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ON t-ALMOST DEDEKIND GRADED DOMAINS

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Abstract. Let Γ be a nonzero torsionless commutative cancellative monoid with quotient group $\langle\Gamma\rangle$, $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain graded by Γ such that $R_\alpha \neq \{0\}$ for all $\alpha \in \Gamma$, H be the set of nonzero homogeneous elements of R, $C(f)$ be the ideal of R generated by the homogeneous components of $f \in R$, and $N(H) = \{f \in R \mid C(f)_v = R\}$. In this paper, we introduce the notion of graded t-almost Dedekind domains. We then show that R is a t-almost Dedekind domain if and only if R is a graded t-almost Dedekind domain and R_H is a t-almost Dedekind domains. We also show that if $R = D[\Gamma]$ is the monoid domain of Γ over an integral domain D, then R is a graded t-almost Dedekind domain if and only if D and Γ are t-almost Dedekind, if and only if $R_{N(H)}$ is an almost Dedekind domain. In particular, if $\langle\Gamma\rangle$ satisfies the ascending chain condition on its cyclic subgroups, then $R = D[\Gamma]$ is a t-almost Dedekind domain if and only if R is a graded t-almost Dedekind domain.

1. Introduction

An *almost Dedekind domain* D is an integral domain in which D_M is a rank-one discrete valuation ring (DVR) for all maximal ideals M of D. As in [15], we say that D is a *t-almost Dedekind domain* if D_P is a rank-one DVR for all maximal t-ideals P of D. (Definitions related with the t-operation and graded integral domains will be reviewed in Section 2.) Clearly, D is an almost Dedekind domain if and only if D is a t-almost Dedekind domain whose nonzero maximal ideals are t-ideals. Also, a Dedekind domain is an almost Dedekind domain, while an almost Dedekind domain need not be a Dedekind domain (see, for example, [16]). It is clear that D is a Dedekind domain (resp., Krull domain) if and only if D is an almost Dedekind domain (resp., a t-almost Dedekind domain) in which each nonzero nonunit is contained in only finitely many maximal ideals (resp., maximal t-ideals) of D. Note that a rank-one DVR has (Krull) dimension one; so if D is an almost (resp., a t-almost) Dedekind domain, then $\dim(D) \leq 1$ (resp., t-$\dim(D) \leq 1$), i.e., each nonzero prime ideal (resp., prime t-ideal) of D is a maximal ideal (resp., maximal t-ideal).

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Let D be an integral domain, Γ be a nonzero torsionless commutative cancellative monoid, and $D[\Gamma]$ be the monoid domain of Γ over D. If $\Gamma = \mathbb{N}_0$ is the additive monoid of nonnegative integers, then $D[\Gamma] = D[X]$, the polynomial ring over D. Clearly, $D[X]$ is an almost Dedekind domain if and only if D is a field, if and only if $D[X]$ is a Dedekind domain. Also, it is known that $D[\Gamma]$ is an almost Dedekind domain if and only if D is a field and Γ is isomorphic to either $\mathbb{Z}_+$ or a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ such that if $\mathrm{char}(D) = p$ is nonzero, then $\frac{1}{p^k} \notin \Gamma$ for some integer $k \geq 1$ [11, Corollary 20.15]. However, note that D is a t-almost Dedekind domain if and only if $D[X]$ is a t-almost Dedekind domain, if and only if $D[X]_{N_v}$ is an almost Dedekind domain, where $N_v = \{f \in D[X] \bigm| (A_f)_v = D\}$ and A_f is the ideal of D generated by the coefficients of $f \in D[X]$ [15, Theorems 4.2 and 4.4]. Hence, it is natural to ask when $D[\Gamma]$ is a t-almost Dedekind domain.

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain graded by Γ such that $R_\alpha \neq \{0\}$ for all $\alpha \in \Gamma$, H be the set of nonzero homogeneous elements of R, $C(f)$ be the ideal of R generated by the homogeneous components of $f \in R$, and $N(H) = \{f \in R \mid C(f)_v = R\}$. In Section 2, we review definitions related with the t-operation and graded integral domains. In Section 3, we first introduce the concept of graded t-almost Dedekind domains. We then show that R is a graded t-almost Dedekind domain if and only if every nonzero homogeneous ideal of R is a w-cancellation ideal. We also prove that R is a t-almost Dedekind domain if and only if R is a graded t-almost Dedekind domain and R_H is a t-almost Dedekind domains. In particular, if R satisfies property (#), then R is a graded t-almost Dedekind domain if and only if $R_{N(H)}$ is an almost Dedekind domain. In Section 4, we study (graded) t-almost Dedekind domain properties of R when $R = D[\Gamma]$. Among other things, we prove that $R = D[\Gamma]$ is a graded t-almost Dedekind domain if and only if D and Γ are t-almost Dedekind, if and only if $R_{N(H)}$ is an almost Dedekind domain. As a corollary, we have that $R = D[\Gamma]$ is a t-almost Dedekind domain if and only if $R_{N(H)}$ is an almost Dedekind domain and $K[\langle\Gamma\rangle]$ is a t-almost Dedekind domain, where $\langle\Gamma\rangle$ is the quotient group of Γ. In particular, if $\langle\Gamma\rangle$ satisfies the ascending chain condition on its cyclic subgroups, then $D[\Gamma]$ is a graded t-almost Dedekind domain if and only if $D[\Gamma]$ is a t-almost Dedekind domain.

2. The t-operation and graded integral domains

Let D be an integral domain with quotient field K and $F(D)$ (resp., $f(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of D. For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$; then $I^{-1} \in F(D)$. The v-operation on D is defined by $I_v = (I^{-1})^{-1}$; the t-operation by $I_t = \bigcup\{J_v \mid J \in f(D)$ and $J \subseteq I\}$; and the w-operation by $I_w = \{x \in K \mid xJ \subseteq I$ for some $J \in f(D)$ with $J_v = D\}$ for all $I \in F(D)$. We say that $I \in F(D)$ is a *v-ideal* (resp.,

t-ideal, w-ideal) if $I_v = I$ (resp., $I_t = I$, $I_w = I$), and a v-ideal (resp., t-ideal, w-ideal) I is a maximal v-ideal (resp., maximal t-ideal, maximal w-ideal) if I is maximal (under inclusion) among proper integral v-ideals (resp., t-ideals, w-ideals). Let v-Max(D) (resp., t-Max(D), w-Max(D)) be the set of maximal v-ideals (resp., t-ideals, w-ideals) of D. As in the case of rank-one nondiscrete valuation domains, v-Max(D) can be empty even when D is not a field. However, it is well known that if $* = t$ or w, then $*$-Max$(D) \neq \emptyset$ when D is not a field; each prime ideal minimal over a $*$-ideal is a $*$-ideal (hence a height-one prime ideal is a $*$-ideal); each proper integral $*$-ideal is contained in a maximal $*$-ideal; $D = \bigcap_{P \in *\text{-Max}(D)} D_P$; and t-Max$(D) = w$-Max(D). An $I \in F(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$, and D is called a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. It is known that D is a PvMD if and only if D_P is a valuation domain for all maximal t-ideals P of D [13, Theorem 5]. We say that a nonzero ideal I of D is a *cancellation ideal* (resp., *w-cancellation ideal*) if $IA = IB$ (resp., $(IA)_w = (IB)_w$) for nonzero ideals A and B of D implies $A = B$ (resp., $A_w = B_w$). The v-, t-, and w-operations are the most well-known examples of so-called star-operations. For more on basic properties of star-operations, see [12, Sections 32 and 34].

Let Γ be a torsionless grading (i.e., commutative, cancellative) monoid (written additively) and $\langle \Gamma \rangle = \{a - b \mid a, b \in \Gamma\}$ be the quotient group of Γ; so $\langle \Gamma \rangle$ is a torsionfree abelian group. It is well known that a cancellative monoid Γ is torsionless if and only if Γ can be given a total order compatible with the monoid operation [17, page 123]. A (Γ-)graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is an integral domain graded by Γ. For every $\alpha \in \Gamma$, a nonzero element $x \in R_\alpha$ is called a *homogeneous element* of degree α, i.e., $\deg(x) = \alpha$, and $\deg(0) = 0$. Thus, every $0 \neq f \in R$ can be written uniquely as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$ with $\deg(x_{\alpha_i}) = \alpha_i$ and $\alpha_1 < \cdots < \alpha_n$. Let $H = \bigcup_{\alpha \in \Gamma} R_\alpha \setminus \{0\}$. Then H is the saturated multiplicative set of nonzero homogeneous elements of R, and $R_H = \bigoplus_{\alpha \in \langle \Gamma \rangle} (R_H)_\alpha$, called the *homogeneous quotient field* of R, is a $\langle \Gamma \rangle$-graded integral domain. Clearly, $(R_H)_\alpha = \{\frac{a}{b} \mid a \in R_\beta, 0 \neq b \in R_\gamma$, and $\alpha = \beta - \gamma\}$ for all $\alpha \in \langle \Gamma \rangle$, $(R_H)_0$ is a field, and every nonzero homogeneous element of R_H is a unit. For a fractional ideal I of R with $I \subseteq R_H$, let I^* denote the fractional ideal of R generated by the homogenous elements in I. We say that I is *homogeneous* if $I^* = I$. A graded integral domain R is a *graded DVR* if R has a unique nonzero prime homogeneous ideal and the prime homogeneous ideal is principal. It is easily shown that a graded DVR is a graded valuation ring. (R is a *graded valuation ring* if for each nonzero homogeneous element $x \in R_H$, either $x \in R$ or $x^{-1} \in R$.) For more on basic properties of graded integral domains, the reader can refer to [5] or [17].

For $f \in R_H$, let $C(f)$ denote the fractional ideal of R generated by the homogenous components of f. Dedekind-Mertens Lemma says that if $f, g \in R$, then $C(f)^{n+1}C(g) = C(f)^n C(fg)$ for some integer $n \geq 1$ [5, Lemma 1.2].

For an ideal I of R, let $C(I) = \sum_{f \in I} C(f)$; so I is homogeneous if and only if $C(f) \subseteq I$ for all $f \in I$. A homogeneous ideal of R is called a *maximal homogeneous ideal* (resp., *maximal homogeneous t-ideal*) of R if it is maximal among proper integral homogeneous ideals (resp., homogeneous t-ideals) of R. Let $N(H) = \{0 \neq f \in R \mid C(f)_v = R\}$ and Ω be the set of maximal t-ideals Q of R with $Q \cap H \neq \emptyset$. Note that if Q is a maximal t-ideal of R, then $Q \cap H \neq \emptyset$ if and only if Q is homogeneous [3, Lemma 1.2]. Hence, Ω is the set of maximal homogeneous t-ideals of R. As in [4], we say that R satisfies property (#) if $I \cap N(H) \neq \emptyset$ when I is a nonzero ideal of R with $C(I)_t = R$; equivalently, $\text{Max}(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$ [4, Proposition 1.4]. It is known that R satisfies property (#) if $R = D[\Gamma]$ or R contains a unit of nonzero degree [4, Example 1.6]. For any undefined definition and notation, see [11].

3. Graded t-almost Dedekind domains

Let Γ be a nonzero torsionless commutative cancellative monoid with quotient group $\langle \Gamma \rangle$, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain graded by Γ such that $R_\alpha \neq \{0\}$ for all $\alpha \in \Gamma$, and H be the saturated multiplicative set of nonzero homogeneous elements of R.

Definition 1. A graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a *graded almost Dedekind domain* (resp., *graded t-almost Dedekind domain*) if R_Q is a rank-one DVR for all maximal homogeneous ideals (resp., maximal homogeneous t-ideals) Q of R.

It is clear that $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded almost Dedekind domain if and only if R is a graded t-almost Dedekind domain in which each nonzero maximal homogeneous ideal is a t-ideal.

Proposition 2. *A graded t-almost Dedekind domain is a PvMD.*

Proof. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$. Then R is a graded PvMD if and only if R_Q is a valuation domain for all nonzero maximal homogeneous t-ideals Q of R [7, Lemma 2.7]. (A graded PvMD is a graded integral domain in which each nonzero finitely generated homogeneous ideal is t-invertible.) Hence, a graded t-almost Dedekind domain R is a graded PvMD, and thus R is a PvMD [1, Theorem 6.4]. □

However, a PvMD need not be a graded t-almost Dedekind domain. (For example, let $V[X]$ be the polynomial ring over a rank-one nondiscrete valuation domain V. Then $V[X]$ is a $\mathbb{N}_0$-graded integral domain with $\deg(aX^n) = n$ for $0 \neq a \in V$ and $n \geq 0$, $V[X]$ is a PvMD, but $V[X]$ is not a graded t-almost

Dedekind domain.) By definitions, we have the following implications:

$$\begin{array}{ccc} \text{Almost Dedekind domain} & \Longrightarrow & \text{Graded almost Dedekind domain} \\ \Big\Downarrow & & \Big\Downarrow \\ t\text{-Almost Dedekind domain} & \Longrightarrow & \text{Graded } t\text{-almost Dedekind domain,} \end{array}$$

while the next examples show that the reverse implications don't hold.

Example 3. (1) Since $\mathbb{Z}$ is a PID, $\mathbb{Z}$ is a t-almost Dedekind domain. Thus, $\mathbb{Z}[X]$ is a t-almost Dedekind domain (and hence a graded t-almost Dedekind domain) [15, Theorems 4.2]. However, note that $(2, X)$ is a maximal homogenous ideal but $\mathbb{Z}[X]_{(2,X)}$ is not a rank-one DVR. Thus, $Z[X]$ is neither a graded almost Dedekind domain nor an almost Dedekind domain.

(2) Let K be a field with $\text{char}(K) = p > 0$ and $\mathbb{Q}$ be the additive group of rational numbers. Then $K[\mathbb{Q}]$ is a Prüfer domain [11, Theorem 13.6] but not an almost Dedekind domain [11, Corollary 20.15]; hence $K[\mathbb{Q}]$ is not a t-almost Dedekind domain. (Note that a Prüfer domain D is an almost Dedekind domain if and only if D is a t-almost Dedekind domain.) Since all of nonzero homogeneous elements of $K[\mathbb{Q}]$ are unit, $K[\mathbb{Q}]$ is a graded almost Dedekind domain (hence a graded t-almost Dedekind domain). Thus, a graded almost (resp., graded t-almost) Dedekind domain need not be an almost (resp., t-almost) Dedekind domain.

The next result is the graded t-almost Dedekind domain analog of [9, Theorem 2.14] that an integral domain D is a t-almost Dedekind domain if and only if each nonzero ideal of D is a w-cancellation ideal. We recall that I is a cancellation (resp., w-cancellation) ideal of D if and only if ID_P is principal for all maximal ideals (resp., maximal t-ideals) P of D [6, Corollary 2.4].

Proposition 4. *The following statements are equivalent for* $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$.

(1) *R is a graded t-almost Dedekind domain.*
(2) *Each nonzero homogeneous ideal of R is a w-cancellation ideal.*
(3) *Each nonzero prime homogeneous ideal of R is a w-cancellation ideal.*
(4) *Each prime homogeneous w-ideal of R is a w-cancellation ideal.*
(5) *Each prime homogeneous t-ideal of R is a w-cancellation ideal.*
(6) *$R_{H\setminus M}$ is a graded DVR for every maximal homogeneous t-ideal M of R.*

Proof. (1) $\Rightarrow$ (2) Let I be a nonzero homogeneous ideal of R. If Q is a maximal t-ideal of R, then $IR_Q = R_Q$ if Q is not homogeneous, and IR_Q is principal if Q is homogeneous by assumption. Thus, I is a w-cancellation ideal.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) Clear.

(5) $\Rightarrow$ (1) Let M be a maximal homogeneous t-ideal of R. Then M is a w-cancellation ideal by assumption, and thus MR_M is principal. So it suffices to show that $\text{ht}M = 1$. Assume $\text{ht}M \geq 2$, and let Q be a prime ideal of R

with $(0) \subsetneq Q \subsetneq M$. We may assume that Q is a t-ideal. If Q is homogeneous, then Q is a w-cancellation ideal by assumption, and hence QR_M is principal, a contradiction because MR_M is principal. Hence, Q is not homogeneous, and we may assume that M does not contain a nonzero prime homogeneous ideal. Let $0 \neq f = x_{\alpha_1} + \cdots + x_{\alpha_n} \in Q$ with $\alpha_1 < \cdots < \alpha_n$. Then $fR_M = x_{\alpha_i}R_M$ for some α_i (because MR_M is principal and $MR_M = \sqrt{x_{\alpha_j}R_M}$ for $j = 1, \ldots, n$) and $\frac{f}{x_{\alpha_i}}$ is a unit in R_M. Hence, $x_{\alpha_i} \in Q$, and so if P is a minimal prime ideal of $x_{\alpha_i}R$ such that $P \subseteq Q$, then P is homogeneous and $P \subsetneq M$, a contradiction. Thus, $\mathrm{ht}M = 1$.

(1) $\Leftrightarrow$ (6) This follows because $R_{H\setminus M}$ is a graded DVR if and only if $R_M = (R_{H\setminus M})_{M_{H\setminus M}}$ is a rank-one DVR [8, Theorem 9]. □

The next result is the graded almost Dedekind domain analog of [12, Theorem 36.5] that an integral domain D is an almost Dedekind domain if and only if each nonzero ideal of D is a cancellation ideal.

Corollary 5. *The following statements are equivalent for* $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$.

(1) *R is a graded almost Dedekind domain.*
(2) *Each nonzero homogeneous ideal of R is a cancellation ideal.*
(3) *Each nonzero prime homogeneous ideal of R is a cancellation ideal.*
(4) *$R_{H\setminus M}$ is a graded DVR for every maximal homogeneous ideal M of R.*

Proof. (1) $\Rightarrow$ (2) Let I be a nonzero homogeneous ideal of R and M be a maximal ideal of R. It suffices to show that IR_M is principal [6, Corollary 2.4]. If $I \nsubseteq M$, then $IR_M = R_M$. Next, assume that $I \subseteq M$, and let $P = M^*$. Then P is a nonzero prime homogeneous ideal of R such that $I \subseteq P \subseteq M$. Hence, R_P is a rank-one DVR by assumption, and thus $IR_P = xR_P$ for some $x \in I$. Clearly, we can choose x in H because I is homogeneous. Let $a \in I \cap H$. Then $a = x\frac{g}{f}$ for some $f \in R \setminus P$ and $g \in R$, and since $f \notin P$, at least one of the homogeneous components of f is not in P. So if α is such a homogeneous element, then $a\alpha = x\beta$ for some $\beta \in H$, and since $P = M^*$, $\alpha \notin M$. Thus, $a = x\frac{\beta}{\alpha} \in xR_M$. Again, since I is homogeneous, $IR_M \subseteq xR_M$, and thus $IR_M = xR_M$.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Note that a cancellation ideal is a w-cancellation t-ideal [6, Corollary 2.5 and Theorem 4.1]; so R is a graded t-almost Dedekind domain whose nonzero maximal homogeneous ideals are t-ideals by Proposition 4. Thus, R is a graded almost Dedekind domain.

(1) $\Leftrightarrow$ (4) See the proof of (1) $\Leftrightarrow$ (6) in Proposition 4. □

It is known that if D is an almost (resp., a t-almost) Dedekind domain, then D_S is an almost (resp., a t-almost) Dedekind domain for a multiplicative subset S of D [12, Corollary 36.3] (resp., [15, Proposition 4.3]). We next give the graded integral domain analog.

Proposition 6. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and S be a saturated multiplicative set of nonzero homogeneous elements of R.*

(1) *If $\Delta = \{\alpha \in \langle\Gamma\rangle \mid \alpha = \beta - \gamma$ for some $\beta, \gamma \in \Gamma$ with $S \cap R_\gamma \neq \emptyset\}$, then Δ is a monoid with $\Gamma \subseteq \Delta \subseteq \langle\Gamma\rangle$.*

(2) *R_S is a Δ-graded integral domain.*

(3) *If R is a graded almost (resp., graded t-almost) Dedekind domain, then R_S is a graded almost (resp., graded t-almost) Dedekind domain.*

Proof. (1) and (2) This follows because R_S is an integral domain.

(3) Let M be a nonzero maximal homogeneous ideal (resp., nonzero maximal homogeneous t-ideal) of R_S, and let $P = M \cap R$. Then P is a nonzero prime homogeneous ideal (resp., nonzero prime homogeneous t-ideal) of R and $M = PR_S$; hence $R_M = (R_S)_{PR_S} = R_P$ is a rank-one DVR by assumption. Thus, R_S is a graded almost (resp., graded t-almost) Dedekind domain. □

Theorem 7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then R is a t-almost Dedekind domain if and only if R is a graded t-almost Dedekind domain and R_H is a t-almost Dedekind domain.*

Proof. Assume that R is a graded t-almost Dedekind domain and R_H is a t-almost Dedekind domain. Let Q be a maximal t-ideal of R. If $Q \cap H \neq \emptyset$, then Q is homogeneous [3, Lemma 1.2], and thus R_Q is a rank-one DVR. Next, if $Q \cap H = \emptyset$, then Q_H is a t-ideal of R_H because R is a PvMD by Proposition 2. Hence, $R_Q = (R_H)_{Q_H}$ is a rank-one DVR. Thus, R is a t-almost Dedekind domain. The converse is clear. □

Corollary 8. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, and assume that R satisfies property (#).*

(1) *R is a graded t-almost Dedekind domain if and only if $R_{N(H)}$ is an almost Dedekind domain.*

(2) *R is a t-almost Dedekind domain if and only if $R_{N(H)}$ is an almost Dedekind domain and R_H is a t-almost Dedekind domains.*

Proof. (1) Recall that R satisfies property (#) if and only if $\mathrm{Max}(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$ [4, Proposition 1.4]. Thus, the result follows directly from the definition of graded t-almost Dedekind domains.

(2) This is an immediate consequence of (1) and Theorem 7. □

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. If R_H is a UFD, then R_H is a t-almost Dedekind domain, and hence by Corollary 8(2), we have:

Corollary 9. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain that satisfies property (#). If R_H is a UFD, then the following statements are equivalent.*

(1) *R is a t-almost Dedekind domain.*

(2) *$R_{N(H)}$ is an almost Dedekind domain.*

(3) *R is a graded t-almost Dedekind domain.*

We end this section with two examples of graded integral domains $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ such that R_H is a UFD.

Example 10. (1) If $\langle\Gamma\rangle = \mathbb{Z}$, then $R_H \cong k[X, X^{-1}]$, where k is a field and X is an indeterminate over k. Hence, R_H is a PID.

(2) If $\langle\Gamma\rangle$ satisfies the ascending chain condition on its cyclic subgroups, then R_H is a UFD [2, Proposition 3.5].

4. *t*-Almost Dedekind domains as monoid domains

Let Γ be a nonzero torsionless commutative cancellative monoid with quotient group $\langle\Gamma\rangle$, D be an integral domain with quotient field K, and $D[\Gamma]$ be the monoid domain of Γ over D. Clearly, $D[\Gamma]$ is a Γ-graded integral domain with $\deg(aX^\alpha) = \alpha$ for $0 \neq a \in D$ and $\alpha \in \Gamma$. Also, $D[\Gamma]$ satisfies property (#). In this section, we study when $D[\Gamma]$ is a graded t-almost Dedekind domain.

Let A_f be the ideal of D generated by the coefficients of $f \in D[\Gamma]$; so $C(f) \subseteq A_f D[\Gamma]$. For a proper prime ideal S of Γ, let $\Gamma_S = \{\alpha - s \mid \alpha \in \Gamma$ and $s \in \Gamma \setminus S\}$; then Γ_S is a monoid with $\Gamma \subseteq \Gamma_S \subseteq \langle\Gamma\rangle$. We say that Γ is a *t-almost Dedekind monoid* if Γ_S is a principal monoid for all maximal t-ideals S of Γ. Clearly, torsionfree abelian groups and unique factorization monoids are t-almost Dedekind monoids. (The t-operation on Γ is defined by the same way as in the case of integral domains. For more on definitions related with monoids, see [14].)

Lemma 11. *Let $D[\Gamma]$ be the monoid domain of Γ over D and $D(\Gamma) = \{\frac{f}{g} \mid f, g \in D[\Gamma], g \neq 0$, and $A_g = D\}$.*

1. *If D is a valuation domain, then $D(\Gamma)$ is a valuation domain whose value group is the same as that of D.*
2. *If Γ is a valuation monoid with maximal ideal S, then $D[\Gamma]_{D[S]}$ is a valuation domain whose value group is the same as that of S.*

Proof. (1) Let $f = a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n} \in D[\Gamma]$ with $\alpha_1 < \cdots < \alpha_n$. Since D is a valuation domain, $A_f = a_iD$ for some i, and thus $fD(\Gamma) = a_iD(\Gamma)$. Thus, $D(\Gamma)$ is a valuation domain whose value group is the same as that of D.

(2) Let f be as in (1). Then, since Γ is a valuation monoid, $\bigcup_{j=1}^n(\alpha_j + \Gamma) = \alpha_i + \Gamma$ for some i, and hence $fD[\Gamma]_{D[S]} = X^{\alpha_i}D[\Gamma]_{D[S]}$. Thus, $D[\Gamma]_{D[S]}$ is a valuation domain whose value group is the same as that of S. □

Lemma 12. *Let $D[\Gamma]$ be the monoid domain of Γ over D and H be the set of nonzero homogeneous elements of $D[\Gamma]$. Then*

$$t\text{-}Max(D[\Gamma]) = \{P[\Gamma] \mid P \in t\text{-}Max(D)\} \cup \{D[S] \mid S \in t\text{-}Max(\Gamma)\} \cup \{Q \in t\text{-}Max(D[\Gamma]) \mid Q \cap H = \emptyset\}.$$

Proof. ($\subseteq$) [3, Lemma 1.2 and Corollary 1.3]. ($\supseteq$) Let $P \in t$-Max(D) and $S \in t$-Max(Γ). Then $P[\Gamma]$ and $D[S]$ are both t-ideals of $D[\Gamma]$ [10, Corollary 2.4], and thus $P[\Gamma]$ and $D[S]$ are maximal t-ideals [3, Corollary 1.3]. □

We next give the main result of this section.

Theorem 13. *Let $D[\Gamma]$ be the monoid domain of Γ over D. Then the following statements are equivalent.*

(1) *$D[\Gamma]$ is a graded t-almost Dedekind domain.*
(2) *D is a t-almost Dedekind domain and Γ is a t-almost Dedekind monoid.*
(3) *$D[\Gamma]_{N(H)}$ is an almost Dedekind domain.*

Proof. (1) $\Rightarrow$ (2) Let P be a maximal t-ideal of D. Then $P[\Gamma]$ is a maximal t-ideal of $D[\Gamma]$, and hence $D_P(\Gamma) = D[\Gamma]_{P[\Gamma]}$ is a rank-one DVR. Note that $D_P(\Gamma) \cap K = D_P$; thus D_P is a rank-one DVR [12, Theorem 19.16]. Next, let S be a maximal t-ideal of Γ. Then, by Lemma 12, $D[S]$ is a maximal t-ideal of $D[\Gamma]$, and hence $D[\Gamma]_{D[S]}$ is a rank-one DVR. Note that $\{\beta \in \langle\Gamma\rangle \mid X^\beta \in D[\Gamma]_{D[S]}\} = \Gamma_S$. Thus, Γ_S is a rank-one discrete valuation monoid.

(2) $\Rightarrow$ (1) Let Q be a nonzero maximal homogeneous t-ideal of $D[\Gamma]$. If $Q \cap D \neq (0)$, then $Q \cap D$ is a maximal t-ideal of D and $Q = (Q \cap D)[\Gamma]$ by Lemma 12. Hence, $D[\Gamma]_Q = D[\Gamma]_{(Q\cap D)[\Gamma]} = D_{Q\cap D}(\Gamma)$, and since $D_{Q\cap D}$ is a rank-one DVR by (2), $D_{Q\cap D}(\Gamma)$ is a rank-one DVR by Lemma 11(1). Next, assume that $Q \cap D = (0)$, and let $S = \{\alpha \in \Gamma \mid X^\alpha \in Q\}$. Then $S \neq \emptyset$, and hence by Lemma 12, S is a maximal t-ideal of Γ and $Q = D[S]$; so Γ_S is a rank-one discrete valuation monoid. Thus, $D[\Gamma]_Q = D[\Gamma]_{D[S]}$ is a rank-one DVR by Lemma 11(2).

(1) $\Leftrightarrow$ (3) This follows directly from Corollary 8(1) because $D[\Gamma]$ satisfies property (#). □

Corollary 14. *Let $D[\Gamma]$ be the monoid domain of Γ over D, and assume that Γ is a group. Then $D[\Gamma]$ is a graded t-almost Dedekind domain if and only if D is a t-almost Dedekind domain.*

Proof. Clearly, a torsionfree abelian group is a t-almost Dedekind monoid. Hence, the result follows directly from Theorem 13. □

Corollary 15. *Let $D[\Gamma]$ be the monoid domain of Γ over D. Then the following statements are equivalent.*

(1) *$D[\Gamma]$ is a t-almost Dedekind domain.*
(2) (i) *D is a t-almost Dedekind domain,* (ii) *Γ is a t-almost Dedekind monoid, and* (iii) *$K[\langle\Gamma\rangle]$ is a t-almost Dedekind domain.*
(3) *$D[\Gamma]_{N(H)}$ is an almost Dedekind domain and $K[\langle\Gamma\rangle]$ is a t-almost Dedekind domain.*
(4) *$D[\Gamma]$ is a graded t-almost Dedekind domain and $K[\langle\Gamma\rangle]$ is a t-almost Dedekind domain.*

Proof. (1) $\Rightarrow$ (2) Since a t-almost Dedekind domain is a graded t-almost Dedekind domain, by Theorem 13, (i) and (ii) are satisfied. For (iii), note that $K[\langle\Gamma\rangle] = D[\Gamma]_H$. Thus, $K[\langle\Gamma\rangle]$ is a t-almost Dedekind domain by Corollary 8.

(2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (1) Theorem 13 and Corollary 8. □

Corollary 16. *Assume that* $\langle\Gamma\rangle$ *satisfies the ascending chain condition on its cyclic subgroups (e.g.,* $\langle\Gamma\rangle \cong \mathbb{Z}$*). Then the following statements are equivalent.*

(1) $D[\Gamma]$ *is a* t*-almost Dedekind domain.*
(2) D *is a* t*-almost Dedekind domain and* Γ *is a* t*-almost Dedekind monoid.*
(3) $D[\Gamma]_{N(H)}$ *is an almost Dedekind domain.*
(4) $D[\Gamma]$ *is a graded* t*-almost Dedekind domain.*

Proof. By Example 10, $K[\langle\Gamma\rangle]$ is a UFD. Thus, the result follows directly from Corollary 15. □

Corollary 17. *Let* D *be a Krull domain and* $\mathbb{Q}$ *be the additive group of rational numbers.*

(1) *If* $char(D) = 0$*, then* $D[\mathbb{Q}]$ *is a* t*-almost Dedekind domain.*
(2) *If* $char(D) \neq 0$*, then* $D[\mathbb{Q}]$ *is a graded* t*-almost Dedekind domain but not a* t*-almost Dedekind domain.*

Proof. Let K be the quotient field of D. Then $K[\mathbb{Q}]$ is a t-almost Dedekind domain if and only if $K[\mathbb{Q}]$ is an almost Dedekind domain, if and only if $\text{char}(D) = 0$ [11, Theorem 13.6 and Corollary 20.15]. Thus, the result follows directly from Theorem 13 and Corollary 15. □

Let $\{X_\alpha\}$ be a nonempty set of indeterminates over an integral domain D and $D[\{X_\alpha\}]$ be the polynomial ring over D. Let N_α be the additive monoid of nonnegative integers for all α, and let $\Gamma = \bigoplus_\alpha N_\alpha$. Clearly, Γ is a unique factorization monoid and $D[\{X_\alpha\}] = D[\Gamma]$. Also, if $N_v = \{f \in D[\{X_\alpha\}] \mid (A_f)_v = D\}$, then N_v is a saturated multiplicative subset of $D[\{X_\alpha\}]$.

Corollary 18. *Let* $D[\{X_\alpha\}]$ *be the polynomial ring over* D*. Then the following statements are equivalent.*

(1) D *is a* t*-almost Dedekind domain.*
(2) $D[\{X_\alpha\}]$ *is a* t*-almost Dedekind domain.*
(3) $D[\{X_\alpha\}]$ *is a graded* t*-almost Dedekind domain.*
(4) $D[\{X_\alpha, X_\alpha^{-1}\}]$ *is a* t*-almost Dedekind domain.*
(5) $D[\{X_\alpha, X_\alpha^{-1}\}]$ *is a graded* t*-almost Dedekind domain.*
(6) $D[\{X_\alpha\}]_{N_v}$ *is an almost Dedekind domain.*

Proof. This result follows from Corollary 15 and the following observation: Let $N_\alpha = \mathbb{N}_0$ be the additive monoid of nonnegative integers and $\Gamma = \bigoplus_\alpha N_\alpha$. Then $D[\Gamma] = D[\{X_\alpha\}]$, $D[\langle\Gamma\rangle] = D[\{X_\alpha, X_\alpha^{-1}\}]$, Γ is a unique factorization monoid, $K[\langle\Gamma\rangle] = K[\{X_\alpha, X_\alpha^{-1}\}]$ is a UFD, and $D[\{X_\alpha\}]_{N_v} = R_{N(H)}$ where $R = D[\langle\Gamma\rangle]$. □

Corollary 19. *Let* G_i *be either* $\mathbb{Z}$ *or* $\mathbb{Q}$ *for* $i = 1, 2$ *and* $G = G_1 \bigoplus G_2$*. If* $char(D) = 0$*, then* D *is a* t*-almost Dedekind domain if and only if* $D[G]$ *is a* t*-almost Dedekind domain.*

Proof. By Corollary 15, it suffices to show that $K[G]$ is a t-almost Dedekind domain. If $G_1 = G_2 = \mathbb{Z}$, then $K[G] \cong K[X, X^{-1}, Y, Y^{-1}]$, a Laurent polynomial ring, and thus $K[G]$ is a t-almost Dedekind domain by Corollary 18. Next, assume that $G_1 = \mathbb{Q}$ and $G_2 = \mathbb{Z}$ or $\mathbb{Q}$. By [11, Theorem 7.1], $K[G] = K[\mathbb{Q} \bigoplus G_2] \simeq (K[\mathbb{Q}])[G_2]$. Hence, if $R = K[\mathbb{Q}]$ and L is the quotient field of R, then R and $L[G_2]$ are t-almost Dedekind domains by Corollaries 17(1) and 18. Thus, by Corollary 15, $K[G]$ is a t-almost Dedekind domain. $\square$

By Corollary 15, if $K[\langle\Gamma\rangle]$ is a t-almost Dedekind domain, then $D[\Gamma]$ is a t-almost Dedekind domain if and only if D and Γ are both t-almost Dedekind. We end this article with the following question.

Question 20. When $K[\langle\Gamma\rangle]$ is a t-almost Dedekind domain ?

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