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WEAK AMENABILITY OF THE LAU PRODUCT OF BANACH ALGEBRAS DEFINED BY A BANACH ALGEBRA MORPHISM

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Abstract. Let A and B be two Banach algebras and $T : B \to A$ be a bounded homomorphism, with $\|T\| \leq 1$. Recently, Dabhi, Jabbari and Haghnejad Azar (*Acta Math. Sin.* (*Engl. Ser.*) **31** (2015), no. 9, 1461–1474) obtained some results about the n-weak amenability of $A \times_T B$. In the present paper, we address a gap in the proof of these results and extend and improve them by discussing general necessary and sufficient conditions for $A \times_T B$ to be n-weakly amenable, for an integer $n \geq 0$.

1. Introduction

Let A and B be Banach algebras and let $T : B \to A$ be a continuous homomorphism with $\|T\| \leq 1$. Then the Cartesian product space $A \times B$ equipped with the norm $\|(a, b)\| = \|a\| + \|b\|$ and the algebra multiplication

$$(a, b)(c, d) = (ac + aT(d) + T(b)c, bd) \quad (a, c \in A, b, d \in B),$$

is a Banach algebra which is called the morphism product of A and B and is denoted by $A \times_T B$. This type of product was first introduced by Bhatt and Dabhi in [2] for the case where A is commutative and was extended by Dabhi, Jabbari and Haghnejad Azar in [3] for the general case; see also [5]. When $T = 0$, this multiplication is the usual coordinatewise product and so $A \times_T B$ is in fact the direct product $A \times B$. Furthermore, let A be unital with the identity element e and let $\theta : B \to \mathbb{C}$ be a non-zero multiplicative linear functional. Define $T_\theta : B \to A$ as $T_\theta(b) = \theta(b)e$ for each $b \in B$. Then the above product with respect to T_θ coincides with the product investigated by Lau in [8], for certain classes of Banach algebras and followed by Sangani Monfared in [9] for the general case. Some aspects of $A \times_T B$ are investigated by many authors in [3, 5, 7, 10].

In [3] Dabhi, Jabbari and Haghnejad Azar, for an integer $n \geq 0$, investigated the n-weak amenability properties of $A \times_T B$. As one of the main results, they

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showed that the n-weak amenability of $A \times_T B$ always implies the n-weak amenability of both A and B, and were able to prove the converse, under a suitable conditions on A and B; [3, Proposition 3.5].

In the present paper, we address a gap in the proof of these results and extend and improve them by discussing general necessary and sufficient conditions for $A \times_T B$ to be n-weakly amenable for an integer $n \geq 0$.

2. Preliminaries

Let A be a Banach algebra, and X be a Banach A-bimodule. A derivation from A into X is a linear mapping $D : A \to X$ satisfying

$$D(ab) = D(a)b + aD(b) \qquad (a, b \in A).$$

An instance of special importance is the inner derivations $d_x(a) = ax - xa$ defined for each $x \in X$. For a Banach A-bimodule X, the dual X^* of X equipped with the module actions $(fa)(x) = f(ax)$ and $(af)(x) = f(xa)$ for all $a \in A$, $x \in X$ and $f \in X^*$, is a Banach A-bimodule. Similarly, the n-th dual $X^{(n)}$ of X is a Banach A-bimodule. In particular, $A^{(n)}$ is a Banach A-bimodule.

The concept of weak amenability was first introduced by Bade, Curtis and Dales in [1] for commutative Banach algebras. A commutative Banach algebra A is called weakly amenable if every bounded derivation from A into every symmetric Banach A-bimodule is zero. This is equivalent to the fact that every bounded derivation from A into the dual Banach module A^* is inner. The latter was used by Johnson in [6] as a definition of weak amenability for the non-commutative case. The concept of n-weak amenability was initiated by Dales, Ghahramani and Grønbæk in [4], where they presented many important properties of this sort of Banach algebra. A Banach algebra A is said to be n-weakly amenable, for an integer $n \geq 0$, if every bounded derivation from A into $A^{(n)}$ is inner, where $A^{(0)} = A$. Trivially, 1-weak amenability is nothing than weak amenability.

Throughout the paper we assume that n is a non-negative integer, A and B are Banach algebras and $T : B \to A$ is a continuous homomorphism with $\|T\| \leq 1$. For brevity of notation we usually identify an element of A with its canonical image in $A^{(2n)}$, as well as an element of A^* with its image in $A^{(2n+1)}$. One can simply identify the underlying space of $(A \times_T B)^{(n)}$ with the Banach space $A^{(n)} \times B^{(n)}$ equipped with the norm $\|(f, g)\| = \|f\| + \|g\|$ when n is even and the norm $\|(f, g)\| = \max\{\|f\|, \|g\|\}$ when n is odd. A direct verification reveals that $(A \times_T B)$-module operations of $(A \times_T B)^{(n)}$ are as follows.

$$(f,g)(a,b) = \begin{cases} \big(fa + fT(b) + T^{(n)}(g)a, gb\big) & n \text{ is even} \\ \big(fa + fT(b), T^{(n)}(fa) + gb\big) & n \text{ is odd} \end{cases}$$

$$(a,b)(f,g) = \begin{cases} \big(af + T(b)f + aT^{(n)}(g), bg\big) & n \text{ is even} \\ \big(af + T(b)f, T^{(n)}(af) + bg\big) & n \text{ is odd} \end{cases}$$

for $a \in A, b \in B, f \in A^{(n)}$ and $g \in B^{(n)}$.

3. $(2n + 1)$-weak amenability

In this section we clarify the relation between $(2n+1)$-weak amenability of $A \times_T B$ and that of A and B. To do this we need the following result which characterize the continuous derivations from $A \times_T B$ into $(A \times_T B)^{(2n+1)}$.

Lemma 3.1. *A mapping $D : A \times_T B \to (A \times_T B)^{(2n+1)}$ is a continuous derivation if and only if*

$$D(a, b) = (D_A(b) + T_A(a), D_B(b) + T_B(a))$$

for all $a \in A$ and $b \in B$, where

- (a) *$D_B : B \to B^{(2n+1)}$ and $T_A : A \to A^{(2n+1)}$ are continuous derivations.*
- (b) *$D_A : B \to A^{(2n+1)}$ is a bounded linear operator such that $D_A(b_1b_2) = (T_A \circ T)(b_1b_2)$ for all $b_1, b_2 \in B$ and $D_A(b)a = (T_A \circ T)(b)a$ and $aD_A(b) = a(T_A \circ T)(b)$ for all $a \in A$ and $b \in B$.*
- (c) *$T_B : A \to B^{(2n+1)}$ is a bounded linear operator such that $(T^{(2n+1)} \circ T_A)(a_1a_2) = T_B(a_1a_2)$ for all $a_1, a_2 \in A$ and $bT_B(a) = b(T^{(2n+1)} \circ T_A)(a)$ and $T_B(a)b = (T^{(2n+1)} \circ T_A)(a)b$ for all $a \in A$ and $b \in B$.*

Moreover, $D = d_{(f,g)}$ for some $f \in A^{(2n+1)}$ and $g \in B^{(2n+1)}$ if and only if $D_B = d_g$, $T_A = d_f$, $T_B = T^{(2n+1)} \circ T_A$ and $D_A = T_A \circ T$.

Proof. A straightforward verification shows that $D : A \times_T B \to (A \times_T B)^{(2n+1)}$ is a bounded linear map if and only if there exist bounded linear mappings $T_A : A \to A^{(2n+1)}, D_B : B \to B^{(2n+1)}, T_B : A \to B^{(2n+1)}$ and $D_A : B \to A^{(2n+1)}$ such that $D(a, b) = (D_A(b) + T_A(a), D_B(b) + T_B(a))$ for all $a \in A$ and $b \in B$. Moreover, D is a derivation if and only if

$$D((a, b)(c, d)) = D(a, b)(c, d) + (a, b)D(c, d)$$

for all $a, c \in A$ and $b, d \in B$. This is holds if and only if

$$\begin{aligned} &T_A(ac + aT(d) + T(b)c) + D_A(bd) \\ &= T_A(a)c + D_A(b)c + T_A(a)T(d) + D_A(b)T(d) \\ &\quad + aT_A(c) + aD_A(d) + T(b)T_A(c) + T(b)D_A(d) \end{aligned}$$

and

$$\begin{aligned} &D_B(bd) + T_B(ac + aT(d) + T(b)c) \\ &= D_B(b)d + T_B(a)d + T^{(2n+1)}(T_A(a)c) + T^{(2n+1)}(D_A(b)c) \\ &\quad + bD_B(d) + bT_B(c) + T^{(2n+1)}(aT_A(c)) + T^{(2n+1)}(aD_A(d)) \end{aligned}$$

for all $a, c \in A$ and $b, d \in B$. By choosing suitable values of a, b, c and d, we deduce that these equations hold if and only if T_A and D_B are derivations, $D_A(bd) = D_A(b)T(d) + T(b)D_A(d)$, $(T_A \circ T)(b)a = D_A(b)a$, $a(T_A \circ T)(b) = aD_A(b)$, $T_B(ac) = (T^{(2n+1)} \circ T_A)(ac)$, $T_B(T(b)c) = bT_B(c) + T^{(2n+1)}(D_A(b)c)$ and $T_B(aT(d)) = T_B(a)d + T^{(2n+1)}(aD_A(d))$ for all $a, c \in A$ and $b, d \in B$. By the appropriate using of [3, Lemma 3.4] and that T_A is a derivation and

$T(B) \subseteq A$, we conclude that the above statements hold if and only if conditions (a)-(c) satisfied.

Now let $D = d_{(f,g)}$ for some $f \in A^{(2n+1)}$ and $g \in B^{(2n+1)}$. Let $a \in A$ then

$$(T_A(a), T_B(a)) = d_{(f,g)}(a, 0) = (af - fa, T^{(2n+1)}(af - fa)).$$

Thus $T_A = d_f$ and $T_B = T^{(2n+1)} \circ T_A$. Similarly, $D_B = d_g$ and $D_A = T_A \circ T$. Conversely, suppose that $T_A = d_f$, $D_B = d_g$ for some $f \in A^{(2n+1)}$ and $g \in B^{(2n+1)}$ and $T_B = T^{(2n+1)} \circ T_A$ and $D_A = T_A \circ T$. Then $D_A(b) = fT(b) - T(b)f$ and $T_B(a) = T^{(2n+1)}(af - fa)$ for all $a \in A$ and $b \in B$. Therefore

$$\begin{aligned} D(a, b) &= (T_A(a) + D_A(b), D_B(b) + T_B(a)) \\ &= (af - fa + fT(b) - T(b)f, bg - gb + T^{(2n+1)}(af - fa)) \\ &= (a, b)(f, g) - (f, g)(a, b) \end{aligned}$$

for all $(a, b) \in A \times_T B$. Consequently, $D = d_{(f,g)}$. □

For Banach algebra A and Banach A-bimodule X, the annihilator of A in X is defined by

$$\mathrm{Ann}_X(A) = \{x \in X : xa = 0 = ax \text{ for all } a \in A\}.$$

It is easy to see that $\mathrm{Ann}_X(A) = \{0\}$ if and only if $\langle AX \cup XA \rangle$, the linear span of $AX \cup XA$, is dense in X. It is shown in [3, Proposition 3.5, part (7)] that if $A \times_T B$ is $(2n + 1)$-weakly amenable, then both A and B are $(2n + 1)$-weakly amenable. It is also shown that, the converse holds if $\mathrm{Ann}_{A^{(2n+1)}}(A)$ and $\mathrm{Ann}_{B^{(2n+1)}}(B)$ are trivial. In the next theorem, we characterize the $(2n + 1)$-weak amenability of $A \times_T B$ in terms of A and B, which also shows that the hypothesis of triviality of $\mathrm{Ann}_{A^{(2n+1)}}(A)$ and $\mathrm{Ann}_{B^{(2n+1)}}(B)$ in [3, Proposition 3.5] is superfluous.

Theorem 3.2. *The Banach algebra $A \times_T B$ is $(2n+1)$-weakly amenable if and only if both A and B are $(2n + 1)$-weakly amenable.*

Proof. To prove the necessity, suppose that $A \times_T B$ is $(2n+1)$-weakly amenable. Let $D : A \to A^{(2n+1)}$ be a continuous derivation. Define $\overline{D} : A \times_T B \to (A \times_T B)^{(2n+1)}$ by $\overline{D}(a, b) = (D(T(b)) + D(a), T^{(2n+1)}(D(a)))$. Then Lemma 3.1 implies that $\overline{D}$ is a continuous derivation, so it is inner. Therefore A is $(2n + 1)$-weakly amenable. Now let $D : B \to B^{(2n+1)}$ be a continuous derivation. Then $\overline{D} : A \times_T B \to (A \times_T B)^{(2n+1)}$ defined by $\overline{D}(a, b) = (0, D(b))$ is a continuous derivation and hence it is inner. Lemma 3.1 implies that D is inner, as required.

To prove the sufficiency, suppose that A and B are $(2n+1)$-weakly amenable. Let $\overline{D} : A \times_T B \to (A \times_T B)^{(2n+1)}$ be a continuous derivation. By Lemma 3.1, $\overline{D}$ enjoys the presentation

$$\overline{D}(a, b) = (D_A(b) + T_A(a), D_B(b) + T_B(a)); \qquad (a \in A, b \in B),$$

in which, the component mappings D_A, D_B, T_A and T_B are satisfying the conditions (a)-(c) Lemma 3.1. So, D_B and T_A are inner derivations. Since A and B are $(2n+1)$-weakly amenable, it follows that A^2 and B^2 are dense in A and B respectively, [4, Proposition 1.2]. From condition (b) and (c) Lemma 3.1, we get $D_A = T_A \circ T$ and $T_B = T^{(2n+1)} \circ T_A$. It follows that, $\overline{D}$ is inner. Therefore, $A \times_T B$ is $(2n+1)$-weakly amenable, as claimed. □

4. $(2n)$-weak amenability

In this section we are concerned with the conditions for the $(2n)$-weak amenability of $A \times_T B$. The next result, which is devoted to the even case, needs a similar argument used in Lemma 3.1.

Lemma 4.1. *A mapping $D : A\times_T B \to (A\times_T B)^{(2n)}$ is a continuous derivation if and only if*

$$D(a, b) = (D_A(b) + T_A(a), D_B(b) + T_B(a))$$

for all $a \in A$ and $b \in B$, where

(a) *$D_B : B \to B^{(2n)}$ is a continuous derivation.*
(b) *$D_A : B \to A^{(2n)}$ is a bounded linear operator such that $D_A(b_1b_2) = T(b_1)D_A(b_2) + D_A(b_1)T(b_2)$ for all $b_1, b_2 \in B$.*
(c) *$T_B : A \to B^{(2n)}$ is a bounded linear operator such that $T_B(a_1a_2) = 0$ for all $a_1, a_2 \in A$ and $bT_B(a) = T_B(a)b = 0$ for all $a \in A$ and $b \in B$.*
(d) *$T_A : A \to A^{(2n)}$ is a bounded linear operator such that $T_A(T(b)a) = T^{(2n)}(D_B(b))a + D_A(b)a + T(b)T_A(a)$ and $T_A(aT(b)) = aT^{(2n)}(D_B(b)) + aD_A(b) + T_A(a)T(b)$ for all $a \in A$ and $b \in B$ and $T_A(a_1a_2) = a_1T_A(a_2) + T_A(a_1)a_2 + T^{(2n)}(T_B(a_1))a_2 + a_1T^{(2n)}(T_B(a_2))$ for $a_1, a_2 \in A$.*

Moreover, $D = d_{(f,g)}$ for some $f \in A^{(2n)}$ and $g \in B^{(2n)}$ if and only if $D_B = d_g$, $T_A = d_f$ $D_A = T_A \circ T - T^{(2n)} \circ D_B$ and $T_B = 0$.

In the next, we gives general necessary and sufficient conditions for $A \times_T B$ to be $(2n)$-weakly amenable.

Theorem 4.2. *The Banach algebra $A \times_T B$ is $(2n)$-weakly amenable if and only if*

(1) *both A and B are $(2n)$-weakly amenable.*
(2) *If $D : B \to A^{(2n)}$ is a bounded linear operator such that $D(b_1b_2) = 0$ for all $b_1, b_2 \in B$ and $aD(b) = D(b)a = 0$ for all $a \in A$ and $b \in B$, then $D = 0$.*
(3) *If $S : A \to B^{(2n)}$ is a bounded linear operator such that $S(a_1a_2) = 0$ for all $a_1, a_2 \in A$ and $S(a)b = bS(a) = 0$ for all $a \in A$ and $b \in B$, then $S = 0$.*

Proof. To prove the necessity, suppose that $A \times_T B$ is $(2n)$-weakly amenable. Let $D : A \to A^{(2n)}$ be a continuous derivation. Then $\overline{D} : A \times_T B \to (A \times_T B)^{(2n)}$ defined by $\overline{D}(a, b) = (D(T(b)) + D(a), 0)$ is a continuous derivation and hence it is inner. It follows from Lemma 4.1, that D is inner. Therefore, A is $(2n)$-weakly amenable. To prove that B is also $(2n)$-weakly amenable, for a continuous derivation $D : B \to B^{(2n)}$, define $\overline{D} : A \times_T B \to (A \times_T B)^{(2n)}$ by $\overline{D}(a, b) = (-T^{(2n)}(D(b)), D(b))$. Then $\overline{D}$ is a continuous derivation and hence it is inner. So D is inner.

Let $D : B \to A^{(2n)}$ be a bounded linear operator such that $D(b_1b_2) = 0$ for all $b_1, b_2 \in B$ and $aD(b) = D(b)a = 0$ for all $a \in A$ and $b \in B$. By Lemma 4.1, we conclude that $\overline{D} : A \times_T B \to (A \times_T B)^{(2n)}$ defined by $\overline{D}(a, b) = (D(b), 0)$ is a continuous derivation and so it is inner. Thus $D = 0$. This proves (2).

To prove (3), we need a similar argument. Indeed, if $S : A \to B^{(2n)}$ be a bounded linear operator such that $S(a_1a_2) = 0$ for all $a_1, a_2 \in A$ and $S(a)b = bS(a) = 0$ for all $a \in A$ and $b \in B$. Then Lemma 4.1 implies that $\overline{D} : A \times_T B \to (A \times_T B)^{(2n)}$ given by $\overline{D}(a, b) = (-T^{(2n)}(S(a)), S(a))$, is a continuous derivation and so it is inner. Hence, $S = 0$. This completes the proof of necessity.

For sufficiency, suppose that $\overline{D} : A \times_T B \to (A \times_T B)^{(2n)}$ is a continuous derivation. By Lemma 4.1, $\overline{D}$ is in the form

$$\overline{D}(a, b) = (D_A(b) + T_A(a), D_B(b) + T_B(a)); \qquad (a \in A, b \in B),$$

in which, the component mappings D_A, D_B, T_A and T_B satisfying the conditions (a)-(d) Lemma 4.1. By condition (3), $T_B = 0$. This implies that T_A is a continuous derivation. By conditions (1), D_B and T_A are inner derivation. Thus there are $f \in A^{2n)}$ and $g \in B^{(2n)}$ such that $T_A = d_f$ and $D_B = d_g$. Now define $D : B \to A^{(2n)}$ by $D = D_A - T_A \circ T + T^{(2n)} \circ D_B$. From condition (d) Lemma 4.1, it follows that D satisfies in condition (2). So, $D = 0$. This shows that $D_A = T_A \circ T - T^{(2n)} \circ D_B$. Therefore $\overline{D}$ is inner, by Lemma 4.1. This proves that $A \times_T B$ is $(2n)$-weakly amenable, as claimed. □

Proposition 4.3. *Condition* (2) *of Theorem 4.2 holds if and only if* $\langle B^2 \rangle$ *is dense in* B *or* $\mathrm{Ann}_{A^{(2n)}}(A) = \{0\}$.

Proof. Take a non-zero $f \in A^{(2n)}$ with $af = fa = 0$ for all $a \in A$, and let $g \in B^*$ be such that $g|_{B^2} = 0$. Then $D : B \to A^{(2n)}$ defined by $D(b) = g(b)f$ satisfies in condition (2) of Theorem 4.2, so it is zero. Thus $g = 0$. This shows that $\langle B^2 \rangle$ is dense in B, as required. □

Using a similar argument, one can obtain the following.

Proposition 4.4. *Condition* (3) *of Theorem 4.2 holds if and only if* $\langle A^2 \rangle$ *is dense in* A *or* $\mathrm{Ann}_{B^{(2n)}}(B) = \{0\}$.

In [3, Proposition 3.5, part (6)], it has been proved that if both A and B are $(2n)$-weakly amenable and $\mathrm{Ann}_{B^{(2n)}}(B) = \{0\}$, then $A \times_T B$ is $(2n)$-weakly amenable. There appear to be some gaps in the proof presented in [3]. In details, for a continuous derivation $D : A \times_T B \to (A \times_T B)^{(2n)}$ with $D = (D_1, D_2)$, it has been shown that $D_1(0, b) = D_1(T(b), 0) - T^{(2n)}(D_2(0, b))$

for all $b \in B$. By a careful look at their proof, we could only conclude that

$$D_1(0,b)a = \Big(D_1(T(b),0) - T^{(2n)}(D_2(0,b))\Big)\, a \quad \text{and}$$
$$aD_1(0,b) = a\Big(D_1(T(b),0) - T^{(2n)}(D_2(0,b))\Big)$$

for all $a \in A$ and $b \in B$. Therefore, we believe that to achieve the equality $D_1(0,b) = D_1(T(b),0) - T^{(2n)}(D_2(0,b))$ for all $b \in B$, more assumptions, such as $\mathrm{Ann}_{A^{(2n)}}(A) = \{0\}$, are needed. If we combine Theorem 4.2 and Propositions 4.7 and 4.8, we have the following theorem which improves [3, Proposition 3.5, part (6)] and also extends [7, Theorem 3.4].

Theorem 4.5. *The Banach algebra $A \times_T B$ is $(2n)$-weakly amenable if and only if*

(1) *both A and B are $(2n)$-weakly amenable.*
(2) *$\langle B^2 \rangle$ is dense in B or $\mathrm{Ann}_{A^{(2n)}}(A) = \{0\}$.*
(3) *$\langle A^2 \rangle$ is dense in A or $\mathrm{Ann}_{B^{(2n)}}(B) = \{0\}$.*

We know from [4, Proposition 1.3] that if A is weakly amenable, then $\langle A^2 \rangle$ is dense in A. Thus as a consequence of Theorem 4.5, we have the next result.

Proposition 4.6. *If $A \times_T B$ is $(2n)$-weakly amenable, then both A and B are also $(2n)$-weakly amenable. The converse holds if any of the following statements holds.*

(i) *$\langle A^2 \rangle$ is dense in A and $\langle B^2 \rangle$ is dense in B.*
(ii) *both A and B are weakly amenable.*
(iii) *$\langle B^2 \rangle$ is dense in B and $\mathrm{Ann}_{B^{(2n)}}(B) = \{0\}$.*
(iv) *$\langle A^2 \rangle$ is dense in A and $\mathrm{Ann}_{A^{(2n)}}(A) = \{0\}$.*
(v) *$\mathrm{Ann}_{B^{(2n)}}(B) = \{0\}$ and $\mathrm{Ann}_{A^{(2n)}}(A) = \{0\}$.*

Proposition 4.7. *Let B be commutative and $(2n)$-weakly amenable. If $T(B)$ is dense in A, then condition (3) of Theorem 4.2 holds.*

Proof. Let $S : A \to B^{(2n)}$ be a bounded linear operator such that $S(a_1a_2) = 0$ for all $a_1, a_2 \in A$ and $S(a)b = bS(a) = 0$ for all $a \in A$ and $b \in B$. Then $S \circ T$ is a continuous derivation from B into $B^{(2n)}$, so it is zero. Therefore, $S(T(b)) = 0$ for all $b \in B$. From density of $T(B)$ in A follows that $S = 0$, as required. □

The next result shows that, under certain condition on T, condition (3) of Theorem 4.2 follows from condition (2).

Proposition 4.8. *If $T^{(2n)}$ is injective and $T(B)$ is dense in A, then condition (3) of Theorem 4.2 follows from condition (2).*

Proof. Let $S : A \to B^{(2n)}$ be a bounded linear operator such that $S(a_1a_2) = 0$ for all $a_1, a_2 \in A$ and $S(a)b = bS(a) = 0$ for all $a \in A$ and $b \in B$. Then $D = T^{(2n)} \circ S \circ T : B \to A^{(2n)}$ is a bounded linear operator such that $D(b_1b_2) = 0$ for all $b_1, b_2 \in B$ and $aD(b) = D(b)a = 0$ for all $a \in A$ and $b \in B$, so it is zero.

Since $T^{(2n)}$ is injective, $S(T(b)) = 0$ for all $b \in B$. Now density of $T(B)$ in A implies that $S = 0$. $\square$

As a consequence of Theorem 3.2 and Proposition 4.5, we have the following result, which extends [7, Corollay 3.5] and [5, Proposition 3.11].

Theorem 4.9. *Let A and B have a bounded left (resp. right) approximate identity. Then $A \times_T B$ is n-weakly amenable if and only if both A and B are n-weakly amenable.*

As a consequence of Theorem 4.9, with $A = \mathbb{C}$ and a non-zero multiplicative linear functional $T : B \to \mathbb{C}$, we have the next result.

Corollary 4.10. *The Banach algebra $\mathbb{C} \times_T B$ is n-weakly amenable if and only if B is n-weakly amenable.*

Let A be unital and $\theta : B \to \mathbb{C}$ be a non-zero multiplicative linear functional. Define $T_\theta(b) := \theta(b)\mathbf{1}$. Then $A \times_{T_\theta} B$ is the θ-Lau product $A \times_\theta B$, [9]. As a consequence of Theorem 4.9, we have the next result which has already proved in [7, Theorem 3.1].

Corollary 4.11. *Let A be unital and θ be a non-zero multiplicative linear functional on B. Then $A \times_\theta B$ is n-weakly amenable if and only if A and B are n-weakly amenable.*

As another consequence of Theorem 4.9, we have the next result.

Corollary 4.12. *Suppose that $a \in A$ be an idempotent and $T_a : \mathbb{C} \to A$ given by $T_a(1) = a$, then $A \times_{T_a} \mathbb{C}$ is n-weakly amenable if and only if A is n-weakly amenable.*

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