

## CLOSED CONVEX SPACELIKE HYPERSURFACES IN LOCALLY SYMMETRIC LORENTZ SPACES

ZHONGYANG SUN

ABSTRACT. In 1997, H. Li [12] proposed a conjecture: if  $M^n (n \geq 3)$  is a complete spacelike hypersurface in de Sitter space  $S_1^{n+1}(1)$  with constant normalized scalar curvature  $R$  satisfying  $\frac{n-2}{n} \leq R \leq 1$ , then is  $M^n$  totally umbilical? Recently, F. E. C. Camargo et al. ([5]) partially proved the conjecture. In this paper, from a different viewpoint, we study closed convex spacelike hypersurface  $M^n$  in locally symmetric Lorentz space  $L_1^{n+1}$  and also prove that  $M^n$  is totally umbilical if the square of length of second fundamental form of the closed convex spacelike hypersurface  $M^n$  is constant, i.e., Theorem 1. On the other hand, we obtain that if the sectional curvature of the closed convex spacelike hypersurface  $M^n$  in locally symmetric Lorentz space  $L_1^{n+1}$  satisfies  $K(M^n) > 0$ , then  $M^n$  is totally umbilical, i.e., Theorem 2.

### 1. Introduction and main results

Let  $L_p^{n+p}$  be an  $(n+p)$ -dimensional connected semi-Riemannian manifold of index  $p (\geq 1)$ . In particular,  $L_1^{n+1}$  is called a Lorentz space. A hypersurface  $M^n$  of a Lorentz space is said to be *spacelike* if the metric on  $M^n$  induced from that of the Lorentz space is positive definite. A Lorentz space is said to be *locally symmetric Lorentz space* if the components  $\bar{R}_{ABCD,E}$  of the covariant derivative of the Lorentz space  $L_1^{n+1}$  curvature tensor  $\bar{R}_{ABCD,E} = 0$ . When the Lorentz space is of constant curvature  $c$ , we call it Lorentz space form. It is well-known that a de Sitter space  $S_1^{n+1}(1)$  is a simply connected Lorentz space form with constant sectional curvature 1.

In 1977, Goddard [9] proposed the conjecture: If  $M^n$  is a complete spacelike hypersurface with constant mean curvature  $H$  in a de Sitter space  $S_1^{n+1}(c)$ , then is  $M^n$  totally umbilical? In general, the conjecture is false. J. Ramanathan [16] proved Goddard's conjecture for  $S_1^3(1)$  and  $0 \leq H \leq 1$ . Moreover, when

---

Received August 19, 2016; Accepted February 24, 2017.

2010 *Mathematics Subject Classification*. Primary 53C50, 53C24, 53C20.

*Key words and phrases*. locally symmetric Lorentz space, second fundamental form, spacelike hypersurface, de Sitter space.

This work was supported by the National Natural Science Foundation of China (No. 11501500).

$H > 1$ , he also showed that the conjecture is false. When  $H^2 \leq c$  if  $n = 2$  or when  $n^2 H^2 < 4(n-1)c$  if  $n \geq 3$ , K. Akutagawa [1] proved that Goddard's conjecture is true. There are also many results such as [3, 11, 14, 15].

On the other hand, concerning the study of spacelike hypersurface with constant scalar curvature in a de Sitter space, H. Li [12] proposed a conjecture:

**Conjecture 1.** *If  $M^n (n \geq 3)$  is a complete spacelike hypersurface in de Sitter space  $S_1^{n+1}(1)$  with constant normalized scalar curvature  $R$  satisfying  $\frac{n-2}{n} \leq R \leq 1$ , then is  $M^n$  totally umbilical?*

F. E. C. Camargo et al. [5] proved that Li's question is true if the mean curvature  $H$  is bounded. Next J. C. Liu and Z. Y. Sun [13] generalized the partial Conjecture's result of F. E. C. Camargo et al. [5] in locally symmetric Lorentz space  $L_1^{n+1}$ . Recently, Z. Y. Sun [18] generalized the partial Conjecture's result of F. E. C. Camargo et al. [5] in Lorentzian manifold. There are also many results such as [4, 6] and [10].

In this paper, first we recall that Choi et al. [8, 17] introduced the class of  $(n+1)$ -dimensional Lorentz spaces  $L_1^{n+1}$  of index 1 which satisfy the following conditions for some constants  $c_1$  and  $c_2$ :

(I) for any spacelike vector  $u$  and any timelike vector  $v$

$$(1.1) \quad \overline{K}(u, v) = -\frac{c_1}{n},$$

(II) for any spacelike vectors  $u$  and  $v$

$$(1.2) \quad \overline{K}(u, v) \geq c_2,$$

where  $\overline{K}$  denotes the sectional curvature on  $L_1^{n+1}$ .

When  $L_1^{n+1}$  satisfies conditions (1.1) and (1.2), we will say that  $L_1^{n+1}$  satisfies condition (\*).

*Remark 1.* The Lorentz space form  $\overline{M}_1^{n+1}(c)$  satisfies condition (\*), where  $-\frac{c_1}{n} = c_2 = c$ .

There are several examples of Lorentz spaces which are not Lorentz space forms and satisfies condition (\*). For instance, semi-Riemannian product manifold  $H_1^k(-\frac{c_1}{n}) \times N^{n+1-k}(c_2)$ ,  $c_1 > 0$ , and  $\mathbf{R}_1^k \times S^{n+1-k}(1)$ . In particular,  $\mathbf{R}_1^1 \times S^n(1)$  is a so-called *Einstein Static Universe*. Also the *Robertson-Walker spacetime*  $N(c, f) = I \times_f N^3(c)$  is another general example of Lorentz space, where  $I$  denotes an open interval of  $\mathbf{R}_1^1$  and  $f > 0$  a smooth function defined on the interval  $I$ ,  $N^3(c)$  a 3-dimensional Riemannian manifold of constant curvature  $c$ .  $N(c, f)$  also satisfy (\*) if we choose an appropriate function  $f$ . For more details, we refer the readers to [8, 17].

In order to prove our theorems, first we introduce some basic facts and notations. Let  $\overline{R}_{CD}$  be the components of the Ricci tensor of  $L_1^{n+1}$  satisfying

(\*), then the scalar curvature  $\bar{R}$  of  $L_1^{n+1}$  is given by

$$\bar{R} = \sum_{A=1}^{n+1} \epsilon_A \bar{R}_{AA} = -2 \sum_{i=1}^n \bar{R}_{(n+1)ii(n+1)} + \sum_{i,j=1}^n \bar{R}_{ijji} = 2c_1 + \sum_{i,j=1}^n \bar{R}_{ijji}.$$

It is well known that  $\bar{R}$  is constant when the Lorentz space  $L_1^{n+1}$  is locally symmetric. Hence  $\sum_{i,j=1}^n \bar{R}_{ijji}$  is constant. From (2.1) in Section 2, we can define a  $P$  such that

$$(1.3) \quad n(n-1)P = n^2H^2 - S = \sum_{i,j=1}^n \bar{R}_{ijji} - n(n-1)R.$$

Hence, when  $M^n$  is a spacelike hypersurface in locally symmetric Lorentz space  $L_1^{n+1}$  satisfying (\*), we conclude from (1.3) that the normalized scalar curvature  $R$  of  $M^n$  is constant if and only if  $P$  is constant.

With the appearance of Simons integral inequality, there are many research results on the classification of the submanifolds with constant square of length of second fundamental form. Chern-do Carmo-Kobayashi [7] gave some results on the classification of the submanifolds with constant square of length of second fundamental form in sphere spaces. Q. L. Wang [19] studied spacelike hypersurfaces with constant square of length of second fundamental form in sphere spaces and obtained two sufficient conditions  $H^2 \geq \frac{n-1}{n^2}S$  or  $\text{Ric}(M^n) \geq n-1$  to be a totally umbilical for this hypersurfaces.

In this paper, we study closed convex spacelike hypersurfaces with constant square of length of second fundamental form in locally symmetric Lorentz spaces satisfying (\*) and obtain the following results.

**Theorem 1.** *Let  $M^n (n \geq 3)$  be a closed convex spacelike hypersurface with constant square of length of second fundamental form in an  $(n+1)$ -dimensional locally symmetric Lorentz space  $L_1^{n+1}$  satisfying (\*). Suppose that the  $P$  defined by (1.3) satisfies  $P \leq \frac{2c}{n}$  and  $c = 2c_2 + \frac{c_1}{n} > 0$ , where  $c_1, c_2$  given as in (\*). Then  $S = nH^2$  and  $M^n$  is totally umbilical.*

In particular, when we take  $L_1^{n+1} = S_1^{n+1}(1)$  in Theorem 1, then  $\frac{c_1}{n} = c_2 = c = 1$  and  $P = 1 - R$ . Meanwhile, the condition  $P \leq \frac{2c}{n}$  reduces to  $R \geq \frac{n-2}{n}$  in Theorem 1. We obtain the following.

**Corollary 1.** *Let  $M^n (n \geq 3)$  be a closed convex spacelike hypersurface with normalized scalar curvature  $R$  satisfying  $R \geq \frac{n-2}{n}$  and constant square of length of second fundamental form in an  $(n+1)$ -dimensional de Sitter space  $S_1^{n+1}(1)$ . Then  $S = nH^2$  and  $M^n$  is totally umbilical.*

*Remark 2.* The condition  $R \geq \frac{n-2}{n}$  in Corollary 1 is better than the condition  $\frac{n-2}{n} \leq R \leq 1$  in Conjecture 1 and we don't need to assume that the normalized scalar curvature  $R$  is constant in Corollary 1.

If the sectional curvature of the hypersurface  $M^n$  satisfies  $K(M^n) > 0$ , then we get:

**Theorem 2.** *Let  $M^n (n \geq 3)$  be a closed convex spacelike hypersurface with constant square of length of second fundamental form in an  $(n+1)$ -dimensional locally symmetric Lorentz space  $L_1^{n+1}$  satisfying (\*). Suppose that the sectional curvature of the hypersurface  $M^n$  satisfies  $K(M^n) > 0$ . Then  $M^n$  is totally umbilical.*

When we take  $L_1^{n+1} = S_1^{n+1}(1)$  in Theorem 2, we have:

**Corollary 2.** *Let  $M^n (n \geq 3)$  be a closed convex spacelike hypersurface with constant square of length of second fundamental form in an  $(n+1)$ -dimensional de Sitter space  $S_1^{n+1}(1)$ . Suppose that the sectional curvature of the hypersurface  $M^n$  satisfies  $K(M^n) > 0$ . Then  $M^n$  is totally umbilical.*

Since the condition  $R \geq \frac{n-2}{n}$  is essential in Conjecture 1, it is not known whether does the conditions  $R \geq \frac{n-2}{n}$  and  $P \leq \frac{2c}{n}$  are essential in Corollary 1 and Theorem 1? Hence, we propose the following.

**Problem.** *Let  $M^n (n \geq 3)$  be a closed spacelike hypersurface in an  $(n+1)$ -dimensional de Sitter space  $S_1^{n+1}(1)$ . Then  $M^n$  is totally umbilical?*

## 2. Preliminaries

In this section, we review some fundamental facts and give estimate the Laplacian  $\Delta S$  of the squared length  $S$  of the second fundamental form for spacelike hypersurfaces in locally symmetric Lorentz spaces  $L_1^{n+1}$  satisfying (\*). We shall make use of the following convention on the ranges of indices:  $1 \leq A, B, C, \dots \leq n+1$ ;  $1 \leq i, j, k, \dots \leq n$ .

We assume that  $M^n$  is a spacelike hypersurface in a Lorentz space  $L_1^{n+1}$ . Choose a local field of pseudo-Riemannian orthonormal frames  $\{e_1, \dots, e_{n+1}\}$  in  $L_1^{n+1}$  such that, restricted to  $M^n$ ,  $\{e_1, \dots, e_n\}$  are tangent to  $M^n$  and  $e_{n+1}$  is normal to  $M^n$ . That is,  $\{e_1, \dots, e_n\}$  are spacelike vectors and  $e_{n+1}$  is a timelike vector. Let  $\{\omega_A\}$  and  $\{\omega_{AB}\}$  be the fields of dual frames and the connection forms of  $L_1^{n+1}$ , respectively. Let  $\epsilon_i = 1, \epsilon_{n+1} = -1$ , then the structure equations of  $L_1^{n+1}$  are given by

$$\begin{aligned} d\omega_A &= -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D \bar{R}_{ABCD} \omega_C \wedge \omega_D. \end{aligned}$$

Here the components  $\bar{R}_{CD}$  of the Ricci tensor and the scalar curvature  $\bar{R}$  of Lorentz spaces  $L_1^{n+1}$  are given, respectively, by

$$\bar{R}_{CD} = \sum_B \epsilon_B \bar{R}_{BCDB}, \quad \bar{R} = \sum_A \epsilon_A \bar{R}_{AA}.$$

The components  $\bar{R}_{ABCD;E}$  of the covariant derivative of the Riemannian curvature tensor  $\bar{R}$  are defined by

$$\sum_E \epsilon_E \bar{R}_{ABCD;E} \omega_E = d\bar{R}_{ABCD} - \sum_E \epsilon_E (\bar{R}_{EBCD} \omega_{EA} + \bar{R}_{AECD} \omega_{EB} + \bar{R}_{ABED} \omega_{EC} + \bar{R}_{ABCE} \omega_{ED}).$$

We restrict these forms to  $M^n$  in  $L_1^{n+1}$ , then  $\omega_{n+1} = 0$ . Hence, we have  $\sum_i \omega_{(n+1)i} \wedge \omega_i = 0$ . Using Cartan's lemma, we know that there are  $h_{ij}$  such that  $\omega_{(n+1)i} = \sum_j h_{ij} \omega_j$  and  $h_{ij} = h_{ji}$ , where the  $h_{ij}$  are the coefficients of the second fundamental form of  $M^n$ . This gives the second fundamental form of  $M^n$ ,  $h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j$ .

The Gauss equation, components  $R_{ij}$  of the Ricci tensor and the normalized scalar curvature  $R$  of  $M^n$  are given, respectively, by

$$\begin{aligned} R_{ijkl} &= \bar{R}_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}), \\ R_{ij} &= \sum_k \bar{R}_{kij k} - nHh_{ij} + \sum_k h_{ik}h_{kj}, \\ (2.1) \quad n(n-1)R &= \sum_{i,j} \bar{R}_{ijji} - n^2H^2 + S, \end{aligned}$$

where  $H = \frac{1}{n} \sum_j h_{jj}$  and  $S = \sum_{i,j} h_{ij}^2$  are the mean curvature and the squared length of the second fundamental form of  $M^n$ , respectively.

Let  $h_{ijk}$  and  $h_{ijkl}$  be the first and the second covariant derivatives of  $h_{ij}$ , respectively, so that

$$\begin{aligned} \sum_k h_{ijk} \omega_k &= dh_{ij} - \sum_k (h_{ik} \omega_{kj} + h_{kj} \omega_{ki}), \\ \sum_l h_{ijkl} \omega_l &= dh_{ijk} - \sum_l (h_{ljk} \omega_{li} + h_{ilk} \omega_{lj} + h_{ijl} \omega_{lk}). \end{aligned}$$

Thus, we have the Codazzi equation and the Ricci identity

$$(2.2) \quad h_{ijk} - h_{ikj} = \bar{R}_{(n+1)ijk},$$

$$(2.3) \quad h_{ijkl} - h_{ijlk} = - \sum_m (h_{im} R_{mjkl} + h_{jm} R_{mikl}).$$

Let  $\bar{R}_{ABCD;E}$  be the covariant derivative of  $\bar{R}_{ABCD}$ . Thus, restricted on  $M^n$ ,  $\bar{R}_{(n+1)ijk;l}$  is given by

$$(2.4) \quad \bar{R}_{(n+1)ijk;l} = \bar{R}_{(n+1)ijkl} + \bar{R}_{(n+1)i(n+1)k} h_{jl} + \bar{R}_{(n+1)ij(n+1)} h_{kl} + \sum_m \bar{R}_{mijk} h_{ml},$$

where  $\bar{R}_{(n+1)ijk;l}$  denotes the covariant derivative of  $\bar{R}_{(n+1)ijk}$  as a tensor on  $M^n$  so that

$$\begin{aligned} \sum_l \bar{R}_{(n+1)ijk;l} \omega_l &= d\bar{R}_{(n+1)ijk} - \sum_l \bar{R}_{(n+1)ljk} \omega_{li} \\ &\quad - \sum_l \bar{R}_{(n+1)ilk} \omega_{lj} - \sum_l \bar{R}_{(n+1)ijl} \omega_{lk}. \end{aligned}$$

Next we compute the Laplacian  $\Delta h_{ij} = \sum_k h_{ijkk}$ . Combining Gauss equation, Codazzi equation, Ricci identity and (2.4), we have (see [13])

$$\begin{aligned} \Delta h_{ij} &= (nH)_{ij} + \sum_k (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j}) \\ &\quad - \sum_k (h_{kk} \bar{R}_{(n+1)ij(n+1)} + h_{ij} \bar{R}_{(n+1)k(n+1)k}) \\ &\quad - \sum_{k,l} (2h_{kl} \bar{R}_{lij} + h_{jl} \bar{R}_{lik} + h_{il} \bar{R}_{lkj}) \\ &\quad - nH \sum_l h_{il} h_{lj} + S h_{ij}. \end{aligned}$$

According to the above equation, the Laplacian  $\Delta S$  of the squared length  $S$  of the second fundamental form  $h_{ij}$  of  $M^n$  is obtained

$$\begin{aligned} \frac{1}{2} \Delta S &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij} \Delta h_{ij} \\ &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} (nH)_{ij} h_{ij} + \sum_{i,j,k} (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j}) h_{ij} \\ (2.5) \quad &\quad - \left( \sum_{i,j} nH h_{ij} \bar{R}_{(n+1)ij(n+1)} + S \sum_k \bar{R}_{(n+1)k(n+1)k} \right) \\ &\quad - 2 \sum_{i,j,k,l} (h_{kl} h_{ij} \bar{R}_{lij} + h_{il} h_{ij} \bar{R}_{lkj}) - nH \sum_{i,j,l} h_{il} h_{lj} h_{ij} + S^2. \end{aligned}$$

Choose a local orthonormal frame field  $\{e_1, \dots, e_n\}$  such that  $h_{ij} = \lambda_i \delta_{ij}$ , where  $\lambda_i, 1 \leq i \leq n$ , are principal curvatures of  $M^n$ . Estimating the right-hand side of (2.5) by using the curvature conditions (\*), the following lemma was obtained by Liu-Sun ([13]).

**Lemma 2.1** ([13, Lemma 2.1]). *Let  $M^n$  be a spacelike hypersurface in a locally symmetric Lorentz space  $L_1^{n+1}$  satisfying (\*). Then*

$$(2.6) \quad \frac{1}{2} \Delta S \geq \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + n c (S - nH^2) + \left( S^2 - nH \sum_i \lambda_i^3 \right),$$

where  $c = 2c_2 + \frac{c_1}{n}$  and  $c_1, c_2$  are given as in (\*).

In the following, we will continue to calculate  $\Delta S$  for spacelike hypersurfaces in locally symmetry Lorentz spaces satisfying (\*). Thus, we need the following algebraic lemma.

**Lemma 2.2** ([2]). *Let  $\mu_1, \dots, \mu_n$  be real numbers such that  $\sum_i \mu_i = 0$  and  $\sum_i \mu_i^2 = B^2$ , where  $B \geq 0$  is constant. Then*

$$\left| \sum_i \mu_i^3 \right| \leq \frac{n-2}{\sqrt{n(n-1)}} B^3$$

and equality holds if and only if at least  $n-1$  of the  $\mu_i$ 's are equal.

Let  $\phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j$  be a symmetric tensor defined on  $M^n$ , where  $\phi_{ij} = h_{ij} - H\delta_{ij}$ . It is easy to check that  $\phi$  is traceless. Choose a local orthonormal frame field  $\{e_1, \dots, e_n\}$  such that  $h_{ij} = \lambda_i \delta_{ij}$  and  $\phi_{ij} = \mu_i \delta_{ij}$ . Let  $|\phi|^2 = \sum_i \mu_i^2$ . A direct computation gets

$$(2.7) \quad |\phi|^2 = S - nH^2 = \frac{1}{2n} \sum_{i,j} (\lambda_i - \lambda_j)^2.$$

Hence,  $|\phi|^2 = 0$  if and only if  $M^n$  is totally umbilical. We also get

$$\sum_i \lambda_i^3 = nH^3 + 3H \sum_i \mu_i^2 + \sum_i \mu_i^3.$$

By applying Lemma 2.2 to the real numbers  $\mu_1, \dots, \mu_n$ , we obtain

$$(2.8) \quad \begin{aligned} -nH \sum_i \lambda_i^3 &= -n^2 H^4 - 3nH^2 \sum_i \mu_i^2 - nH \sum_i \mu_i^3 \\ &\geq 2n^2 H^4 - 3nSH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H| (S - nH^2)^{\frac{3}{2}}. \end{aligned}$$

Substituting (2.7) and (2.8) into (2.6), we obtain the following.

**Lemma 2.3.** *Let  $M^n$  be a spacelike hypersurface in a locally symmetric Lorentz space  $L_1^{n+1}$  satisfying (\*), then*

$$(2.9) \quad \frac{1}{2} \Delta S \geq \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + |\phi|^2 L_{|H|}(|\phi|),$$

where  $|\phi|^2 = S - nH^2$ ,  $L_{|H|}(|\phi|) = |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H| |\phi| + nc - nH^2$ ,  $c = 2c_2 + \frac{c_1}{n}$  and  $c_1, c_2$  are given as in (\*).

### 3. The proofs of main theorems

The proof of Theorem 1. Since  $S$  is constant, from Lemma 2.3 we have

$$(3.1) \quad 0 = \frac{1}{2} \Delta S \geq \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + |\phi|^2 \left( |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H| |\phi| + nc - nH^2 \right).$$

Since  $M^n$  is a closed convex hypersurface, we take a point  $p \in M^n$  such that  $(nH)(p) = \min_{x \in M^n} (nH)(x)$ , where  $nH = \sum_i \lambda_i$ . By the maximum principle, we get

$$(3.2) \quad (nH)_{ii}(p) \geq 0, \quad i = 1, \dots, n.$$

Substituting (3.2) into (3.1), note that  $M^n$  is a convex hypersurface, i.e.,  $\lambda_i \geq 0$ , we have

$$(3.3) \quad 0 \geq |\phi|^2(p) \left( |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H| |\phi| + nc - nH^2 \right) (p).$$

Combining (1.3) and (2.7), we have

$$(3.4) \quad |\phi|^2 = n(n-1)(H^2 - P).$$

Next, we will consider the following polynomial given by

$$L_H(x) = x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H| x + nc - nH^2.$$

We claim that

$$L_H(|\phi|) > 0.$$

**Case 1,**  $H^2 < \frac{4(n-1)}{n^2}c$ . Then the discriminant of  $L_{|H|}(x)$  is negative. Therefore, we have  $L_{|H|}(|\phi|) > 0$ .

**Case 2,**  $H^2 \geq \frac{4(n-1)}{n^2}c$ . Let  $\alpha$  be the biggest root of the equation  $L_{|H|}(x) = 0$ , which is positive. We know that  $\alpha$  is the only one root of  $L_{|H|}(x)$  if  $H^2 = \frac{4(n-1)}{n^2}c$ .

If we can prove that  $(|\phi|)^2 = |\phi|^2 > \alpha^2$ , then we have  $|\phi| > \alpha$ . Hence,  $L_{|H|}(|\phi|) > 0$ . Since  $P \leq \frac{2c}{n}$  and  $c > 0$ , it follows from (3.4) that

$$(3.5) \quad |\phi|^2 \geq (n-1)(nH^2 - 2c).$$

By virtue of (3.5), it is straightforward to verify that

$$|\phi|^2 - \alpha^2 \geq \frac{n-2}{2(n-1)} \left( n^2 H^2 - nH \sqrt{n^2 H^2 - 4(n-1)c} - 2(n-1)c \right).$$

Thus,  $|\phi|^2 - \alpha^2 > 0$  if and only if

$$(3.6) \quad n^2 H^2 - nH \sqrt{n^2 H^2 - 4(n-1)c} - 2(n-1)c > 0.$$

Taking into account that the inequality (3.6) is equivalent to  $4(n-1)^2 c^2 > 0$ , which is true because of  $c > 0$ . Hence,  $|\phi|^2 - \alpha^2 > 0$  which proves our claim.

From  $L_H(|\phi|) > 0$  and (3.3), we have

$$(3.8) \quad |\phi|^2(p) \left( |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H| |\phi| + nc - nH^2 \right) (p) = 0.$$

Combining  $L_H(|\phi|) > 0$  and (3.8), we have

$$(3.9) \quad |\phi|^2(p) = 0.$$



For any point  $q \in M^n$ , combining  $S(q) = S(p)$ , (2.7) and (3.9), note that  $(nH)(p) = \min_{x \in M^n} (nH)(x) \geq 0$ , we have

$$\begin{aligned} |\phi|^2(q) &= S(q) - nH^2(q) \\ &= S(p) - nH^2(p) \\ &= |\phi|^2(p) = 0. \end{aligned}$$

Therefore, for any point  $q \in M^n$ , we have  $|\phi|^2(q) = 0$ . That is,  $|\phi|^2 = 0$  which shows  $M^n$  is totally umbilical. This completes the proof of Theorem 1.  $\square$

*The proof of Theorem 2.* Combining (2.2) and (2.3), we get

$$\begin{aligned} \Delta h_{ij} &= \sum_k h_{ikjk} + \bar{R}_{(n+1)jkk} \\ &= \sum_k \left( h_{kikj} - \sum_l (h_{kl}R_{lij}k + h_{il}R_{lkj}k) + \bar{R}_{(n+1)ijkk} \right). \end{aligned}$$

Since  $h_{kikj} = h_{kkij} + \bar{R}_{(n+1)kikj}$ , we obtain

$$(3.10) \quad \Delta h_{ij} = (nH)_{ij} + \sum_k (\bar{R}_{(n+1)ijkk} + \bar{R}_{(n+1)kikj}) - \sum_{k,l} (h_{kl}R_{lij}k + h_{il}R_{lkj}k).$$

From (3.10), we have

$$\begin{aligned} (3.11) \quad \frac{1}{2} \Delta S &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij} \Delta h_{ij} \\ &= \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \sum_k (\bar{R}_{(n+1)ijkk} + \bar{R}_{(n+1)kikj}) h_{ij} \\ &\quad - \sum_{i,j,k,l} h_{ij} (h_{kl}R_{lij}k + h_{il}R_{lkj}k) \\ &= \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \sum_k (\bar{R}_{(n+1)ijkk} + \bar{R}_{(n+1)kikj}) h_{ij} \\ &\quad + \sum_{i < k} (h_{kk} - h_{ii})^2 R_{kii}. \end{aligned}$$

Since  $L_1^{n+1}$  is locally symmetric, we have  $\sum_{i,j,k} (\bar{R}_{(n+1)ijkk} + \bar{R}_{(n+1)kikj}) h_{ij} = 0$ . Combining (3.11), we have

$$(3.12) \quad \frac{1}{2} \Delta S = \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \sum_{i < k} (h_{kk} - h_{ii})^2 R_{kii}.$$

Since  $S$  is constant, from (3.12) we have

$$(3.13) \quad 0 \geq \sum_i \lambda_i (nH)_{ii} + \sum_{i < k} (h_{kk} - h_{ii})^2 R_{kii}.$$

By taking the similar processing as in the proof of Theorem 1 on the inequality  $(nH)_{ii}(p) \geq 0$ , we can arrive to  $(nH)_{ii}(p) \geq 0$ ,  $i = 1, \dots, n$ .

Combining  $(nH)_{ii}(p) \geq 0$  and (3.13), note that  $M^n$  is convex hypersurface, i.e.,  $\lambda_i \geq 0$ , we have

$$(3.14) \quad 0 \geq \sum_{i < k} (h_{kk} - h_{ii})^2 R_{kiiik}.$$

Since  $R_{kiiik} > 0$ , from (3.14) we have  $h_{ii} = h_{jj}$ , i.e.,  $M^n$  is totally umbilical. This completes the proof of Theorem 2.  $\square$

**Acknowledgement.** The author would like to thank Professor Kefeng Liu and Professor Hongwei Xu for long-time encouragement.

### References

- [1] K. Akutagawa, *On spacelike hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z. **196** (1987), no. 1, 12–19.
- [2] H. Alencar and M. do Carmo, *Hypersurfaces with constant mean curvature in spheres*, Proc. Amer. Math. Soc. **120** (1994), no. 4, 1223–1229.
- [3] J. O. Baek, Q. M. Cheng, and Y. J. Suh, *Complete space-like hypersurface in locally symmetric Lorentz spaces*, J. Geom. Phys. **49** (2004), no. 2, 231–247.
- [4] A. Brasil Jr., A. G. Colares, and O. Palmas, *A gap theorem for complete constant scalar curvature hypersurfaces in the De Sitter space*, J. Geom. Phys. **37** (2001), no. 3, 237–250.
- [5] F. E. C. Camargo, R. M. B. Chaves, and L. A. M. Sousa Jr., *Rigidity theorems for complete spacelike hypersurfaces with constant scalar curvature in de Sitter space*, Differential Geom. Appl. **26** (2008), no. 6, 592–599.
- [6] Q. M. Cheng and S. Ishikawa, *Spacelike hypersurfaces with constant scalar curvature*, Manuscripta Math. **95** (1998), no. 4, 499–505.
- [7] S. S. Chern, M. do Carmo, and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. Chicago, Chicago, Ill., 1968) pp. 59–75 Springer, New York, 1970.
- [8] S. M. Choi, S. M. Lyu, and Y. J. Suh, *Complete space-like hypersurfaces in a Lorentz manifold*, Math. J. Toyama Univ. **22** (1999), 53–76.
- [9] A. J. Goddard, *Some remarks on the existence of spacelike hypersurfaces of constant mean curvature*, Math. Proc. Cambridge Philos. Soc. **82** (1977), no. 3, 489–495.
- [10] Z. Hu, M. Scherfner, and S. Zhai, *On spacelike hypersurfaces with constant scalar curvature in the de Sitter space*, Differential Geom. Appl. **25** (2007), no. 6, 594–611.
- [11] U. H. Ki, H. J. Kim, and H. Nakagawa, *On space-like hypersurfaces with constant mean curvature of a Lorentz space form*, Tokyo J. Math. **14** (1991), no. 1, 205–216.
- [12] H. Li, *Global rigidity theorems of hypersurfaces*, Ark. Mat. **35** (1997), no. 2, 327–351.
- [13] J. C. Liu and Z. Y. Sun, *On spacelike hypersurfaces with constant scalar curvature in locally symmetric Lorentz spaces*, J. Math. Anal. Appl. **364** (2010), no. 1, 195–203.
- [14] S. Montiel, *An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature*, Indiana Univ. Math. J. **37** (1988), no. 4, 909–917.
- [15] V. Oliker, *A priori estimates of the principal curvatures of spacelike hypersurfaces in the de Sitter space with applications to hypersurfaces in hyperbolic space*, Amer. J. Math. **114** (1992), no. 3, 605–626.
- [16] J. Ramanathan, *Complete spacelike hypersurfaces of constant mean curvature in de Sitter space*, Indiana Univ. Math. J. **36** (1987), no. 2, 349–359.

- [17] Y. J. Suh, Y. S. Choi, and H. Y. Yang, *On space-like hypersurfaces with constant mean curvature in a Lorentz manifold*, Houston J. Math. **28** (2002), no. 1, 47–70.
- [18] Z. Y. Sun, *Partial generalizations of some conjectures in Lorentzian manifolds*, Kodai Math. J. **36** (2013), no. 3, 508–526.
- [19] Q. L. Wang and C. Y. Xia, *Rigidity theorems for closed hypersurfaces in a unit sphere*, J. Geom. Phys. **55** (2005), no. 3, 227–240.

ZHONGYANG SUN  
SCHOOL OF MATHEMATICS SCIENCE  
HUAIBEI NORMAL UNIVERSITY  
HUAIBEI, ANHUI, 235000, P. R. CHINA  
AND  
CENTER OF MATHEMATICAL SCIENCES  
ZHEJIANG UNIVERSITY  
HANGZHOU, ZHEJIANG, 310027, P. R. CHINA  
*E-mail address:* [sunzhongyang12@163.com](mailto:sunzhongyang12@163.com)