Bull. Korean Math. Soc. **54** (2017), No. 6, pp. 2029–2041
https://doi.org/10.4134/BKMS.b160698
pISSN: 1015-8634 / eISSN: 2234-3016

FLAG-TRANSITIVE POINT-PRIMITIVE SYMMETRIC DESIGNS AND THREE DIMENSIONAL PROJECTIVE SPECIAL UNITARY GROUPS

Ashraf Daneshkhah and Sheyda Zang Zarin

Abstract. The main aim of this article is to study symmetric (v, k, λ) designs admitting a flag-transitive and point-primitive automorphism group G whose socle is $\mathrm{PSU}(3, q)$. We indeed show that such designs must be complete.

1. Introduction

A *symmetric* (v, k, λ) *design* is an incidence structure $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ consisting of a set $\mathcal{V}$ of v *points* and a set $\mathcal{B}$ of v *blocks* such that every point is incident with exactly k blocks, and every pair of blocks is incident with exactly λ points. A *nontrivial* symmetric design is one in which $2 < k < v - 1$. A symmetric $(v, v-1, v-2)$ design is called *complete*. A *flag* of $\mathcal{D}$ is an incident pair (α, B) where α and B are a point and a block of $\mathcal{D}$, respectively. An *automorphism* of a symmetric design $\mathcal{D}$ is a permutation of the points permuting the blocks and preserving the incidence relation. An automorphism group G of $\mathcal{D}$ is called *flag-transitive* if it is transitive on the set of flags of $\mathcal{D}$. If G is primitive on the point set $\mathcal{V}$, then G is said to be *point-primitive*. A group G is said to be *almost simple* with socle X if $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$ where X is a nonabelian simple group. Further notation and definitions in both design theory and group theory are standard and can be found, for example, in [5, 10, 13].

Symmetric designs with λ small have been of most interest. Kantor [11] classified flag-transitive symmetric $(v, k, 1)$ designs (projective planes) of order n and showed that either $\mathcal{D}$ is a Desarguesian projective plane and $\mathrm{PSL}(3, n) \trianglelefteq G$, or G is a sharply flag-transitive Frobenius group of odd order $(n^2+n+1)(n+1)$, where n is even and $n^2 + n + 1$ is prime. Regueiro [17] gave a complete classification of biplanes ($\lambda = 2$) with flag-transitive automorphism groups apart from those admitting a 1-dimensional affine group (see also [18, 19, 20, 21]). Zhou and Dong studied nontrivial symmetric $(v, k, 3)$ designs (triplanes) and

Received August 24, 2016; Revised October 16, 2016; Accepted December 26, 2016.
2010 *Mathematics Subject Classification.* 05B05, 05B25, 20B25.
Key words and phrases. symmetric design, point-primitive, flag-transitive, automorphism group.

proved that if $\mathcal{D}$ is a nontrivial symmetric $(v, k, 3)$ design with a flag-transitive and point-primitive automorphism group G, then $\mathcal{D}$ has parameters $(11, 6, 3)$, $(15, 7, 3)$, $(45, 12, 3)$ or G is a subgroup of $\mathrm{A\Gamma L}(1, q)$ where $q = p^m$ with $p \geqslant 5$ prime [7, 27, 28, 29, 30]. Nontrivial symmetric $(v, k, 4)$ designs admitting flag-transitive and point-primitive almost simple automorphism group whose socle is an alternating group or $\mathrm{PSL}(2, q)$ have also been investigated [6, 31]. It is known [24] that if a nontrivial symmetric (v, k, λ) design $\mathcal{D}$ with $\lambda \leqslant 100$ admitting a flag-transitive, point-primitive automorphism group G, then G must be an affine or almost simple group. Therefore, it is interesting to study such designs whose socle is of almost simple type or affine type.

In this paper, however, we are interested in large λ. In this direction, it is recently shown in [2] that there are only four possible symmetric (v, k, λ) designs admitting a flag-transitive and point-primitive automorphism group G satisfying $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$ where $X = \mathrm{PSL}(2, q)$, see also [26]. In the case where X is a sporadic simple group, there also exist four possible parameters (see [25]). This study for $X := \mathrm{PSL}(3, q)$ gives rise to one nontrivial design (up to isomorphism) which is a Desarguesian projective plane $\mathrm{PG}(2, q)$ and $\mathrm{PSL}(3, q) \leqslant G$ (see [1]). This paper is devoted to studying symmetric designs admitting a flag-transitive and point-primitive almost simple automorphism group G whose socle is $X := \mathrm{PSU}(3, q)$. Indeed, the situation for $\mathrm{PSU}(3, q)$ is rather different and trivial design is the only symmetric design admitting such automorphism group G. We prove Theorem 1.1 below in Section 3.

Theorem 1.1. *Let $\mathcal{D}$ be a symmetric (v, k, λ) design, and let G be an automorphisms group of $\mathcal{D}$ with socle $X = \mathrm{PSU}(3, q)$. If G is flag-transitive and point-primitive, then $\mathcal{D}$ is a complete design.*

In order to prove Theorem 1.1, we need to know the complete list [3, Table 8.5] of maximal subgroups of almost simple groups with socle $\mathrm{PSU}(3, q)$ (see Lemma 2.4 below). We frequently apply Lemma 2.1 below as a key tool and use GAP [8] for computations.

In the case where G is imprimitive, Praeger and Zhou [22] studied point-imprimitive symmetric (v, k, λ) designs, and determined all such possible designs for $\lambda \leqslant 10$. This motivates Praeger and Reichard [14] to classify flag-transitive symmetric $(96, 20, 4)$ designs. As a result of their work, the only examples for flag-transitive, point-imprimitive symmetric $(v, k, 4)$ designs are $(15, 8, 4)$ and $(96, 20, 4)$ designs. In a recent study of imprimitive flag-transitive designs [4], Cameron and Praeger gave a construction of a family of designs with a specified point-partition, and determine the subgroup of automorphisms leaving invariant the point-partition. They gave necessary and sufficient conditions for a design in the family to possess a flag-transitive group of automorphisms preserving the specified point-partition. Consequently, they gave examples of flag-transitive designs in the family, including a new symmetric $(1480, 336, 80)$ design with automorphism group $2^{12} : ((3 \cdot \mathrm{M}_{22}) : 2)$, and a construction of one of the families of the symmetric designs exhibiting a flag-transitive, point-imprimitive automorphism group.

2. Preliminaries

In this section, we state some useful facts in both design theory and group theory. The following Lemma 2.1 is a key result in our approach to prove Theorem 1.1:

Lemma 2.1. *Let $\mathcal{D}$ be a symmetric (v, k, λ) design, and let G be a flag-transitive automorphism group of $\mathcal{D}$. If α is a point in $\mathcal{V}$ and $M := G_\alpha$, then*

(a) $k(k-1) = \lambda(v-1)$;
(b) $k \mid |M|$ *and* $\lambda v < k^2$;
(c) $k \mid \gcd(\lambda(v-1), |M|)$;
(d) $k \mid \lambda d$, *for all subdegrees d of G.*

Proof. The proof follows from [2, Lemma 2.1], see also [31, Lemma 2.2]. □

Recall that a group G is called almost simple if $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$ where X is a (nonabelian) simple group. If M is a maximal subgroup of an almost simple group G with socle X, then $G = MX$, and since we may identify X with $\mathrm{Inn}(X)$, the group of inner automorphisms of X, we also conclude that $|M|$ divides $|\mathrm{Out}(X)| \cdot |X \cap M|$. This implies the following elementary and useful fact:

Lemma 2.2. *Let G be an almost simple group with socle X, and let M be maximal in G not containing X. Then*

(a) $G = MX$;
(b) $|M|$ *divides* $|\mathrm{Out}(X)| \cdot |X \cap M|$.

Lemma 2.3. *Suppose that $\mathcal{D}$ is a symmetric (v, k, λ) design admitting a flag-transitive and point-primitive almost simple automorphism group G with socle X of Lie type in odd characteristic p. Suppose also that the point-stabiliser G_α, not containing X, is not a parabolic subgroup of G. Then* $\gcd(p, v-1) = 1$.

Proof. Note that G_α is maximal in G, then by Tits' Lemma [23, (1.6)], p divides $|G : G_\alpha| = v$, and so $\gcd(p, v-1) = 1$. □

If a group G acts primitively on a set $\mathcal{V}$ and $\alpha \in \mathcal{V}$ (with $|\mathcal{V}| \geqslant 2$), then the point-stabiliser G_α is maximal in G [5, Corollary 1.5A]. Therefore, in our study, we need a list of all maximal subgroups of almost simple group G with socle $X := \mathrm{PSU}(3, q)$. Note that if M is a maximal subgroup of G, then $M_0 := M \cap X$ is not necessarily maximal in X in which case M is called a *novelty.* By [3, Tables 8.5 and 8.6], the complete list of maximal subgroups of an almost simple group G with socle $\mathrm{PSU}(3, q)$ are known, and in this case, there arose only four novelties, see [3, 9, 12, 16].

Lemma 2.4 ([3, Tables 8.5 and 8.6])**.** *Let G be a group such that $X = \mathrm{PSU}(3,q) \trianglelefteq G \leqslant \mathrm{Aut}(X)$, and let M be a maximal subgroup of G not containing X. Then $M_0 = X \cap M$, is (isomorphic to) one of the following subgroups*:

(a) $\hat{}[q]^{1+2} : (q^2-1)$;
(b) $\hat{}\mathrm{GU}(2,q)$;
(c) $\hat{}(q^2-q+1) : 3$ *with* $q \neq 3, 5$ (*novelty if* $q = 5$);
(d) $\hat{}(q+1)^2 : \mathrm{S}_3$ (*novelty if* $q = 5$);
(e) $\mathrm{SO}_3(q)$ *with* $q \geqslant 7$, *q odd*;
(f) $\hat{}\mathrm{SU}(3,q_0) \cdot \gcd(3, \frac{q+1}{q_0+1})$, *where* $q = q_0^r$, *r odd and prime*;
(g) $3^2 : \mathrm{Q}_8$ *with* $p = q \equiv 2 \pmod 3$, $q \geqslant 11$ (*novelty if* $q = 5$);
(h) $\mathrm{PSL}(2,7)$ *with* $q \neq 5$, $p = q \equiv 3, 5, 6 \pmod 7$ (*novelty if* $q = 5$);
(i) A_6 *with* $p = q \equiv 11, 14 \pmod{15}$;
(j) $\mathrm{A}_6 \cdot 2_3$ *with* $q = 5$;
(k) A_7 *with* $q = 5$.

3. Proof of Theorem 1.1

In this section, suppose that $\mathcal{D}$ is a symmetric (v,k,λ) design and G is an almost simple automorphism group G with simple socle $X := \mathrm{PSU}(3,q)$, where $q = p^f$ (p prime), that is to say, $X \lhd G \leqslant \mathrm{Aut}(X)$. Suppose also that V is the underlying vector space of X over the finite field $\mathbb{F}_{q^2}$.

Let now G be a flag-transitive and point-primitive automorphism group of $\mathcal{D}$. Then the point-stabiliser $M := G_\alpha$ is maximal in G [5, Corollary 1.5A]. Set $M_0 := X \cap M$. So M_0 is (isomorphic to) one of the subgroups listed in Lemma 2.4(a)-(k). Moreover, by Lemma 2.2,

$$v = \frac{|X|}{|M_0|} = \frac{q^3(q^2-1)(q^3+1)}{\gcd(3,q+1)\cdot|M_0|}. \tag{3.1}$$

Note that $|\mathrm{Out}(X)| = 2f \cdot \gcd(3, q+1)$. Therefore, by Lemma 2.1(b) and Lemma 2.2(b),

$$k \mid 2f \cdot \gcd(3,q+1) \cdot |M_0|. \tag{3.2}$$

In what follows, considering possible structure for the subgroup M_0 as in Lemma 2.4(b)-(k), we prove that none of these cases could occur.

Lemma 3.1. *The subgroup M_0 cannot be* $\hat{}\mathrm{GU}(2,q)$.

Proof. Let V be the underlying vector space of $X = \mathrm{PSU}(3,q)$ over the finite field $\mathbb{F}_{q^2}$. By (3.1), we have that $v = q^2(q^2-q+1)$. It follows from [15, Lemma 3.9] and Lemma 2.1(d) that k divides $\lambda(q^2-1)(q+1)$, and by Lemma 2.1(c), k divides $\lambda \gcd(q(q^2-1)(q+1), q^4-q^3+q^2-1) = 16\lambda(q-1)$, so $k \mid 16\lambda(q-1)$. Let now m be a positive integer such that $mk = 16\lambda(q-1)$. By Lemma 2.1(a), $k(k-1) = \lambda(v-1)$, and so

$$k = \frac{m(q^3+q+1)}{16} + 1. \tag{3.3}$$

Moreover, $k \mid 6fq(q^2-1)(q+1)$, and by (3.3), we have $k \mid 6f(q+1)(m(q^3+q+1)+16)$. Thus

$$k \mid 6fm(2q^2+3q+1)+96f(q+1), \tag{3.4}$$

and so $16k < 96fm(2q^2+3q+1)+16\cdot 96fm(q+1)$. By (3.3), we have that $m(q^3+q+1)+16 < 96fm(2q^2+19q+17)$. Therefore $q/2 < 96f+5$. This inequality holds when

$$\begin{array}{ll} p=2, & f \leqslant 11;\\ p=3, & f \leqslant 6;\\ p=5, & f \leqslant 4;\\ p=7, & f \leqslant 3;\\ p \in \{11,13,17,19\}, & f \leqslant 2;\\ 23 \leqslant p \leqslant 193 \text{ (prime)}, & f=1. \end{array} \tag{3.5}$$

The possible values of k and v are listed in Table 1 below. For such parameters k and v as in Table 1, by straightforward calculation, we observe that Lemma 2.1(a) does not hold, which is a contradiction. □

Lemma 3.2. *The subgroup M_0 cannot be* $\hat{}(q^2-q+1):3$.

Proof. Here, by (3.1), we have $v = q^3(q^2-1)(q+1)/3$. Note that $|\mathrm{Out}(X)| = 2\cdot\gcd(3,q+1)\cdot f$. Then by (3.2), we conclude that k divides $6f(q^2-q+1)$. By [20, 30], we may assume that $\lambda \geqslant 4$, and so Lemma 2.1(b) yields

$$\frac{4q^3(q^2-1)(q+1)}{3} \leqslant \lambda v < k^2 \leqslant 36f^2(q^2-q+1)^2.$$

Then $q^3(q^2-1)(q+1) < 27f^2(q^2-q+1)^2$. It is easy to observe that $q^2 < \frac{q^3(q^2-1)(q+1)}{(q^2-q+1)^2}$ for $q \geqslant 2$. Then $q^2 < 27f^2$. This inequality holds when

$$\begin{array}{ll} p=2, & f \leqslant 4;\\ p=3, & f \leqslant 2;\\ p=5, & f=1. \end{array} \tag{3.6}$$

Recall that k is a divisor of $6f(q^2-q+1)$. Then, for each $q=p^f$ with p and f as in (3.6), the possible values of k and v are listed in Table 2 below. It is a contradiction as for each k and v as in Table 2, $v-1$ dose not divide $k(k-1)$. □

Lemma 3.3. *The subgroup M_0 cannot be* $\hat{}(q+1)^2:\mathrm{S}_3$.

Proof. The argument here is the same as proof of Lemma 3.2. By (3.1), we have $v = q^3(q-1)(q^2-q+1)/6$, and since $|\mathrm{Out}(X)| = 2f\cdot\gcd(3,q+1)$, it follows from (3.2) that k divides $12f(q+1)^2$. As λ is at least 4, by Lemma 2.1(b), we must have

$$\frac{4q^3(q-1)(q^2-q+1)}{6} \leqslant \lambda v < k^2 \leqslant 144f^2(q+1)^4.$$

Table 1. Possible value for k and v when p and f are as in (3.5).

q	v	k divides	q	v	k divides
2	12	108	97	87626017	536594688
3	63	576	101	103040301	630482400
4	208	3600	103	111468763	681797376
5	525	4320	107	129866007	793758528
7	2107	16128	109	139875013	854647200
8	3648	81648	113	161617233	986864256
9	5913	86400	121	212601961	2593388160
11	13431	95040	125	242203125	4429404000
13	26533	183456	127	258112387	1573060608
16	61696	1664640	128	266354688	11361676032
17	78897	528768	131	292268991	1780384320
19	123823	820800	137	349722777	2128966848
23	268203	1748736	139	370634743	2255803200
25	375625	4867200	149	489598653	2977020000
27	512487	9906624	151	516465451	3139833600
29	683733	4384800	157	603727957	3668509728
31	894691	5713920	163	701607583	4261294656
32	1016832	32408640	167	773166747	4694554368
37	1824877	11540448	169	810932473	9846345600
41	2758521	17357760	173	890597253	5405355936
43	3341143	20978496	179	1020922383	6193972800
47	4778067	29887488	181	1067386141	6475079520
49	5649553	70560000	191	1323931971	8026767360
53	7744413	48218976	193	1380336193	8367837696
59	11915463	73915200	243	3472494543	105032220480
61	13622581	84414240	256	4278255616	206960578560
64	16519168	613267200	289	6951703393	83997734400
67	19854847	122683968	343	13801051243	249867410688
71	25058811	154586880	361	16936647481	204365738880
73	28014553	172691136	512	68585521152	3718085317632
79	38463283	236620800	625	152344140625	3667959360000
81	42521841	1045716480	729	282042647433	10181391292800
83	46893423	288138816	1024	1098438934528	66035059200000
89	62045193	380635200	2048	17583600304128	1161650937655296

This implies that $q^3(q-1)(q^2-q+1) < 216f^2(q+1)^4$, and since $q+1 \leqslant (3/2)q$, we have

$$q^3(q-1)(q^2-q+1) < 216f^2(q+1)^4 \leqslant (\frac{3^7}{2})f^2q^4.$$

It follows that

$$(q-1)(q^2-q+1) < (\frac{3^7}{2})f^2q,$$

TABLE 2. Possible value for k and v when $q = p^f$ with p and f as in (3.6).

q	2	3	4	5	8	9	16
v	24	288	1600	6000	96768	194400	5918720
k divides	18	42	156	126	1026	876	5784

and so $q^2 - 2q + 2 - 1/q < (3^7/2)f^2$. Therefore, $(q-1)^2 < (3^7/2)f^2$, and since $(16/81)(q-1)^2 < q^3(q-1)(q^2-q+1)/(q+1)^4$, we must have $q = p^f < (27\sqrt{6}/2)f + 1$. This is true only when

$$\begin{array}{ll} p = 2, & f \leqslant 8; \\ p = 3, & f \leqslant 4; \\ p = 5, 7, & f \leqslant 2; \\ 11 \leqslant p \leqslant 31, & f = 1. \end{array} \tag{3.7}$$

TABLE 3. Possible value for k and v when $q = p^f$ is as in (3.7).

q	v	k divides	q	v	k divides
2	4	108	23	22618453	6912
3	63	192	25	37562500	16224
4	416	600	27	59960979	28224
5	1750	432	29	92531866	10800
7	14749	768	31	138677105	12288
8	34048	2916	32	168116224	65340
9	70956	2400	49	2214624776	60000
11	246235	1728	64	11100880896	304200
13	689858	2352	81	45923588280	322752
16	2467840	13872	128	721643634688	1397844
17	3576664	3888	256	46547421102080	6340704
19	7057911	4800			

Since k is a divisor of $12f(q+1)^2$, for each $q = p^f$ with p and f as in (3.7), the possible values of k and v are listed in Table 3. This leads us to a contradiction as, for parameters k and v as in Table 3, the fraction $k(k-1)/(v-1)$ is not integer. $\square$

Lemma 3.4. *The subgroup M_0 cannot be* $\mathrm{SO}_3(q)$ *with $q \geqslant 7$, odd.*

Proof. By (3.1), we have that $v = q^2(q^3+1)/d$ with $d = \gcd(3, q+1)$. It follows from (3.2) that k divides $2dfq(q^2-1)$, and so k is a divisor of $6fq(q^2-1)$.

Moreover, Lemma 2.1(a) implies that k divides $\lambda(v-1)$. Note by Lemma 2.3 that $v-1$ is coprime to q. Thus k divides $6\lambda f \gcd(q^2-1, v-1)$. Let $d = 1$, then $v - 1 = q^5 + q^2 - 1$, and so $\gcd(q^2-1, v-1) = \gcd(q^2-1, q) = 1$. Thus, in this case k divides $6\lambda f$. Let now $d = 3$. Then $v - 1 = (q^5 + q^2 - 3)/3$, and so $\gcd(q^2-1, 3(v-1)) = \gcd(q^2-1, q-2) = \gcd(q-2, 3) = 1$ or 3. Since k

TABLE 4. Possible value for k and v when $q = p^f$ is as in (3.9).

q	v	k divides	q	v	k divides
7	16856	672	17	473382	29376
9	59130	2880	25	9766250	62400
11	53724	7920	27	4783212	117936
13	371462	4368			

divides $6\lambda f \gcd(q^2-1, v-1)$, we conclude that k divides $18\lambda f$. Therefore in either of case, k is a divisor of $18\lambda f$. Then there exists a positive integer m such that $mk = 18\lambda f$. Since $k(k-1) = \lambda(v-1)$, it follows that

$$\frac{18\lambda f}{m}(k-1) = \frac{\lambda(q^5+q^2-d)}{d},$$

where $d = \gcd(3, q+1)$. Thus

$$k = \frac{m(q^5+q^2-d)}{18df} + 1. \tag{3.8}$$

Since $d = 1$ or 3, we have by (3.2) that $k \mid 6fq(q^2-1)$. Then (3.8) yields $m(q^5+q^2-d) \leqslant 108df^2q(q^2-1)$. Since also $m \geqslant 1$ and $d \leqslant 3$, we have that $q^2 < q^5+q^2-3/q(q^2-1) \leqslant 324f^2$. This inequality only holds for

$$q \in \{7, 9, 11, 13, 17, 25, 27\}. \tag{3.9}$$

For these values of q, since k divides $2dfq(q^2-1)$, the possible values of k can be found as in Table 4. This leads us to a contradiction as for each value of v and k as in Table 4, the fraction $k(k-1)/(v-1)$ is not integer. □

Lemma 3.5. *The subgroup M_0 cannot be $\hat{}\mathrm{SU}(3,q_0)\cdot c$, where $q = q_0^r$, r odd prime and $c := \gcd\left(3, \frac{q+1}{q_0+1}\right)$.*

Proof. In this case, $|M_0| = c \cdot q_0^3(q_0^2-1)(q_0^3+1)/\gcd(3, q+1)$. It follows from (3.1) that

$$v = \frac{1}{c} \cdot \frac{q_0^{3r}(q_0^{2r}-1)(q_0^{3r}+1)}{q_0^3(q_0^2-1)(q_0^3+1)}. \tag{3.10}$$

Note by (3.2) that k divides $6fq_0^3(q_0^2-1)(q_0^3+1)$. We may assume that $\lambda \geqslant 4$ by [20, 30]. Moreover, $c \in \{1, 3\}$, and $f^2 \leqslant q_0^r$ as $q = q_0^r$. Since $\lambda v < k^2$ by Lemma 2.1(b), we must have

$$\begin{aligned}\frac{4q_0^{3r}(q_0^{2r}-1)(q_0^{3r}+1)}{3q_0^3(q_0^2-1)(q_0^3+1)} \leqslant \lambda v < k^2 &\leqslant 36f^2q_0^6(q_0^2-1)^2(q_0^3+1)^2\\ &\leqslant 36q_0^{6+r}(q_0^2-1)^2(q_0^3+1)^2.\end{aligned}$$

Therefore $q_0^{3r}(q_0^{2r}-1)(q_0^{3r}+1) < 27q_0^{9+r}(q_0^2-1)^3(q_0^3+1)^3$. Since $q_0^{8r-1} \leqslant q_0^{3r}(q_0^{2r}-1)(q_0^{3r}-1)$ and $q_0^{9+r}(q_0^2-1)^3(q_0^3-1)^3 \leqslant q_0^{24+r}$, we have that $q_0^{8r-1} <$

$27q_0^{24+r}$, and so $q_0^{7r-25} < 27$. But $q_0 \geqslant 2$ and r is odd. Then $r = 3$. Therefore, by (3.10), we have that

$$v = \frac{1}{c} \cdot \frac{q_0^6(q_0^3-1)(q_0^9+1)}{q_0^2-1}, \tag{3.11}$$

where $c := \gcd\left(3, \frac{q+1}{q_0+1}\right)$. By (3.2), k divides $6fq_0^3(q_0^2-1)(q_0^3+1)$. It follows from Lemma 2.1(b), that

$$\lambda \frac{q_0^6(q_0^3-1)(q_0^9+1)}{c(q_0^2-1)} < k^2 \leqslant 36f^2q_0^6(q_0^2-1)^2(q_0^3+1)^2.$$

Therefore

$$\lambda < 36cf^2\frac{(q_0^2-1)^3(q_0^3+1)^2}{(q_0^3-1)(q_0^9+1)} \leqslant 108f^2. \tag{3.12}$$

We now observe by Lemma 2.3 that $v-1$ and q are coprime, and since k divides both $6fq_0^3(q_0^2-1)(q_0^3+1)$ and $\lambda(v-1)$, again by Lemma 2.1(b), we must have

$$\lambda \frac{q_0^6(q_0^3-1)(q_0^9+1)}{c(q_0^2-1)} < k^2 \leqslant 36\lambda^2f^2(q_0^2-1)^2(q_0^3+1)^2,$$

and so

$$\frac{q_0^6(q_0^3-1)(q_0^9+1)}{(q_0^2-1)^3(q_0^3+1)^2} < 36\lambda f^2 c. \tag{3.13}$$

Since $c \leqslant 3$ and $\lambda \leqslant 108f^2$ by (3.12), it follows that $q_0^6 < 23328f^4$. Since also q_0 is at least 2, we conclude that $2^{2f} < 23328 \cdot f^4$, and this holds for $f \leqslant 15$. Then $q_0 \leqslant 32$. Considering (3.13), q_0 is one of the numbers: 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 25, 27, 29, 31, 32, 1923. Inspecting each such value of q_0, we observe by (3.10) that $v = 896$ is the only possible value when $q_0 = 2$. In this case, k divides $6fq_0^3(q_0^2-1)(q_0^3+1) = 1296$. But by straightforward calculation, $v-1 = 895$ is not a divisor of $k(k-1)$, for each divisor k of 1296, contradicting Lemma 2.1(a). □

Lemma 3.6. *The subgroup M_0 cannot be $3^2 : \mathrm{Q}_8$ with $q \geqslant 11$ and $p = q \equiv 2 \pmod 3$.*

Proof. By (3.1), we have that

$$v = \frac{q^3(q^2-1)(q^3+1)}{72 \cdot \gcd(3, q+1)}. \tag{3.14}$$

Note that $|\mathrm{Out}(X)| = 2f \cdot \gcd(3, q+1)$. Then by (3.2), we conclude that k divides $432f$. Since $\lambda \geqslant 4$, Lemma 2.1(b) implies that

$$\frac{4q^3(q^2-1)(q^3+1)}{216} \leqslant \lambda v < k^2 \leqslant 432^2f^2.$$

Therefore $q^3(q^2-1)(q^3+1) < 10077696f^2$ and this implies that $q \in \{3, 5, 7\}$ but this violates $q \geqslant 11$. □

Lemma 3.7. *The subgroup M_0 cannot be* $\mathrm{PSL}(2,7)$ *with* $p=q\equiv 3,5,6 \pmod 7$.

Proof. Note that $f=1$ as $q=p$ by (3.1), we have that

$$v=\frac{q^3(q^2-1)(q^3+1)}{168\cdot \gcd(3,q+1)}. \tag{3.15}$$

As $f=1$, by (3.2) that k divides 1008. Moreover, since $\lambda \geqslant 4$, by Lemma 2.1(b),

$$\frac{4q^3(q^2-1)(q^3+1)}{504}\leqslant \lambda v<k^2\leqslant 1008^2.$$

Then $q^3(q^2-1)(q^3+1)<128024064$. Note that $q\equiv 3,5,6 \pmod 7$. Thus $q\in\{3,11\}$. Note that $q\neq 11$ as v given in (3.15) must be integer. If $q=3$, then $v=36$, but $v-1=35$ does not divides $k(k-1)$ which is also a contradiction. □

Lemma 3.8. *The subgroup M_0 cannot be* A_6, *with* $p=q\equiv 11,14 \pmod{15}$.

Proof. By (3.1), we have that

$$v=\frac{q^3(q^2-1)(q^3+1)}{360\cdot \gcd(3,q+1)}. \tag{3.16}$$

Note by (3.2) that k divides $2160f$. By [20, 30], we may only focus on $\lambda\geqslant 4$, and so Lemma 2.1(b) yields

$$\frac{4q^3(q^2-1)(q^3+1)}{1080}\leqslant \lambda v<k^2\leqslant 2160^2f^2.$$

This follows that

$$q^3(q^2-1)(q^3+1)<1259712000f^2. \tag{3.17}$$

Since $q^8<2q^3(q^2-1)(q^3+1)$ and $q=p^f$ is odd, (3.17) implies that $q\in\{3,5,7,9,11,13\}$. Since also the fraction (3.16) must be integer, $q\in\{5,9,11\}$, and since $p=q\equiv 11,14 \pmod{15}$, the only acceptable value for q is $q=11$. So $v=196988$ and k divides 2160. We then easily observe that, for each divisor k of 2160, the fraction $k(k-1)/(v-1)$ is not integer, which is a contradiction. □

Lemma 3.9. *The subgroup M_0 cannot be* $\mathrm{A}_6\cdot 2_3$ *with* $q=5$.

Proof. By (3.1), we have that

$$v=\frac{q^3(q^2-1)(q^3+1)}{720\cdot \gcd(3,q+1)}=175.$$

It follows from (3.2) that k divides 4320. In this case, for each possible value of k the fraction $k(k-1)/(v-1)$ is not integer, which is a contradiction. □

Lemma 3.10. *The subgroup M_0 cannot be* A_7 *with* $q=5$.

Proof. By (3.1), we have that

$$v = \frac{q^3(q^2-1)(q^3+1)}{2520 \cdot \gcd(3, q+1)} = 50.$$

Note by (3.2) that k divides 15120. Moreover, Lemma 2.1(a) implies that k divides $\lambda(v-1)$. Then k divides $\gcd(15120, \lambda(v-1))$, and so k divides 7λ. Thus there exists a positive integer m such that $mk = 7\lambda$. Since $k(k-1) = \lambda(v-1)$, it follows that $k = 7m+1$. Since k divides 15120 and $k < v$, we have $k = 15$. This is a contradiction as $v - 1 = 49$ does not divide $k(k-1)$. □

3.1. Proof of Theorem 1.1

Suppose that $\mathcal{D}$ is a symmetric (v, k, λ) design and G is an almost simple automorphism group with simple socle $X = \mathrm{PSU}(3, q)$. If G is a flag-transitive and point-primitive automorphism group of $\mathcal{D}$, then the point-stabiliser $M := G_\alpha$ is maximal in G, and so $M_0 := X \cap M$ is isomorphic to one of the subgroups in Lemma 2.4. It follows from Lemmas 3.1–3.10 that $M_0 = \hat{}[q]^{1+2} : (q^2-1)$. In this case, by (3.1), we have that $v = q^3+1$. Then by [15, Lemma 3.9] and Lemma 2.1(c), k divides λq^3. Let now m be a positive integer such that $mk = \lambda q^3$. Since $\lambda < k$, we have that $m < q^3$. By Lemma 2.1(a), $k(k-1) = \lambda(v-1)$, and so $\lambda q^3(k-1)/m = \lambda q^3$. Thus, $k = m+1$ and $\lambda = (m^2+m)/q^3$ which the latter statement implies that $q^3 \mid m^2+m$. Thus, q^3 divides either m, or $m+1$. Since $m < q^3$, it follows that q^3 divides $m+1$, and so $q^3 = m+1$. Therefore, $\lambda = q^3 - 1 = k - 1$ and $v = q^3+1$, that is to say, $\mathcal{D}$ is a $(v, v-1, v-2)$ design with $b = \binom{v}{k}$, which is a complete design.

References

[1] S. H. Alavi and M. Bayat, *Flag-transitive point-primitive symmetric designs and three dimensional projective special linear groups*, Bull. Iranian Math. Soc. **42** (2016), no. 1, 201–221.

[2] S. H. Alavi, M. Bayat, and A. Daneshkhah, *Symmetric designs admitting flag-transitive and point-primitive automorphism groups associated to two dimensional projective special groups*, Des. Codes Cryptogr. **79** (2016), no. 2, 337–351.

[3] J. N. Bray, D. F. Holt, and C. M. Roney-Dougal, *The maximal subgroups of the low-dimensional finite classical groups*, volume 407 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2013.

[4] P. J. Cameron and C. E. Praeger, *Constructing flag-transitive, point-imprimitive designs*, J. Algebraic Combin. **43** (2016), no. 4, 755–769.

[5] J. D. Dixon and B. Mortimer, *Permutation groups*, volume 163 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1996.

[6] H. Dong and S. Zhou, *Alternating groups and flag-transitive* 2-$(v, k, 4)$ *symmetric designs*, J. Combin. Des. **19** (2011), no. 6, 475–483.

[7] ———, *Affine groups and flag-transitive triplanes*, Sci. China Math. **55** (2012), no. 12, 2557–2578.

[8] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.9*, 2015.

[9] R. W. Hartley, *Determination of the ternary collineation groups whose coefficients lie in the* GF(2^n), Ann. of Math. (2) **27** (1925), no. 2, 140–158.

[10] D. R. Hughes and F. C. Piper, *Design Theory*, Cambridge University Press, Cambridge, 1985.
[11] W. M. Kantor, *Primitive permutation groups of odd degree, and an application to finite projective planes*, J. Algebra **106** (1987), no. 1, 15–45.
[12] P. B. Kleidman, *The subgroup structure of some finite simple groups*, Ph.D. Thesis, University of Cambridge, 1987.
[13] E. S. Lander, *Symmetric designs: an algebraic approach*, volume 74 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1983.
[14] M. Law, C. E. Praeger, and S. Reichard, *Flag-transitive symmetric* 2-$(96, 20, 4)$*-designs*, J. Combin. Theory Ser. A **116** (2009), no. 5, 1009–1022.
[15] M. W. Liebeck, J. Saxl, and G. M. Seitz, *On the overgroups of irreducible subgroups of the finite classical groups*, Proc. London Math. Soc. (3) **55** (1987), no. 3, 507–537.
[16] H. H. Mitchell, *Determination of the ordinary and modular ternary linear groups*, Trans. Amer. Math. Soc. **12** (1911), no. 2, 207–242.
[17] E. O'Reilly-Regueiro, *Flag-transitive symmetric designs*, PhD thesis, University of London, 2003.
[18] ———, *Biplanes with flag-transitive automorphism groups of almost simple type, with alternating or sporadic socle*, European J. Combin. **26** (2005), no. 5, 577–584.
[19] ———, *On primitivity and reduction for flag-transitive symmetric designs*, J. Combin. Theory Ser. A **109** (2005), no. 1, 135–148.
[20] ———, *Biplanes with flag-transitive automorphism groups of almost simple type, with classical socle*, J. Algebraic Combin. **26** (2007), no. 4, 529–552.
[21] ———, *Biplanes with flag-transitive automorphism groups of almost simple type, with exceptional socle of Lie type*, J. Algebraic Combin. **27** (2008), no. 4, 479–491.
[22] C. E. Praeger and S. Zhou, *Imprimitive flag-transitive symmetric designs*, J. Combin. Theory Ser. A **113** (2006), no. 7, 1381–1395.
[23] G. M. Seitz, *Flag-transitive subgroups of Chevalley groups*, Ann. of Math. (2) **97** (1973), 27–56.
[24] D. Tian and S. Zhou, *Flag-transitive point-primitive symmetric* (v, k, λ) *designs with* λ *at most 100*, J. Combin. Des. **21** (2013), no. 4, 127–141.
[25] ———, *Flag-transitive 2-*(v, k, λ) *symmetric designs with sporadic socle*, J. Combin. Des. **23** (2015), no. 4, 140–150.
[26] ———, *Classification of flag-transitive primitive symmetric* (v, k, λ) *designs with* $\mathrm{PSL}(2, q)$ *as socle*, J. Math. Res. Appl. **36** (2016), no. 2, 127–139.
[27] S. Zhou and H. Dong, *Sporadic groups and flag-transitive triplanes*, Sci. China Ser. A **52** (2009), no. 2, 394–400.
[28] ———, *Alternating groups and flag-transitive triplanes*, Des. Codes Cryptogr. **57** (2010), no. 2, 117–126.
[29] ———, *Exceptional groups of Lie type and flag-transitive triplanes*, Sci. China Math. **53** (2010), no. 2, 447–456.
[30] S. Zhou, H. Dong, and W. Fang, *Finite classical groups and flag-transitive triplanes*, Discrete Math. **309** (2009), no. 16, 5183–5195.
[31] S. Zhou and D. Tian, *Flag-transitive point-primitive 2-*$(v, k, 4)$ *symmetric designs and two dimensional classical groups*, Appl. Math. J. Chinese Univ. Ser. B **26** (2011), no. 3, 334–341.

Ashraf Daneshkhah
Department of Mathematics
Faculty of Science
Bu-Ali Sina University
Hamedan 65178-38695, Iran
E-mail address: adanesh@basu.ac.ir, daneshkhah.ashraf@gmail.com

Sheyda Zang Zarin
Department of Mathematics
Faculty of Science
Bu-Ali Sina University
Hamedan 65178-38695, Iran
E-mail address: zarinsheyda@gmail.com