

GRAPHICALITY, C^0 CONVERGENCE, AND THE CALABI HOMOMORPHISM

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ABSTRACT. Consider a sequence of compactly supported Hamiltonian diffeomorphisms ϕ_k of an exact symplectic manifold, all of which are “graphical” in the sense that their graphs are identified by a Darboux-Weinstein chart with the image of a one-form. We show by an elementary argument that if the ϕ_k C^0 -converge to the identity, then their Calabi invariants converge to zero. This generalizes a result of Oh, in which the ambient manifold was the two-disk and an additional assumption was made on the Hamiltonians generating the ϕ_k . We discuss connections to the open problem of whether the Calabi homomorphism extends to the Hamiltonian homeomorphism group. The proof is based on a relationship between the Calabi invariant of a C^0 -small Hamiltonian diffeomorphism and the generalized phase function of its graph.

1. Introduction

Let (M, λ) be an exact symplectic manifold, that is, M is a smooth, necessarily noncompact manifold of some even dimension $2n$ endowed with a 1-form λ such that $d\lambda$ is nondegenerate. A compactly supported smooth function $H: [0, 1] \times M \rightarrow \mathbb{R}$ defines a time-dependent vector field $\{X_{H_t}\}_{t \in [0, 1]}$ by the prescription that $d\lambda(X_{H_t}, \cdot) = d(H(t, \cdot))$; the (compactly-supported) Hamiltonian diffeomorphism group of the symplectic manifold $(M, d\lambda)$, which we will write as $Ham^c(M, d\lambda)$, is by definition the group consisting of the time-one maps of such Hamiltonian vector fields. We will denote the time- t map of the Hamiltonian vector field associated to $H: [0, 1] \times M \rightarrow \mathbb{R}$ as ϕ_H^t .

In [4], Calabi introduced a homomorphism $\text{Cal}: Ham^c(M, d\lambda) \rightarrow \mathbb{R}$ which is determined by the formula

$$(1) \quad \text{Cal}(\phi_H^1) = \int_{[0, 1] \times M} H dt \wedge (d\lambda)^n.$$

The fundamental importance of this homomorphism is reflected in the fact that, as follows easily from the simplicity of $\ker(\text{Cal})$ [1, Théorème II.6.2] and

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the obvious fact that $\text{Ham}^c(M, d\lambda)$ has trivial center, any non-injective homomorphism $f: \text{Ham}^c(M, d\lambda) \rightarrow G$ to any group G must factor through Cal as $f = g \circ \text{Cal}$ for some homomorphism $g: \mathbb{R} \rightarrow G$.

At least since the proof of the Eliashberg-Gromov rigidity theorem [5] it has been understood that many features of symplectic geometry survive under appropriate C^0 limits; it is natural to ask whether this principle applies to the Calabi homomorphism. Specifically, we may consider the following question:

Question 1.1. Let (M, λ) be an exact symplectic manifold and let $\{H_k\}_{k=1}^\infty$ be a sequence of smooth functions all having support in some compact subset $K \subset M$, such that $\phi_{H_k}^1 \xrightarrow{C^0} 1_M$ and such that there is $H \in L^{(1, \infty)}([0, 1] \times M)$ such that $H_k \xrightarrow{L^{(1, \infty)}} H$. Must it be true that $\text{Cal}(\phi_{H_k}^1) \rightarrow 0$? (In other words, must it be true that $\int_{[0, 1] \times M} H dt \wedge (d\lambda)^n = 0$?)

Here the $L^{(1, \infty)}$ norm on the space of compactly supported smooth functions $C_c^\infty([0, 1] \times M)$ is defined by $\|H\|_{L^{(1, \infty)}} = \int_0^1 \max |H(t, \cdot)| dt$ and $L^{(1, \infty)}([0, 1] \times M)$ is the completion of $C_c^\infty([0, 1] \times M)$ with respect to this norm. It would also be reasonable to replace the $L^{(1, \infty)}$ norm in Question 1.1 by the C^0 norm; our choice of the $L^{(1, \infty)}$ norm is mainly for consistency with [7], [8]. However it should be emphasized that *some* control over the Hamiltonian functions H_k and not just on the time-one maps $\phi_{H_k}^1$ is necessary in Question 1.1: the reader should not find it difficult to construct sequences of Hamiltonian diffeomorphisms $\phi_{H_k}^1$ all having $\text{Cal}(\phi_{H_k}^1) = 1$ that C^0 -converge to the identity by taking the supports of the H_k to be small (see also Example 1.4 below). This need for something like C^0 control on the Hamiltonian functions is consistent with other aspects of the theory of C^0 Hamiltonian dynamics as in [8].

In [8], the authors introduce the group $\text{Hameo}(M, d\lambda)$ of “Hamiltonian homeomorphisms” of the symplectic manifold $(M, d\lambda)$, and prove that it is a normal subgroup of the group $\text{Sympeo}^c(M, d\lambda)$ of compactly supported symplectic homeomorphisms of $(M, d\lambda)$, *i.e.*, of the C^0 -closure of the group of compactly supported symplectic diffeomorphisms in the group of compactly supported homeomorphisms of M . As follows from a straightforward generalization of the discussion in [6, Section 7], an affirmative answer to Question 1.1 would imply that $\text{Hameo}(M, d\lambda)$ is a proper subgroup of $\text{Sympeo}^c(M, d\lambda)$, and hence that the latter group is not simple. This is particularly interesting for $(M, d\lambda)$ equal to the open two-dimensional disk D^2 with its standard symplectic form, in which case $\text{Sympeo}^c(M, d\lambda)$ coincides with the compactly supported area-preserving homeomorphism group of D^2 , whose simplicity or non-simplicity is a longstanding open problem. In this case Question 1.1 is equivalent to a slightly stronger version of [6, Conjecture 6.8], stating that Cal extends continuously to a homomorphism $\text{Hameo}(D^2, dx \wedge dy) \rightarrow \mathbb{R}$. (To obtain Oh’s conjecture precisely one would strengthen the hypothesis of Question 1.1 to the statement that the isotopies $\{\phi_k^t\}_{t \in [0, 1]}$ converge uniformly as $k \rightarrow \infty$

to some loop of homeomorphisms, instead of just assuming that their time-one maps converge to the identity.)

The main purpose of this note is to generalize a result of [7] related to Question 1.1, which we will recall presently after setting up Oh's notation and terminology. Given our exact symplectic manifold (M, λ) , let us write $\Lambda = \lambda \oplus (-\lambda) \in \Omega^1(M \times M)$, so that $(M \times M, \Lambda)$ is likewise an exact symplectic manifold. By the Darboux-Weinstein Theorem, there is a symplectomorphism $\Psi: \mathcal{U}_\Delta \rightarrow \mathcal{V}$ where \mathcal{U}_Δ is a neighborhood of the diagonal $\Delta \subset M \times M$ and \mathcal{V} is a neighborhood of the zero-section in $T^*\Delta$ which is equipped with the symplectic form $-d\theta_{can}$ where $\theta_{can} \in \Omega^1(T^*\Delta)$ is the canonical one-form, with $\Psi|_\Delta$ equal to the inclusion of Δ as the zero-section. We fix such a Darboux-Weinstein chart Ψ . If $\phi: M \rightarrow M$ is a compactly supported symplectic diffeomorphism, then its graph $\Gamma_\phi = \{(\phi(x), x) \mid x \in M\}$ is a Lagrangian submanifold of $(M \times M, d\Lambda)$, and if ϕ is sufficiently C^0 -close to 1_M , then Γ_ϕ will lie in the domain \mathcal{U}_Δ of Ψ . One says that ϕ is Ψ -graphical if additionally the Lagrangian submanifold $\Psi(\Gamma_\phi) \subset T^*\Delta$ coincides with the image of a section of the cotangent bundle $T^*\Delta \rightarrow \Delta$. In particular this would hold if ϕ were assumed to be C^1 -close to the identity; on the other hand it is possible for a diffeomorphism to be Ψ -graphical while still being fairly far away from the identity in the C^1 -sense.

One formulation of the main result of [7] is the following:

Theorem 1.2 ([7, Theorem 1.10]). *For $M = D^2$, let $H_k: [0, 1] \times D^2 \rightarrow \mathbb{R}$ be a sequence as in Question 1.1, and assume moreover that all of the diffeomorphisms $\phi_{H_k}^1$ are Ψ -graphical. Then $\text{Cal}(\phi_{H_k}^1) \rightarrow 0$.*

Thus, at least for $M = D^2$ the answer to Question 1.1 is affirmative under an additional hypothesis; one might then hope to answer Question 1.1 by finding a way to drop this hypothesis. However we will prove the following generalization of Theorem 1.2, which we interpret as suggesting that such a strategy raises more questions than might have been anticipated.

Theorem 1.3. *Let (M, λ) be an exact symplectic manifold, and let $\{\phi_k\}_{k=1}^\infty$ be a sequence in $\text{Ham}^c(M, d\lambda)$ with all ϕ_k generated by Hamiltonians that are supported in a fixed compact subset, such that $\phi_k \xrightarrow{C^0} 1_M$. Assume moreover that each ϕ_k is Ψ -graphical. Then $\text{Cal}(\phi_k) \rightarrow 0$.*

We emphasize that no assumption is made on the convergence of the Hamiltonians H_k generating the ϕ_k . On the other hand, if one drops the graphicality hypothesis, then an assumption similar to that in Question 1.1 is certainly needed, see Example 1.4. This author believes that a major impediment to answering Question 1.1 affirmatively is the current lack of a precise idea of what role the convergence of the Hamiltonian functions might play in the convergence of the Calabi invariants; Theorem 1.3 shows that this role only becomes essential when the ϕ_k are no longer graphical.

Example 1.4. On an arbitrary $2n$ -dimensional exact symplectic manifold (M, λ) , for a sufficiently small $\delta > 0$ we may consider a symplectically embedded copy of the cube $C^{2n}(\delta) = [0, \delta]^{2n} \subset M$ (with $C^{2n}(\delta)$ carrying the standard symplectic form $\sum_i dx_i \wedge dy_i$). Define a sequence of smooth functions $F_k: M \rightarrow \mathbb{R}$ as follows. Divide $C^{2n}(\delta)$ into k^{2n} equal subcubes $C_{\vec{i}}^{(k)}$ for $\vec{i} = (i_1, \dots, i_{2n}) \in \{1, \dots, k\}^{2n}$, by taking

$$C_{\vec{i}}^{(k)} = \prod_{j=1}^{2n} \left[\frac{i_j - 1}{k} \delta, \frac{i_j}{k} \delta \right],$$

and take F_k to be a smooth function which is supported in the union of the interiors of the $C_{\vec{i}}^{(k)}$, obeys $0 \leq F_k \leq 1$ everywhere, and which, for each \vec{i} , is equal to 1 on a subset of $C_{\vec{i}}^{(k)}$ having measure at least $(1 - 1/k)(\delta/k)^{2n}$. Define $H_k: [0, 1] \times M \rightarrow \mathbb{R}$ by $H_k(t, x) = F_k(x)$.

Evidently the sequence H_k converges to the indicator function of $[0, 1] \times C^{2n}(\delta)$ in every L^p norm for $p < \infty$. In particular $\text{Cal}(\phi_{H_k}^1) = \int_M H_k dt \wedge (d\lambda)^n \rightarrow \delta^{2n} > 0$ as $k \rightarrow \infty$. Meanwhile each $\phi_{H_k}^1$ acts as the identity on $M \setminus C^{2n}(\delta)$ and maps each of the subcubes $C_{\vec{i}}^{(k)}$ to themselves; since these subcubes have diameter $\sqrt{2n}\delta/k$ it follows that $\phi_{H_k}^1 \xrightarrow{C^0} 1_M$.

Thus, if the answer to Question 1.1 is to be affirmative, then its assumptions must be rather sharp: it is not sufficient for the Hamiltonian functions to converge in L^p for any finite p , or to be uniformly bounded—one would need uniform convergence in the space variable. Given the formula (1) for the Calabi homomorphism, one might naively have expected that L^1 convergence would be sufficient, but this is not the case.

Remark 1.5. The uniqueness theorem for the Hamiltonians that generate Hamiltonian homeomorphisms [10], [3] is somewhat reminiscent of Question 1.1. Indeed this theorem (when M is noncompact as it is in our case) can be phrased as stating that if $H_k: [0, 1] \times M \rightarrow \mathbb{R}$ are compactly supported smooth functions such that $H_k \xrightarrow{L^{(1, \infty)}} H$ and if $\phi_{H_k}^t \xrightarrow{C^0} 1_M$ uniformly in t , then $H \equiv 0$. Thus, in comparison to this uniqueness theorem, Question 1.1 asks whether one can obtain the weaker conclusion that $\int_M H dt \wedge (d\lambda)^n = 0$ from the weaker hypothesis that only the time-one map $\phi_{H_k}^1 \xrightarrow{C^0} 1_M$. Note that Example 1.4 shows that the uniqueness theorem would fail to hold if we instead were to only assume that the functions H_k converge in L^p for some finite p , or that the H_k are uniformly bounded, since in Example 1.4 we clearly have $\phi_{H_k}^t \xrightarrow{C^0} 1_M$ uniformly in t . Thus the (potential) sharpness of the hypotheses in Question 1.1 has some precedent in prior results.

Remark 1.6. En route to proving Theorem 1.3 we will prove a related result, Corollary 2.4, that does not make any graphicality assumptions and suggests a general viewpoint on Question 1.1. Namely, given that $\phi_{H_k}^1 \xrightarrow{C^0} 1_M$, Corollary 2.4 shows that the statement that $\text{Cal}(\phi_{H_k}^1) \rightarrow 0$ is equivalent to the statement that the integrals of suitable pullbacks of *generalized phase functions* S_k for the Lagrangian submanifolds $\Psi(\Gamma_{\phi_{H_k}^1})$ converge to zero. Here (as in [2]) a generalized phase function for a Lagrangian submanifold $L \subset T^*\Delta$ is a compactly supported smooth function $S_L: L \rightarrow \mathbb{R}$ with $dS_L = \theta_{\text{can}}|_L$. One way of constructing a generalized phase function for L is to begin with a generating function $S: M \times \mathbb{R}^N \rightarrow \mathbb{R}$ (as in [9]) with fiber critical set $\Sigma_S \subset M \times \mathbb{R}^N$ and canonical embedding $\iota_L: \Sigma_S \rightarrow L$, and then define $S_L = S \circ \iota_L^{-1}$. Thus in view of Corollary 2.4, Question 1.1 leads to a question about the relationship between the behavior of a Hamiltonian function H and that of the generating function of the graph of the time-one map of H .

The following section contains the proofs of Corollary 2.4 and Theorem 1.3.

2. The Calabi invariant and generalized phase functions

As in the introduction, we work in a fixed exact symplectic manifold (M, λ) , and we fix a Darboux-Weinstein chart $\Psi: \mathcal{U}_\Delta \rightarrow \mathcal{V} \subset T^*\Delta$ where $\Delta \subset (M \times M, d\Lambda)$ is the diagonal (and $\Lambda = \pi_1^*\lambda - \pi_2^*\lambda \in \Omega^1(M \times M)$ where $\pi_1, \pi_2: M \times M \rightarrow M$ are the projections to the two factors). The graph of ϕ is $\Gamma_\phi = \{(\phi(x), x) \mid x \in M\} \subset M \times M$. (Throughout the paper our sign and ordering conventions are chosen to be consistent with [7].) Assuming that $\Gamma_\phi \subset \mathcal{U}_\Delta$ we let

$$L_\phi = \Psi(\Gamma_\phi) \subset T^*\Delta.$$

Thus ϕ is graphical in the sense of the introduction if and only if there is $\alpha \in \Omega^1(\Delta)$ such that $L_\phi = \{(x, \alpha_x) \mid x \in \Delta\}$. Such an α is necessarily closed, and we will argue below that it is exact. (This is not completely obvious since, if $\phi = \phi_H^1$, we allow the possibility that some $\Gamma_{\phi_H^t}$ is not contained in \mathcal{U}_Δ .)

The following is well-known.

Lemma 2.1. *Let $\phi \in \text{Ham}^c(M, d\lambda)$. Then there is $f_{\lambda, \phi} \in C_c^\infty(M)$ such that $df_{\lambda, \phi} = \phi^*\lambda - \lambda$. Moreover*

$$(2) \quad \text{Cal}(\phi) = \frac{1}{n+1} \int_M f_{\lambda, \phi} (d\lambda)^n.$$

Proof. Assume that $\phi = \phi_H^1$ for some $H \in C_c^\infty([0, 1] \times M)$. Then

$$\begin{aligned} \phi^*\lambda - \lambda &= \int_0^1 \frac{d}{dt} \phi_H^{t*} \lambda dt = \int_0^1 \phi_H^{t*} (d\iota_{X_{H_t}} \lambda + \iota_{X_{H_t}} d\lambda) dt \\ &= d \left(\int_0^1 (\iota_{X_{H_t}} \lambda + H(t, \cdot)) \circ \phi_H^t dt \right) \end{aligned}$$

so we may take $f_{\lambda,\phi} = \int_0^1 (\iota_{X_{H_t}} \lambda + H(t, \cdot)) \circ \phi_H^t dt$. The formula (2) is then a standard calculation involving repeated applications of Stokes' theorem, see e.g. [1, Proposition II.4.3]. \square

Lemma 2.2. *Assume that the domain of the Darboux-Weinstein chart $\Psi: \mathcal{U}_\Delta \rightarrow \mathcal{V}$ has Δ as a deformation retract. Then there is a smooth function $R: \mathcal{V} \rightarrow \mathbb{R}$ such that $R|_\Delta \equiv 0$ and $-\theta_{can} = (\Psi^{-1})^* \Lambda + dR$.*

Proof. Since Ψ is a symplectomorphism $(\mathcal{U}_\Delta, d\Lambda) \rightarrow (\mathcal{V}, -d\theta_{can})$, we have $d(\Psi^* \theta_{can} + \Lambda) = 0$. Of course, since $\Lambda|_\Delta = 0$ while θ_{can} vanishes on the zero-section of $T^*\Delta$, and since Ψ maps Δ to the zero-section, we have $(\Psi^* \theta_{can} + \Lambda)|_\Delta = 0$. So $\Psi^* \theta_{can} + \Lambda$ represents a class in the relative de Rham cohomology $H^1(\mathcal{U}_\Delta, \Delta)$, which is trivial by the assumption that Δ is a deformation retract of \mathcal{U}_Δ . So there is $g \in C^\infty(\mathcal{U}_\Delta)$ with $g|_\Delta = 0$ such that $\Psi^* \theta_{can} + \Lambda = dg$. So the lemma holds with $R = -g \circ \Psi^{-1}$. \square

Putting together the two preceding lemmas gives the following.

Proposition 2.3. *Assume that $\phi \in Ham^c(M, d\lambda)$ has $\Gamma_\phi \subset \mathcal{U}_\Delta$ where $\Psi: \mathcal{U}_\Delta \rightarrow \mathcal{V}$ is a Darboux-Weinstein chart whose domain \mathcal{U}_Δ has Δ as a deformation retract. Let $f_{\lambda,\phi}$ be as in Lemma 2.1, and let R be as in Lemma 2.2. Where $L_\phi = \Psi(\Gamma_\phi)$, define $S_\phi: L_\phi \rightarrow \mathbb{R}$ by*

$$(3) \quad S_\phi = R + f_{\lambda,\phi} \circ \pi_2 \circ \Psi^{-1}.$$

Then $S_\phi: L_\phi \rightarrow \mathbb{R}$ is a compactly supported smooth function satisfying $dS_\phi = -\theta_{can}|_{L_\phi}$.

Proof. That S_ϕ is compactly supported follows from the facts that L_ϕ coincides outside of a compact subset with Δ , on which R vanishes identically, that $\pi_2 \circ \Psi^{-1}$ maps L_ϕ diffeomorphically to M , and that $f_{\lambda,\phi}$ is compactly supported.

By the defining property of R we have

$$(4) \quad (dR + \theta_{can})|_{L_\phi} = (-\Psi^{-1*} \Lambda)|_{L_\phi} = -\Psi^{-1*}(\Lambda|_{\Gamma_\phi}).$$

Meanwhile since (by the definition of Γ_ϕ as $\{(\phi(x), x) | x \in M\}$) we have $\phi \circ \pi_2|_{\Gamma_\phi} = \pi_1|_{\Gamma_\phi}$, we see that

$$(5) \quad d(f_{\lambda,\phi} \circ \pi_2|_{\Gamma_\phi}) = (\pi_2|_{\Gamma_\phi})^*(\phi^* \lambda - \lambda) = (\pi_1^* \lambda - \pi_2^* \lambda)|_{\Gamma_\phi} = \Lambda|_{\Gamma_\phi}.$$

Combining (4) and (5) gives

$$\Psi^* ((dR + \theta_{can})|_{L_\phi}) = -d(f_{\lambda,\phi} \circ \pi_2|_{\Psi^{-1}(L_\phi)}),$$

from which the result follows immediately. \square

The key point now is that the conditions that S_ϕ be compactly supported and that $dS_\phi = -\theta_{can}|_{L_\phi}$ uniquely determine the smooth function $S_\phi: L_\phi \rightarrow \mathbb{R}$. A function satisfying these properties is (perhaps after a sign reversal) sometimes called a ‘‘generalized phase function’’; the formula (3) for such a function together with the formula (2) thus relate generalized phase functions

to the Calabi homomorphism. The relation is especially simple in the C^0 -small, graphical case, but first we note a consequence that does not require graphicality.

Corollary 2.4. *Assume that $\{\phi_k\}_{k=1}^\infty$ is a sequence in $Ham(M, d\lambda)$ with each ϕ_k supported in a fixed compact set, such that $\phi_k \xrightarrow{C^0} 1_M$. Construct the Lagrangian submanifolds $L_{\phi_k} = \Psi(\Gamma_{\phi_k}) \subset T^*\Delta$ (for sufficiently large k) as above, and suppose that $S_k: L_{\phi_k} \rightarrow \mathbb{R}$ are compactly supported smooth functions obeying $dS_k = -\theta_{can}|_{L_{\phi_k}}$. Then $Cal(\phi_k) \rightarrow 0$ if and only if*

$$(6) \quad \int_M (S_k \circ \Psi \circ (\phi_k \times 1_M))(d\lambda)^n \rightarrow 0.$$

Proof. By shrinking the domain of $\Psi: \mathcal{U}_\Delta \rightarrow \mathcal{V}$ and perhaps removing an initial segment of the sequence $\{\phi_k\}_{k=1}^\infty$ we may assume that \mathcal{U}_Δ has Δ as a deformation retract and that each $\Gamma_{\phi_k} \subset \mathcal{U}_\Delta$ so that we are in the setting of Proposition 2.3.

Note that $\pi_2 \circ \Psi^{-1}: L_{\phi_k} \rightarrow M$ is a diffeomorphism with inverse $\Psi \circ (\phi_k \times 1_M)$. So (3) gives

$$(7) \quad f_{\lambda, \phi_k} = (S_k - R) \circ \Psi \circ (\phi_k \times 1_M).$$

But R is a smooth function, independent of k , which vanishes along the zero section of $T^*\Delta$, so $R \circ \Psi$ vanishes along the diagonal $\Delta \subset M \times M$. So the assumption that $\phi_k \xrightarrow{C^0} 1_M$ implies that $\max |R \circ \Psi \circ (\phi_k \times 1_M)| \rightarrow 0$ as $k \rightarrow \infty$, and the assumption that there is a compact set simultaneously containing the supports of all of the ϕ_k implies that the support of each $R \circ \Psi \circ (\phi_k \times 1_M)$ is contained in this same compact set. Hence $\int_M R \circ \Psi \circ (\phi_k \times 1_M)(d\lambda)^n \rightarrow 0$, in view of which the corollary follows directly from (7) and (2). \square

Proof of Theorem 1.3. We work in the same setting as Corollary 2.4, with the additional assumption that each ϕ_k is Ψ -graphical. As noted at the start of the proof of Corollary 2.4, we may assume that \mathcal{U}_Δ deformation retracts to Δ .

Since ϕ_k is Ψ -graphical, let $\alpha_k \in \Omega^1(\Delta)$ have the property that L_{ϕ_k} is the image of α_k (when the latter is viewed as a map $\Delta \rightarrow T^*\Delta$). Recall that, by the definition of the canonical one-form θ_{can} , $\alpha_k^* \theta_{can} = \alpha_k$. Denoting by $\pi_\Delta: T^*\Delta \rightarrow \Delta$ the bundle projection, $\pi_\Delta|_{L_{\phi_k}}: L_{\phi_k} \rightarrow \Delta$ is a diffeomorphism with inverse α_k , so it follows that $\theta_{can}|_{L_{\phi_k}} = (\pi_\Delta|_{L_{\phi_k}})^* \alpha_k$. So Proposition 2.3 shows that the function $S_{\phi_k}: L_{\phi_k} \rightarrow \mathbb{R}$ defined therein obeys $dS_{\phi_k} = -\theta_{can}|_{L_{\phi_k}} = -(\pi_\Delta|_{L_{\phi_k}})^* \alpha_k$ and hence

$$(8) \quad \alpha_k = -d(S_{\phi_k} \circ \alpha_k).$$

In particular α_k is the derivative of a compactly supported smooth function.

By assumption each ϕ_k is generated by a Hamiltonian H_k with support contained in some fixed compact set $[0, 1] \times K$ where $K \subset M$. As is clear from the formula for f_{λ, ϕ_k} in the proof of Lemma 2.1, the functions f_{λ, ϕ_k} likewise

have support contained in K . So since L_{ϕ_k} coincides with the zero-section outside of $\Psi(\mathcal{U}_\Delta \cap (K \times K))$, we see from (3) that S_{ϕ_k} vanishes at all points of form $\Psi(x, x)$ for $x \notin K$. So (identifying K with its image in Δ by the diagonal embedding) the functions $-S_{\phi_k} \circ \alpha_k$ are all likewise supported in K .

Now let us fix a Riemannian metric on Δ and denote by A the diameter of K with respect to g . Also let $x_0 \in \Delta \setminus K$ be of distance less than 1 from K , so for any $x \in \Delta$ there is a path $\gamma_x: [0, 1] \rightarrow \Delta$ having $|\gamma'_x(t)|_g \leq A + 1$ for all t . Since $x_0 \in \Delta \setminus K$, we have $S_{\phi_k} \circ \alpha_k(x_0) = 0$. Thus, for any $x \in K$, using (8) we have

$$\begin{aligned} S_{\phi_k} \circ \alpha_k(x) &= \int_0^1 \frac{d}{dt} (S_{\phi_k} \circ \alpha_k(\gamma_x(t))) dt \leq (A + 1) \max_{t \in [0, 1]} |d(S_{\phi_k} \circ \alpha_k)(\gamma_x(t))|_g \\ &\leq (A + 1) \max_{\Delta} |\alpha_k|_g. \end{aligned}$$

Since for $x \notin K$ we have $S_k \circ \alpha_k(x) = 0$, and since $\alpha_k: \Delta \rightarrow L_{\phi_k}$ is surjective, this proves that

$$\max_{L_\phi} |S_{\phi_k}| \leq (A + 1) \max_{\Delta} |\alpha_k|_g.$$

But now the result is immediate from Corollary 2.4: the integrand of (6) vanishes outside of K , so the integral is bounded above by

$$(A + 1) \max_{\Delta} |\alpha_k|_g \int_K (d\lambda)^n.$$

The assumption that $\phi_k \xrightarrow{C^0} 1_M$ shows that $\max_{\Delta} |\alpha_k|_g \rightarrow 0$, so $\text{Cal}(\phi_k) \rightarrow 0$ as desired. \square

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