now in a position to prove our main existence result.

Proof of Theorem 1. Let $0 < T \leq +\infty$, $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$ and the generator g satisfy (H1) and (H2) with $f_\cdot$, $u_\cdot$, $h(\cdot)$, $v_\cdot$, $\psi(\cdot)$ and $\varphi(\cdot)$. We only prove the existence of the minimal solution. The maximal solution case can be proved similarly. Denote

$$M := \left(\|\xi\|_\infty + \left\|\int_0^T f_t\,\mathrm{d}t\right\|_\infty\right) e^{\|\int_0^T u_t\,\mathrm{d}t\|_\infty}.$$

Consider a continuous function $\kappa : \mathbf{R} \mapsto [-M, M]$ such that

1) $\kappa(x) = -M$ if $x < -M$; 2) $\kappa(x) = x$ if $|x| \le M$; 3) $\kappa(x) = M$ if $x > M$,

and for each $(\omega, t, y, z) \in \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d$, define

$$g_\kappa(\omega, t, y, z) := g(\omega, t, \kappa(y), z) \quad \text{and} \quad \gamma^\kappa := 2\left(\max_{|x|\le M} h(x) + 1\right).$$

In view of (H1) and (H2), we know that $\mathrm{d}P \times \mathrm{d}t - a.e.$, for each $y \in \mathbf{R}$ and $z \in \mathbf{R}^d$,

$$\begin{aligned} \operatorname{sgn}(y) g_\kappa(\omega, t, y, z) &\le f_t(\omega) + u_t(\omega)|\kappa(y)| + h(\kappa(y))|z|^2 \\ &\le f_t(\omega) + u_t(\omega)|y| + \frac{\gamma^\kappa}{2}|z|^2 \end{aligned} \tag{19}$$

and

$$|g_\kappa(\omega, t, y, z)| \le v_t(\omega)\left(\max_{|x|\le M} \psi(x)\right) + \left(\max_{|x|\le M} \varphi(x)\right)|z|^2.$$

It then follows from Lemma 4 that BSDE (ξ, T, g_κ) admits a minimal bounded solution $(y_\cdot^\kappa, z_\cdot^\kappa)$.

In the sequel, since $(y_\cdot^\kappa, z_\cdot^\kappa)$ is a bounded solution of BSDE (ξ, T, g_κ) and (19) holds for g_κ, it follows from Lemma 2 (see also Remark 2 for details) that for each $t \in [0,T]$, $\mathrm{d}P - a.s.$,

$$|y_t^\kappa| \le \left(\|\xi\|_\infty + \left\|\int_0^T f_t\,\mathrm{d}t\right\|_\infty\right) e^{\|\int_0^T u_t\,\mathrm{d}t\|_\infty} = M. \tag{20}$$

Then, in view of the definitions of κ and g_κ, $(y_\cdot^\kappa, z_\cdot^\kappa)$ is also a bounded solution of BSDE (ξ, T, g).

Finally, we show that $(y_\cdot^\kappa, z_\cdot^\kappa)$ is just the minimal bounded solution of BSDE (ξ, T, g). In fact, let $(\bar{y}_\cdot, \bar{z}_\cdot)$ be any bounded solution of BSDE (ξ, T, g) with $|\bar{y}_\cdot| \le A$ for some constant $A > 0$. Consider a continuous function $\bar{\kappa} : \mathbf{R} \mapsto [-A, A]$ such that

1) $\bar{\kappa}(x) = -A$ if $x < -A$; 2) $\bar{\kappa}(x) = x$ if $|x| \le A$; 3) $\bar{\kappa}(x) = A$ if $x > A$.

and for each $(\omega, t, y, z) \in \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d$, define

$$g_{\bar{\kappa}}(\omega, t, y, z) := g(\omega, t, \bar{\kappa}(y), z).$$

In view of the definitions of functions $\bar{\kappa}$ and $g_{\bar{\kappa}}$, it is not hard to verify that $(\bar{y}_\cdot, \bar{z}_\cdot)$ is a bounded solution of BSDE $(\xi, T, g_{\bar{\kappa}})$. Moreover, noticing that (19) holds also true with $\gamma^{\bar{\kappa}} := 2(\max_{|x|\le A} h(x) + 1)$ instead of γ^κ when κ is replaced with $\bar{\kappa}$, from Lemma 2 again we can conclude that for each $t \in [0,T]$,

$$\mathrm{d}P - a.s., \quad |\bar{y}_t| \le M,$$

which means that, in view of the definitions of functions κ and g_κ, $(\bar{y}_\cdot, \bar{z}_\cdot)$ is also a bounded solution of BSDE (ξ, T, g_κ). Thus, since $(y^\kappa_\cdot, z^\kappa_\cdot)$ is the minimal bounded solution of BSDE (ξ, T, g_κ), we know that for each $t \in [0,T]$, $\mathrm{d}P-a.s.$, $y^\kappa_t \leq \bar{y}_t$, which is the desired result. The proof is then complete. □

By virtue of Theorem 4.1 in [17] and the proof of Theorem 1 in this paper, and in view of the convolution in Lemma 3, it is not very hard to verify the following comparison theorem of the minimal and maximal bounded solutions.

Theorem 2. *Let $0 < T \leq +\infty$, $\xi, \xi' \in L^\infty(\Omega, \mathcal{F}_T, P)$ and both generators g and g' satisfy* (H1) *and* (H2). *Assume that $(y_\cdot, z_\cdot)$ and $(y'_\cdot, z'_\cdot)$ are, respectively, the minimal* (*resp. maximal*) *bounded solution of BSDE (ξ, T, g) and BSDE (ξ', T, g')* (*recall Theorem 1*). *If $\mathrm{d}P-a.s.$, $\xi \leq \xi'$ and $\mathrm{d}P \times \mathrm{d}t-a.e.$, $g(t,y,z) \leq g'(t,y,z)$ for each $y \in \mathbf{R}$ and $z \in \mathbf{R}^d$, then for each $t \in [0,T]$, $y_t \leq y'_t$, $\mathrm{d}P-a.s.$*

The following Example 1 demonstrates that Theorem 1 generalizes, at some extent, the corresponding results in some existing works.

Example 1. Assume that $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$ and $u_\cdot \in L^\infty(\Omega; L^1([0,T]; \mathbf{R}_+))$. Consider BSDE (ξ, T, g) with generator

$$g(\omega, t, y, z) := u_t(\omega)(|y| + e^{-y}) + \sqrt{u_t(\omega)}|z| + e^y|z|^2.$$

It is not difficult to verify that g satisfies (H1) and (H2). It follows from Theorem 1 that BSDE (ξ, T, g) admits both a minimal and a maximal bounded solutions. Note by Remark 1 that this g satisfies neither the conditions of Theorem 3.1 in [10] nor the conditions of Theorem 2 in [16]. The above conclusion cannot be obtained by these results.

Finally, the following Theorem 3 demonstrates that all conclusions obtained in this paper hold still true when T is a $(\mathcal{F}_t)$-stopping time.

Theorem 3. *Let $T = +\infty$, τ is a $(\mathcal{F}_t)$-stopping time valued in $[0,+\infty)$, and $\xi \in L^\infty(\Omega, \mathcal{F}_\tau, P)$. Define $g_\tau(\omega,t,y,z) = 1_{t\leq\tau} g(\omega,t,y,z)$ for $(\omega,t,y,z) \in \Omega \times [0,+\infty) \times \mathbf{R} \times \mathbf{R}^d$. If $(y_t, z_t)_{t\in[0,+\infty)} \in \mathcal{S}^\infty \times \mathrm{M}^2$ is a solution of BSDE $(\xi, +\infty, g_\tau)$, then $y_t \mathbf{1}_{t\geq\tau} = \xi$ for each $t \in [0,+\infty)$, $z_t \mathbf{1}_{t\geq\tau} = 0$, $\mathrm{d}P \times \mathrm{d}t - a.e.$, and $(y_t, z_t)_{t\in[0,+\infty)}$ solves the following BSDE (ξ, τ, g):*

$$y_t = \xi + \int_t^\tau g(s, y_s, z_s)\,\mathrm{d}s - \int_t^\tau z_s \cdot \mathrm{d}B_s, \qquad t \in [0,\tau].$$

Conversely, if $(y_t, z_t)_{t\in[0,+\infty)} \in \mathcal{S}^\infty \times \mathrm{M}^2$ satisfies BSDE (ξ, τ, g), then $(\bar{y}_t, \bar{z}_t)_{t\in[0,+\infty)}$ is a solution of BSDE $(\xi, +\infty, g_\tau)$ in $\mathcal{S}^\infty \times \mathrm{M}^2$, where $\bar{y}_t := y_t \mathbf{1}_{t\leq\tau} + \xi \mathbf{1}_{t>\tau}$ and $\bar{z}_t := z_t \mathbf{1}_{t\leq\tau}$ for $t \in [0,+\infty)$.

Proof. If $(y_t, z_t)_{t\in[0,+\infty)} \in \mathcal{S}^\infty \times \mathrm{M}^2$ is a solution of BSDE $(\xi, +\infty, g_\tau)$, i.e.,

$$y_t = \xi + \int_t^{+\infty} \mathbf{1}_{s\leq\tau} g(s, y_s, z_s)\,\mathrm{d}s - \int_t^{+\infty} z_s \cdot \mathrm{d}B_s, \qquad t \in [0,+\infty), \tag{21}$$

then

$$y_\tau = \xi + \int_\tau^{+\infty} \mathbf{1}_{s\le\tau} g(s, y_s, z_s)\,\mathrm{d}s - \int_\tau^{+\infty} z_s \cdot \mathrm{d}B_s = \xi - \int_\tau^{+\infty} z_s \cdot \mathrm{d}B_s.$$

Taking the conditional expectation with respect to $\mathcal{F}_\tau$ in the above identity yields that $y_\tau = \mathbf{E}[\xi|\mathcal{F}_\tau] = \xi$, and then

$$\int_0^{+\infty} z_s \mathbf{1}_{s\ge\tau} \cdot \mathrm{d}B_s = \int_\tau^{+\infty} z_s \cdot \mathrm{d}B_s = 0,$$

which implies that $\mathbf{E}[\int_0^{+\infty} |z_s \mathbf{1}_{s\ge\tau}|^2 ds] = 0$. Hence, $z_t \mathbf{1}_{t\ge\tau} = 0$, $\mathrm{d}P \times \mathrm{d}t - a.e.$, and by (21) we know that $y_t \mathbf{1}_{t\ge\tau} = \xi$ for each $t \in [0, +\infty)$ and $(y_t, z_t)_{t\in[0,+\infty)}$ solves BSDE (ξ, τ, g). Conversely, the conclusion can be verified directly so that we omit its proof. The proof is then complete. □

Acknowledgement. The authors would like to thank the anonymous referee for his/her careful reading and helpful suggestions.

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