Bull. Korean Math. Soc. **54** (2017), No. 6, pp. 1951–1967
https://doi.org/10.4134/BKMS.b160560
pISSN: 1015-8634 / eISSN: 2234-3016

AN EXTENSION OF THE EXTENDED HURWITZ-LERCH ZETA FUNCTIONS OF TWO VARIABLES

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Abstract. We aim to introduce a further extension of a family of the extended Hurwitz-Lerch Zeta functions of two variables. We then systematically investigate several interesting properties of the extended function such as its integral representations which provide extensions of various earlier corresponding results of two and one variables, its summation formula, its Mellin-Barnes type contour integral representations, its computational representation and fractional derivative formulas. A multi-parameter extension of the extended Hurwitz-Lerch Zeta function of two variables is also introduced. Relevant connections of certain special cases of the main results presented here with some known identities are pointed out.

1. Introduction and preliminaries

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$ and $\mathbb{N}_0$ be the sets of complex numbers, positive real numbers, non-positive and non-negative integers, respectively, and $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$. The *Hurwitz-Lerch Zeta function* $\Phi(z, s, a)$ is defined by (see, e.g., [5, p. 27, Eq. 1.11(1)]; see also [19, p. 121] and [20, p. 194]):

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1}$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1\big).$$

For various properties and the special cases of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$, one may refer to [19] and [20]. Many generalizations of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ have been investigated (see, e.g., [1–3, 5–7, 9, 11, 18, 21, 23, 25]). In particular, Srivastava et al. [25, p. 491, Eq. (1.20)] introduced

Received July 5, 2016; Revised February 3, 2017; Accepted February 20, 2017.
2010 *Mathematics Subject Classification.* Primary 11M25, 11M99, 33B15, 33C60; Secondary 11M35, 11B68, 33C05, 33C90.
Key words and phrases. Hurwitz-Lerch Zeta function, extended Hurwitz-Lerch Zeta functions, Gauss hypergeometric function, Fox-Wright hypergeometric function, Mellin-Barnes contour integral representations, analytic continuation.

and investigated the following extended Hurwitz-Lerch Zeta function:

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma;\kappa)}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n} n!} \frac{z^n}{(n+a)^s} \tag{2}$$

$\big(\lambda, \mu \in \mathbb{C};\ a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \rho, \sigma, \kappa \in \mathbb{R}^+;\ \kappa - \rho - \sigma > -1$ when $s, z \in \mathbb{C};$ $\kappa - \rho - \sigma = -1$ and $s \in \mathbb{C}$ when $|z| < \delta^* := \rho^{-\rho}\sigma^{-\sigma}\kappa^{\kappa};\ \kappa - \rho - \sigma = -1$ and $\Re(s + \nu - \lambda - \mu) > 1$ when $|z| = \delta^*\big)$, where $(\lambda)_\nu$ is the well-known and useful Pochhammer symbol defined (for λ, $\nu \in \mathbb{C}$) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N}), \end{cases} \tag{3}$$

it being understood conventionally that $(0)_0 = 1$, and Γ is the familiar Gamma function.

Motivated mainly by various extensions of the Hurwitz-Lerch Zeta function in one and two variables, we aim to give a further extension of the generalized Hurwitz-Lerch Zeta function of two variables as in (4) and then systematically investigate its several interesting properties such as various integral representations, summation formula, Mellin-Barnes contour integral representations (see [15]), fractional derivatives, and analytic continuation formula. A multi-parameter extension of the extended Hurwitz-Lerch Zeta function of two variables is also introduced. Relevant connections of certain special cases of the main results presented here with some known identities are further pointed out.

2. Extended Hurwitz-Lerch Zeta function of two variables

Here we introduce a further extension of the generalized Hurwitz-Lerch Zeta function of two variables as in (4) and consider some of its special and limiting cases.

An extension

$$\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}(z,t,s,a) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{\delta m}(\beta)_{\eta m}(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\gamma)_{\tau m}(\nu)_{\kappa n} m! n!} \frac{z^m t^n}{(m+n+a)^s} \tag{4}$$

$(\alpha, \beta, \lambda, \mu \in \mathbb{C}; a, \gamma, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \delta, \eta, \tau, \rho, \sigma, \kappa \in \mathbb{R}^+; \tau - \delta - \eta > -1, \kappa - \rho - \sigma > -1$ when $s, z, t \in \mathbb{C};\ \tau - \delta - \eta = -1,\ \kappa - \rho - \sigma = -1$ and $s \in \mathbb{C}$ when $|z| < \Delta := \delta^{-\delta}\eta^{-\eta}\tau^{\tau}$, $|t| < \Delta^* := \rho^{-\rho}\sigma^{-\sigma}\kappa^{\kappa};\ \tau - \delta - \eta = -1,\ \kappa - \rho - \sigma = -1$ and $\Re(s + \gamma + \nu - \alpha - \beta - \mu - \lambda) > 0$ when $|z| = \Delta$, $|t| = \Delta^*\big)$.

Special and limiting cases

Case 1. If we set $\alpha = \lambda = \delta = \rho = 1$ in (4), we obtain an extended Hurwitz-Lerch Zeta function of two variables:

$$\begin{aligned}\Phi^{(1,\eta,\tau:1,\sigma,\kappa)}_{1,\beta;\gamma:1,\mu;\nu}(z,t,s,a) &:= \Phi^{(\eta,\tau:\sigma,\kappa)}_{\beta;\gamma:\mu;\nu}(z,t,s,a) \\ &= \sum_{m,n=0}^{\infty} \frac{(\beta)_{\eta m}(\mu)_{\sigma n}}{(\gamma)_{\tau m}(\nu)_{\kappa n}} \frac{z^m t^n}{(m+n+a)^s}\end{aligned} \tag{5}$$

$\big(\beta, \mu \in \mathbb{C};\ a, \gamma, \nu \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ \eta, \tau, \sigma, \kappa \in \mathbb{R}^+;\ \eta < \tau,\ \sigma < \kappa$ when $s, z, t \in \mathbb{C};\ \eta = \tau$, $\sigma = \kappa$ and $s \in \mathbb{C}$ when $|z| < \eta^{-\eta}\tau^{\tau},\ |t| < \sigma^{-\sigma}\kappa^{\kappa};\ \Re(s + \gamma + \nu - \beta - \mu) > 0$ when $|z| = \eta^{-\eta}\tau^{\tau},\ |t| = \sigma^{-\sigma}\kappa^{\kappa}\big)$.

Case 2. The special case of (4) when $\delta = \rho = \eta = \tau = \sigma = \kappa = 1$ yields the known generalized Hurwitz-Lerch Zeta function introduced by Daman and Pathan [4, p. 253, Eq. (10)]:

$$\begin{aligned}\Phi^{(1,1,1:1,1,1)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a) &:= \Phi_{\alpha,\beta;\gamma;\lambda,\mu;\nu}(z,t,s,a) \\ &= \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\beta)_m(\lambda)_n(\mu)_n}{(\gamma)_m(\nu)_n m!n!} \frac{z^m t^n}{(m+n+a)^s}\end{aligned} \tag{6}$$

$\big(\alpha, \beta, \lambda, \mu \in \mathbb{C};\ \gamma, \nu, a \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s \in \mathbb{C}$ when $|z| < 1;\ \Re(s+\gamma+\nu-\alpha-\beta-\mu-\lambda) > 0$ when $|z| = 1$ and $|t| = 1\big)$.

Case 3. The special case of (4) when $\eta = \tau$, $\sigma = \kappa$ and $\delta = \rho = 1$ with $\gamma = \beta$ and $\mu = \nu$ reduces to yield another known generalized Hurwitz-Lerch Zeta function studied by Pathan and Daman [16]:

$$\Phi^{(1,1,1:1,1,1)}_{\alpha,\beta;\beta:\lambda,\nu;\nu}(z,t,s,a) := \Phi^{*}_{\alpha:\lambda}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\lambda)_n}{m!n!} \frac{z^m t^n}{(m+n+a)^s} \tag{7}$$

$\big(\mu, \lambda \in \mathbb{C};\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C}$ when $|z| < 1$ and $|t| < 1;\ \Re(s - \mu - \lambda) > 0$ when $|z| = 1$ and $|t| = 1\big)$.

Case 4. We use the following limiting case of the extended Hurwitz-Lerch Zeta function (4):

$$\begin{aligned}\Phi^{*\,(\eta,\sigma:\tau,\kappa)}_{\beta;\gamma:\mu;\nu}(z,s,a) &= \lim_{\min\{|\alpha|,|\lambda|\}\to\infty} \left\{ \Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}\left(\frac{z}{\alpha^{\delta}}, \frac{t}{\lambda^{\rho}}, s, a\right) \right\} \\ &:= \sum_{m,n=0}^{\infty} \frac{(\beta)_{\eta m}(\mu)_{\rho n}}{(\gamma)_{\tau m}(\nu)_{\kappa n} m!n!} \frac{z^m t^n}{(m+n+a)^s}\end{aligned} \tag{8}$$

$\big(\beta, \mu \in \mathbb{C};\ a, \gamma, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \eta, \tau, \sigma, \kappa \in \mathbb{R}^+;\ s \in \mathbb{C}$ when $|z| < \eta^{-\eta}\tau^{\tau},\ |t| < \sigma^{-\sigma}\kappa^{\kappa};\ \Re(s + \gamma + \nu - \beta - \mu) > 0$ when $|z| = \eta^{-\eta}\tau^{\tau},\ |t| = \sigma^{-\sigma}\kappa^{\kappa}\big)$.

Case 5. Another limiting case of the extended Hurwitz-Lerch Zeta function (4) is given as follows:

$$\Phi^{*\,(\eta:\sigma)}_{\beta:\mu}(z,t,s,a) := \lim_{\min\{|\alpha|,|\gamma|,|\lambda|,|\nu|\}\to\infty} \left\{ \Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}\left(\frac{z\gamma^{\tau}}{\beta^{\eta}}, \frac{t\nu^{\kappa}}{\lambda^{\rho}}, s, a\right) \right\}$$

$$:= \sum_{m,n=0}^{\infty} \frac{(\beta)_{\eta m}(\mu)_{\sigma n}}{m!n!} \frac{z^m t^n}{(m+n+a)^s} \tag{9}$$

$\big(\beta, \mu \in \mathbb{C};\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ 0 < \eta < 1,\ 0 < \sigma < 1$ and $s, z, t \in \mathbb{C};\ \eta = 1,\ \sigma = 1$ and $s \in \mathbb{C}$ when $|z| < \eta^{-\eta}$, $|t| < \sigma^{-\sigma}$ and $\Re(s - \beta - \mu) > 0$ when $|z| = \eta^{-\eta}$, $|t| = \sigma^{-\sigma}\big)$.

3. Integral representations of $\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}(z,t,s,a)$

We first recall the Fox-Wright function ${}_p\Psi_q(z)$ (p, $q \in \mathbb{N}_0$) or its normalization ${}_p\Psi_q^*$ (p, $q \in \mathbb{N}_0$) with p numerator and q denominator parameters defined for $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$ and $\beta_1, \ldots, \beta_q \in \mathbb{C} \setminus \mathbb{Z}_0^-$ by (see, for details, [5, 10, 12–14]; see also [17, 22, 24]):

$$\begin{aligned} &{}_p\Psi_q^*\left[\begin{matrix} (\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots, (\beta_q, B_q); \end{matrix}\ z\right] \\ &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_{A_1 n} \cdots (\alpha_p)_{A_p n}}{(\beta_1)_{B_1 n} \cdots (\beta_q)_{B_q n}} \frac{z^n}{n!} \\ &= \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_q)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_p)}\, {}_p\Psi_q\left[\begin{matrix} (\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots, (\beta_q, B_q); \end{matrix}\ z\right] \end{aligned} \tag{10}$$

$$\left(A_j \in \mathbb{R}^+ \ (j = 1, \ldots, p);\ B_j \in \mathbb{R}^+ \ (j = 1, \ldots, q);\ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0\right),$$

where the equality in the convergence condition holds true for

$$|z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j}\right) \cdot \left(\prod_{j=1}^{q} B_j^{B_j}\right).$$

In particular, when $A_j = B_k = 1$ ($j = 1, \ldots, p;\quad k = 1, \ldots, q$), (10) reduces immediately to the generalized hypergeometric function ${}_pF_q$ (p, $q \in \mathbb{N}_0$) (see, *e.g.*, [22]):

$$\begin{aligned} {}_p\Psi_q^*\left[\begin{matrix} (\alpha_1, 1), \ldots, (\alpha_p, 1); \\ (\beta_1, 1), \ldots, (\beta_q, 1); \end{matrix}\ z\right] &= {}_pF_q\left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix}\ z\right] \\ &= \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_q)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_p)}\, {}_p\Psi_q\left[\begin{matrix} (\alpha_1, 1), \ldots, (\alpha_p, 1); \\ (\beta_1, 1), \ldots, (\beta_q, 1); \end{matrix}\ z\right]. \end{aligned} \tag{11}$$

Theorem 3.1. *The following integral representation for* $\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}(z,t,s,a)$ *in* (4) *holds true*:

$$\begin{aligned} \Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-ax}\, &{}_2\Psi_1^*\left[\begin{matrix} (\alpha, \delta), (\beta, \eta); \\ (\gamma, \tau); \end{matrix}\ ze^{-x}\right] \\ \times\, &{}_2\Psi_1^*\left[\begin{matrix} (\lambda, \rho), (\mu, \sigma); \\ (\nu, \kappa); \end{matrix}\ te^{-x}\right] \mathrm{d}x \end{aligned} \tag{12}$$

$\big(\min\{\Re(s), \Re(a)\} > 0,\ \delta, \eta, \tau, \rho, \sigma, \kappa \in \mathbb{R}^+$ *and* $\tau - \delta - \eta \geqq -1,\ \kappa - \rho - \sigma \geqq -1$ *when* $|z| < \Delta := \delta^{-\delta}\eta^{-\eta}\tau^{\tau},\ |t| < \Delta^* := \rho^{-\rho}\sigma^{-\sigma}\kappa^{\kappa}\big)$.

Proof. Using the following Eulerian integral

$$\frac{1}{(m+n+a)^s} := \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(m+n+a)t}\,\mathrm{d}t$$
$$(\min\{\Re(s),\ \Re(a)\} > 0;\ m,\ n \in \mathbb{N}_0)$$

in (4), and interchanging the order of summation and integration which is guaranteed under the condition stated in Theorem 3.1, and using (10), we are led to the desired integral representation. □

If, in Theorem 3.1, we set $\alpha = \lambda = \delta = \rho = 1$ and apply the relationship (5), we obtain an integral representation for the function $\Phi^{(\eta;\tau:\sigma;\kappa)}_{\beta;\gamma:\mu;\nu}(z,t,s,a)$ asserted by Corollary 3.2.

Corollary 3.2. *The following integral representation for the function* $\Phi^{(\eta,\tau:\sigma,\kappa)}_{\beta;\gamma:\mu;\nu}$ (z,t,s,a) *in* (5) *holds true*:

$$\Phi^{(\eta,\tau:\sigma,\kappa)}_{\beta;\gamma:\mu;\nu}(z,t,s,a) = \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-ax}\ {}_2\Psi_1^*\left[\begin{array}{c}(1,1),(\beta,\eta);\\ (\gamma,\tau);\end{array} ze^{-x}\right]$$
$$\times\ {}_2\Psi_1^*\left[\begin{array}{c}(1,1),(\mu,\sigma);\\ (\nu,\kappa);\end{array} te^{-x}\right]\mathrm{d}x \tag{13}$$

$\big(\min\{\Re(s), \Re(a)\} > 0,\ \tau > \eta > 0,\ \kappa > \sigma > 0$ *when* $z, t \in \mathbb{C}$; $\tau \geqq \eta > 0$, $\kappa \geqq \sigma > 0$ *when* $|z| = \eta^{-\eta}\tau^{\tau},\ |t| < \rho^{-\rho}\kappa^{\kappa}\big)$.

Remark 3.3. The special case $\delta = \rho = \eta = \tau = \sigma = \kappa = 1$ of the result in Theorem 3.1 yields the following known integral representation [4, p. 253, Eq. (11)]:

$$\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)$$
$$:= \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-ax}\ {}_2F_1(\alpha,\beta;\gamma;ze^{-x})\ {}_2F_1(\lambda,\mu;\nu;te^{-x})\,\mathrm{d}x \tag{14}$$

$\big(\Re(a) > 0;\ \Re(s) > 0$ when $|z| \leqq 1\,(z \neq 1),\ |t| \leqq 1\,(t \neq 1);\ \Re(s) > 1$ when $z = 1,\ t = 1\big)$, which, upon applying Euler's transformation formula (see, *e.g.*, [5, p. 64, Eq. 2.1.4 (23)]):

$${}_2F_1\,(a,\, b;\, c;\, z) = (1-z)^{c-a-b}\ {}_2F_1\ (c-a,\, c-b;\, c;\, z) \quad (|\arg(1-z)| < \pi),$$

gives another integral representation for the function (6):

$$\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)$$
$$:= \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}e^{-ax}}{(1-ze^{-x})^{\alpha+\beta-\gamma}(1-te^{-x})^{\lambda+\mu-\nu}} \tag{15}$$
$${}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;ze^{-x})\ {}_2F_1(\nu-\lambda,\nu-\mu;\nu;te^{-x})\,\mathrm{d}x$$

$\big(\Re(a) > 0;\ \Re(s) > 0$ when $|z| \leqq 1\,(z \neq 1)$, $|t| \leqq 1\,(t \neq 1)$; $\Re(s) > 1$ when $z = 1,\ t = 1\big)$. The special case of (15) when $\beta = \gamma$ and $\mu = \nu$ immediately reduces to the following known result for the function (7) (see [4, p. 252, Eq. (7)]):

$$\Phi^*_{\alpha:\lambda}(z,t,s,a) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{(1-ze^{-x})^{\alpha}(1-te^{-x})^{\lambda}}\,\mathrm{d}x \tag{16}$$

$\big(\Re(a) > 0;\ \Re(s) > 0$ when $|z| \leqq 1\,(z \neq 1)$, $|t| \leqq 1\,(t \neq 1)$; $\Re(s) > 1$ when $z = 1,\ t = 1\big)$, which, upon setting $\alpha = \lambda = 1$, yields the following integral representation for the function:

$$\Phi(z,t,s,a) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{(1-ze^{-x})(1-te^{-x})}\,\mathrm{d}x \tag{17}$$

$\big(\Re(a) > 0;\ \Re(s) > 0$ when $|z| \leqq 1\,(z \neq 1)$, $|t| \leqq 1\,(t \neq 1)$; $\Re(s) > 1$ when $z = 1,\ t = 1\big)$.

Remark 3.4. It is interesting to see that the special cases for $z = 0$ or $t = 0$ in (12) to (17) yield the known integral representations earlier studied by Srivastava et al. [25, p. 494, Eq. (2.4)], Lin and Srivastava [11, p. 728, Eq. (14)], Garg et al. [6, p. 313, Eq. (2.1)], Goyal and Laddha [7, p. 100, Eq. (1.6)] and the familiar Hurwitz-Lerch Zeta function [5, p. 27, Eq. 1.11(3)] which are, respectively, given as follows:

$$\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu;\nu}(t,s,a) := \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1}e^{-ax}\ {}_2\Psi_1^*\left[\begin{array}{c}(\lambda,\rho),(\mu,\sigma);\\(\nu,\kappa);\end{array}\ te^{-x}\right]\mathrm{d}x \tag{18}$$

$$\big(\min\{\Re(s),\ \Re(a)\} > 0,\ \rho,\ \sigma,\ \kappa \in \mathbb{R}^+ \text{ and } \kappa-\rho-\sigma \geqq -1 \text{ when } |t| < \delta^* := \rho^{-\rho}\sigma^{-\sigma}\kappa^{\kappa}\big);$$

$$\Phi^{(\rho,\kappa)}_{(\mu;\nu)}(t,s,a) := \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1}e^{-ax}\ {}_2\Psi_1^*\left[\begin{array}{c}(1,1),(\mu,\rho);\\(\nu,\kappa);\end{array}\ te^{-x}\right]\mathrm{d}x \tag{19}$$

$$\big(\min\{\Re(s),\ \Re(a)\} > 0,\ \kappa > \rho > 0 \text{ when } t \in \mathbb{C};\ \kappa \geqq \rho > 0 \text{ when } |t| < \rho^{-\rho}\kappa^{\kappa}\big);$$

$$\Phi_{\lambda,\mu;\nu}(t,s,a) := \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1}e^{-ax}\ {}_2F_1(\lambda,\mu;\nu;te^{-x})\,\mathrm{d}x \tag{20}$$

$$\big(\Re(a) > 0;\ \Re(s) > 0 \text{ when } |t| \leqq 1\,(t \neq 1);\ \Re(s) > 1 \text{ when } t = 1\big);$$

$$\Phi^*_{\mu}(t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{(1-te^{-x})^{\mu}}\,\mathrm{d}x \tag{21}$$

$$\big(\Re(a) > 0;\ \Re(s) > 0 \text{ when } |t| \leqq 1\,(t \neq 1);\ \Re(s) > 1 \text{ when } t = 1\big);$$

$$\Phi(t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{1-te^{-x}}\,\mathrm{d}x \tag{22}$$

$$\big(\Re(a) > 0;\ \Re(s) > 0 \text{ when } |t| \leqq 1\,(t \neq 1);\ \Re(s) > 1 \text{ when } t = 1\big).$$

Theorem 3.5. *The following double and triple integral representations for* $\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)$ *in* (4) *hold true*:

$$\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a) = \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)\Gamma(\lambda)\Gamma(\nu-\lambda)} \int_0^\infty\int_0^\infty \frac{w^{\alpha-1}y^{\lambda-1}}{(1+w)^\gamma(1+y)^\nu} \times \Phi^{*(\eta,\sigma;\tau-\delta,\kappa-\rho)}_{\beta,\mu;\gamma-\alpha,\nu-\lambda}\left(\frac{zw^\delta}{(1+w)^\tau}, \frac{ty^\rho}{(1+y)^\kappa}, s, a\right) \mathrm{d}w\,\mathrm{d}y \tag{23}$$

$\big(\Re(\gamma) > \Re(\alpha) > 0,\ \Re(\nu) > \Re(\lambda) > 0;\ \tau \geqq \delta > 0,\ \kappa \geqq \rho > 0;\ \eta,\sigma \in \mathbb{R}^+\big)$ *and*

$$\begin{aligned} &\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a) \\ &= \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(s)\Gamma(\alpha)\Gamma(\gamma-\alpha)\Gamma(\lambda)\Gamma(\nu-\lambda)} \int_0^\infty\int_0^\infty\int_0^\infty \frac{x^{s-1}e^{-ax}w^{\alpha-1}y^{\lambda-1}}{(1+w)^\gamma(1+y)^\nu} \\ &\times {}_1\Psi_1^*\left[\begin{matrix} (\beta,\eta); \\ (\gamma-\alpha,\tau-\delta); \end{matrix}\ \frac{ze^{-x}w^\delta}{(1+w)^\tau}\right] \\ &\times {}_1\Psi_1^*\left[\begin{matrix} (\mu,\sigma); \\ (\nu-\lambda,\kappa-\rho); \end{matrix}\ \frac{te^{-x}y^\rho}{(1+y)^\kappa}\right] \mathrm{d}x\,\mathrm{d}w\,\mathrm{d}y \end{aligned} \tag{24}$$

$\big(\Re(\gamma) > \Re(\alpha) > 0,\ \Re(\nu) > \Re(\lambda) > 0;\ \tau \geqq \delta > 0,\ \kappa \geqq \rho > 0;\ \eta,\sigma \in \mathbb{R}^+;\ \min\{\Re(s),\Re(a)\} > 0\big)$.

Proof. Setting $a = \alpha + \delta m$ and $b = \gamma + \tau m$, and $a = \lambda + \rho n$ and $b = \nu + \kappa n$, respectively, in the Eulerian Beta-function formula [5, p. 9, Eq. 1.5(2)]:

$$B(a, b-a) = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)} = \int_0^\infty \frac{y^{a-1}}{(1+y)^b}\,\mathrm{d}y \qquad (\Re(b) > \Re(a) > 0), \tag{25}$$

we find

$$\frac{(\alpha)_{\delta m}}{(\gamma)_{\tau m}} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \frac{1}{(\gamma-\alpha)_{(\tau-\delta)m}} \int_0^\infty \frac{w^{\alpha+\delta m-1}}{(1+w)^{\gamma+\tau m}}\,\mathrm{d}w \tag{26}$$

$$(\Re(\gamma) > \Re(\alpha) > 0;\ \tau \geqq \delta;\ m \in \mathbb{N}_0)$$

and

$$\frac{(\lambda)_{\rho n}}{(\nu)_{\kappa n}} = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)} \frac{1}{(\nu-\lambda)_{(\kappa-\rho)n}} \int_0^\infty \frac{y^{\lambda+\rho n-1}}{(1+y)^{\nu+\kappa n}}\,\mathrm{d}y \tag{27}$$

$$(\Re(\nu) > \Re(\lambda) > 0;\ \kappa \geqq \sigma;\ n \in \mathbb{N}_0),$$

which, by appealing to the definition (8), immediately yields the assertion (23). Moreover, by (12), (25) and (26), we also obtain

$$\begin{aligned} &\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a) \\ &:= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1}e^{-ax} \sum_{m=0}^\infty \frac{(\alpha)_{\delta m}(\beta)_{\eta m}}{(\gamma)_{\tau m}} \frac{(ze^{-x})^m}{m!} \sum_{n=0}^\infty \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{(te^{-x})^n}{n!}\,\mathrm{d}x \end{aligned}$$

$$= \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(s)\Gamma(\alpha)\Gamma(\gamma-\alpha)\Gamma(\lambda)\Gamma(\nu-\lambda)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{s-1}e^{-ax}w^{\alpha-1}y^{\lambda-1}}{(1+w)^\gamma(1+y)^\nu}$$
$$\times \sum_{m=0}^\infty \frac{(\beta)_{\eta m}}{(\gamma-\alpha)_{(\tau-\delta)m}m!}\left(\frac{ze^{-x}w^\delta}{(1+w)^\tau}\right)^n$$
$$\times \sum_{n=0}^\infty \frac{(\mu)_{\sigma n}}{(\nu-\lambda)_{(\kappa-\rho)n}n!}\left(\frac{te^{-x}y^\rho}{(1+y)^\kappa}\right)^n \mathrm{d}x\,\mathrm{d}w\,\mathrm{d}y,$$

which, in view of (10), leads us to the second assertion (24) of Theorem 3.5. □

If, in Theorem 3.5, we set $\tau = \delta$ and $\kappa = \rho$ and using the definition (9) and (10), we obtain the following special cases for the function $\Phi^{(\delta,\eta,\delta:\rho,\sigma,\rho)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)$, respectively, asserted by Corollary 3.6 below.

Corollary 3.6. *Each of the following integral representations for* $\Phi^{(\delta,\eta,\delta:\rho,\sigma,\rho)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}$ (z,t,s,a) *holds true*:

(28)
$$\Phi^{(\delta,\eta,\delta:\rho,\sigma,\rho)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a) = \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)\Gamma(\lambda)\Gamma(\nu-\lambda)} \int_0^\infty \int_0^\infty \frac{w^{\alpha-1}y^{\lambda-1}}{(1+w)^\gamma(1+y)^\nu} \times \Phi^{*(\eta,\sigma)}_{\beta,\mu}\left(\frac{zx^\delta}{(1+y)^\delta}, \frac{ty^\rho}{(1+y)^\rho}, s, a\right) \mathrm{d}w\,\mathrm{d}y$$
$$\big(\Re(\gamma) > \Re(\alpha) > 0,\ \Re(\nu) > \Re(\lambda) > 0;\ \eta,\, \sigma \in \mathbb{R}^+\big)$$

and

(29)
$$\Phi^{(\delta,\eta,\delta:\rho,\sigma,\rho)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a) = \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(s)\Gamma(\alpha)\Gamma(\gamma-\alpha)\Gamma(\lambda)\Gamma(\nu-\lambda)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{s-1}e^{-ax}w^{\alpha-1}y^{\lambda-1}}{(1+w)^\gamma(1+y)^\nu} \times {}_1\Psi_0^*\left[\begin{matrix}(\beta,\eta)\ ; & \dfrac{ze^{-x}w^\delta}{(1+w)^\delta} \\ -\ ; & \end{matrix}\right] {}_1\Psi_0^*\left[\begin{matrix}(\mu,\sigma)\ ; & \dfrac{te^{-x}y^\rho}{(1+y)^\rho} \\ -\ ; & \end{matrix}\right] \mathrm{d}x\,\mathrm{d}w\,\mathrm{d}y$$
$$\big(\Re(\gamma) > \Re(\alpha) > 0,\ \Re(\nu) > \Re(\lambda) > 0;\ \eta,\, \sigma \in \mathbb{R}^+;\ \min\{\Re(s),\, \Re(a)\} > 0\big).$$

Further, if we set $\delta = \eta = \rho = \sigma = 1$ in (29), we obtain a new integral representation for $\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)$ in (6):

(30)
$$\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a) = \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(s)\Gamma(\alpha)\Gamma(\gamma-\alpha)\Gamma(\lambda)\Gamma(\nu-\lambda)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{s-1}e^{-ax}w^{\alpha-1}y^{\lambda-1}}{(1+w)^\gamma(1+y)^\nu} \times \left(1 - \frac{zwe^{-x}}{1+w}\right)^{-\beta} \left(1 - \frac{tye^{-x}}{1+y}\right)^{-\mu} \mathrm{d}x\,\mathrm{d}w\,\mathrm{d}y$$
$$\big(\Re(\gamma) > \Re(\alpha) > 0,\ \Re(\nu) > \Re(\lambda) > 0;\ \min\{\Re(s),\, \Re(a)\} > 0\big).$$

Theorem 3.7. *The following summation formula for* $\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)$ *in* (4) *holds true*:

$$\sum_{k=0}^{\infty}\frac{(s)_k}{k!}\,\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s+k,a)\,x^n=\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a-x) \tag{31}$$

$$(\lambda\in\mathbb{C};\ |x|<|a|;\ s\neq 1).$$

Proof. Using (4) in the right-hand side of the assertion (31), we have

(32)

$$\begin{aligned}
&\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a-x)\\
&=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{\delta m}(\beta)_{\eta m}(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\gamma)_{\tau m}(\nu)_{\kappa n}}\,\frac{z^m t^n}{m!n!(m+n+a-x)^s}\\
&=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{\delta m}(\beta)_{\eta m}(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\gamma)_{\tau m}(\nu)_{\kappa n}}\,\frac{z^m t^n}{m!n!(m+n+a)^s}\left(1-\frac{x}{m+n+a}\right)^{-s}\\
&=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{\delta m}(\beta)_{\eta m}(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\gamma)_{\tau m}(\nu)_{\kappa n}}\,\frac{z^m t^n}{m!n!(m+n+a)^s}\left\{\sum_{k=0}^{\infty}\frac{(s)_k}{k!}\,\frac{x^k}{(m+n+a)^k}\right\}\\
&=\sum_{k=0}^{\infty}\frac{(s)_k}{k!}\left\{\sum_{m,n=0}^{\infty}\frac{(\alpha)_{\delta m}(\beta)_{\eta m}(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\gamma)_{\tau m}(\nu)_{\kappa n}}\,\frac{z^m t^n}{m!n!(m+n+a)^{s+k}}\right\}x^k,
\end{aligned}$$

which, upon using (4), leads to the right-hand side of (31). □

4. Mellin-Barnes contour integral representations

We begin by presenting the following Mellin-Barnes contour integral representation of the extended Hurwitz-Lerch Zeta function $\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)$ of two variables as in Theorem 4.1.

Theorem 4.1. *The following Mellin-Barnes integral representation for* $\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)$ *in* (4) *holds true*:

(33)

$$\begin{aligned}
&\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)\\
&=\frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\Gamma(-\xi)\Gamma(-\zeta)\\
&\qquad\times\frac{\Gamma(\alpha+\delta\xi)\Gamma(\beta+\eta\xi)\Gamma(\lambda+\rho\zeta)\Gamma(\mu+\sigma\zeta)(-z)^{\xi}(-t)^{\zeta}}{\Gamma(\gamma+\tau\xi)\Gamma(\nu+\kappa\zeta)(\xi+\zeta+a)^s}\,\mathrm{d}\xi\,\mathrm{d}\zeta
\end{aligned}$$

$$(|\arg(-z)|<\pi,\ |\arg(-t)|<\pi,\ \min\{\Re(a),\,\Re(s),\,\Re(\gamma),\,\Re(\nu)\}>0),$$

where it is assumed that the poles of the integrand in (33) *are simple. The contours of integration are so described that the poles of* $\Gamma(-\xi)$, $\Gamma(-\zeta)$ *are* *separated from the poles of* $\Gamma(\alpha+\delta\xi)$, $\Gamma(\beta+\eta\xi)$, $\Gamma(\lambda+\rho\zeta)$ *and* $\Gamma(\mu+\sigma\zeta)$ *with indentations, if necessary.*

Proof. If we calculate the integral (33) as sum of the residues at the simple poles of $\Gamma(-\xi)$ at the points $\xi = m$ $(m \in \mathbb{N}_0)$ and of the $\Gamma(-\zeta)$ $(n \in \mathbb{N}_0)$ at the points $\zeta = n$, respectively, we immediately obtain the following series expansion:

$$\begin{aligned}(34)\qquad &\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)\\ &= \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\\ &\quad\times \sum_{m,n=0}^{\infty} \frac{\Gamma(\alpha+\delta m)\Gamma(\beta+\eta m)\Gamma(\lambda+\rho n)\Gamma(\mu+\sigma n)}{\Gamma(\gamma+\tau m)\Gamma(\nu+\kappa n)m!n!}\frac{z^m t^n}{(m+n+a)^s}\end{aligned}$$

for the extended Hurwitz-Lerch Zeta function of two variables. This completes the proof. □

Suitably specializing the Mellin-Barnes integral representation in (33), we can obtain Mellin-Barnes contour integral representations of the extended Hurwitz-Lerch Zeta functions in (5), (7), (8) and (9), whose explicit expressions are left to the interested reader. Here we consider the following special case (33) when $\delta=\rho=\eta=\tau=\sigma=\kappa=1$ which is a Mellin-Barnes contour integral representation of the Hurwitz-Lerch Zeta function of two variables in (6) as in Corollary 4.2.

Corollary 4.2. *The following contour integral representations for* $\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}$ (z,t,s,a) *in* (6) *holds true*:

$$\begin{aligned}\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a) &= \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\Gamma(-\xi)\Gamma(-\zeta)\\ &\quad\times\frac{\Gamma(\alpha+\xi)\Gamma(\beta+\xi)\Gamma(\lambda+\zeta)\Gamma(\mu+\zeta)(-z)^{\xi}(-t)^{\zeta}}{\Gamma(\gamma+\xi)\Gamma(\nu+\zeta)(\xi+\zeta+a)^s}\,\mathrm{d}\xi\,\mathrm{d}\zeta.\end{aligned}\tag{35}$$

5. Computational representation for $\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)$

Here we investigate an analytic continuation for the extended Hurwitz-Lerch function of two variables in (4). This result will enable us to derive corresponding continuation formulas for most of members of the extended Hurwitz-Lerch Zeta functions of two variables and its related functions.

Theorem 5.1. *The following computational representation for* $\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}$ (z,t,s,a) *holds true*:

$$\begin{aligned}&\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)\\ &= \tfrac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}(-z)^{-\frac{\alpha}{\delta}}(-t)^{-\frac{\lambda}{\rho}}\end{aligned}$$

$$\times \sum_{m,n=0}^{\infty} \frac{\Gamma(\frac{\alpha+m}{\delta})\Gamma(\frac{\lambda+n}{\rho})\Gamma\big(\beta-\eta(\frac{\alpha+m}{\delta})\big)\Gamma(\mu-\sigma(\frac{\lambda+n}{\rho}))(-(-z)^{-\frac{1}{\delta}})^m(-(-t)^{-\frac{1}{\rho}})^n}{\Gamma(\gamma-\tau(\frac{\alpha+m}{\delta}))\Gamma(\nu-\kappa(\frac{\lambda+n}{\rho}))\big(a-(\frac{\alpha+m}{\delta})-(\frac{\lambda+n}{\rho})\big)^s m!n!}$$

$$+ \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}(-z)^{-\frac{\beta}{\eta}}(-t)^{-\frac{\mu}{\sigma}}$$

$$\times \sum_{m,n=0}^{\infty} \frac{\Gamma(\frac{\beta+m}{\eta})\Gamma(\frac{\mu+n}{\sigma})\Gamma\big(\alpha-\delta(\frac{\beta+m}{\eta})\big)\Gamma(\lambda-\rho(\frac{\mu+n}{\sigma}))(-(-z)^{-\frac{1}{\eta}})^m(-(-t)^{-\frac{1}{\sigma}})^n}{\Gamma(\gamma-\tau(\frac{\beta+m}{\eta}))\Gamma(\nu-\kappa(\frac{\mu+n}{\sigma}))\big(a-(\frac{\beta+m}{\eta})-(\frac{\mu+n}{\sigma})\big)^s m!n!} \tag{36}$$

$(\min\{\Re(a), \Re(s)\} > 0, \max\{| \arg(-z) |, | \arg(-t) |\} < \pi, \; | x |\to \infty, | t |\to \infty)$.

Proof. To obtain the analytic continuation formula for the extended Hurwitz-Lerch Zeta function in (33), we first calculate the residue of the $\Gamma(\alpha+\delta\xi)$ at the simple poles:

$$\xi = -\frac{\alpha+m}{\delta} \quad (m \in \mathbb{N}_0)$$

and the residues of $\Gamma(\lambda+\rho\zeta)$ at the simple poles:

$$\zeta = -\frac{\lambda+n}{\rho} \quad (n \in \mathbb{N}_0)$$

as (say I_1)

(37)

$$I_1 = \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}$$
$$\times \sum_{m,n=0}^{\infty} \frac{\Gamma(\frac{\alpha+m}{\delta})\Gamma(\frac{\lambda+n}{\rho})\Gamma\big(\beta-\eta(\frac{\alpha+m}{\delta})\big)\Gamma(\mu-\sigma(\frac{\lambda+n}{\rho}))(-z)^{-(\frac{\alpha+m}{\delta})}(-t)^{-(\frac{\lambda+n}{\rho})}(-1)^{m+n}}{\Gamma(\gamma-\tau(\frac{\alpha+m}{\delta}))\Gamma(\nu-\kappa(\frac{\lambda+n}{\rho}))\big(a-(\frac{\alpha+m}{\delta})-(\frac{\lambda+n}{\rho})\big)^s m!n!}$$
$$= \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}(-z)^{-\frac{\alpha}{\delta}}(-t)^{-\frac{\lambda}{\rho}}$$
$$\times \sum_{m,n=0}^{\infty} \frac{\Gamma(\frac{\alpha+m}{\delta})\Gamma(\frac{\lambda+n}{\rho})\Gamma\big(\beta-\eta(\frac{\alpha+m}{\delta})\big)\Gamma(\mu-\sigma(\frac{\lambda+n}{\rho}))(-(-z)^{-\frac{1}{\delta}})^m(-(-t)^{-\frac{1}{\rho}})^n}{\Gamma(\gamma-\tau(\frac{\alpha+m}{\delta}))\Gamma(\nu-\kappa(\frac{\lambda+n}{\rho}))\big(a-(\frac{\alpha+m}{\delta})-(\frac{\lambda+n}{\rho})\big)^s m!n!}.$$

Similarly, the residues of the $\Gamma(\beta+\eta\xi)$ at the simple poles:

$$\xi = -\frac{\beta+m}{\eta} \quad (m \in \mathbb{N}_0)$$

and the residues of $\Gamma(\mu+\sigma\zeta)$ at the simple poles:

$$\zeta = -\frac{\mu+n}{\sigma} \quad (n \in \mathbb{N}_0)$$

as (say I_2)

(38)

$$I_2 = \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}(-z)^{-\frac{\beta}{\eta}}(-t)^{-\frac{\mu}{\sigma}}$$

$$\times \sum_{m,n=0}^{\infty} \frac{\Gamma(\frac{\beta+m}{\eta})\Gamma(\frac{\mu+n}{\sigma})\Gamma\big(\alpha-\delta(\frac{\beta+m}{\eta})\big)\Gamma(\lambda-\rho(\frac{\mu+n}{\sigma}))}{\Gamma(\gamma-\tau(\frac{\beta+m}{\eta}))\Gamma(\nu-\kappa(\frac{\mu+n}{\sigma}))m!n!} \frac{(-(-z)^{-\frac{1}{\eta}})^m(-(-t)^{-\frac{1}{\sigma}})^n}{\Big(a-(\frac{\beta+m}{\eta})-(\frac{\mu+n}{\sigma})\Big)^s}.$$

Thus, by combining the residues (37) and (38), we are led to the representation (36). □

Corollary 5.2. *Under the hypothesis of* Theorem 5.1*, the following asymptotic formula holds true*:

$$\Phi^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a) \sim A(-z)^{-\frac{\alpha}{\delta}}(-t)^{-\frac{\lambda}{\rho}} + B(-z)^{-\frac{\beta}{\eta}}(-t)^{-\frac{\mu}{\sigma}}, \tag{39}$$

where

$$A := \frac{\Gamma(\gamma)\Gamma(\nu)\Gamma(\frac{\alpha}{\delta})\Gamma(\frac{\lambda}{\rho})\Gamma\left(\beta-\eta(\frac{\alpha}{\delta})\right)\Gamma(\mu-\sigma(\frac{\lambda}{\rho}))}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)\Gamma(\gamma-\tau(\frac{\alpha}{\delta}))\Gamma(\nu-\kappa(\frac{\lambda}{\rho}))\left(a-(\frac{\alpha}{\delta})-(\frac{\lambda}{\rho})\right)^s}$$

and

$$B := \frac{\Gamma(\gamma)\Gamma(\nu)\Gamma(\frac{\beta}{\eta})\Gamma(\frac{\mu}{\sigma})\Gamma\left(\alpha-\delta(\frac{\beta}{\eta})\right)\Gamma(\lambda-\rho(\frac{\mu}{\sigma}))}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)\Gamma(\gamma-\tau(\frac{\beta}{\eta}))\Gamma(\nu-\kappa(\frac{\mu}{\sigma}))\left(a-(\frac{\beta}{\eta})-(\frac{\mu}{\sigma})\right)^s}$$

$$(\min\{\Re(\alpha),\ \Re(\beta),\ \Re(\lambda),\ \Re(\mu),\ \Re(a),\ \Re(s)\} > 0,$$
$$\max\{|\arg(-z)|,\ |\arg(-t)|\} < \pi;\ |x| \to \infty, |t| \to \infty).$$

Analytic continuation formulas of the extended Hurwitz-Lerch Zeta functions in (5), (7), (8) and (9) can be easily obtained as suitable special cases of the analytic continuation formula in (36). In particular, the special case of (36) when $\eta = \tau = \sigma = \kappa = 1$ reduces to yield the following representation for the Hurwitz-Lerch Zeta function of two variable due to Daman and Pathan [4, p. 253, Eq. (11)] as in Corollary 5.3.

Corollary 5.3. *The following computational representation for* $\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)$ *holds true*:

$$\begin{aligned}
&\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)\\
&= \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}(-z)^{-\alpha}(-t)^{-\lambda}\\
&\quad\times \sum_{m,n=0}^{\infty} \frac{\Gamma(\alpha+m)\Gamma(\lambda+n)\Gamma(\beta-(\alpha+m))\Gamma(\mu-(\lambda+n))(-(-z)^{-m})(-(-t)^{-n})}{\Gamma(\gamma-(\alpha+m))\Gamma(\nu-(\lambda+n))(a-(\alpha+m)-(\lambda+n))^s m!n!}\\
&\quad+ \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\mu)}(-z)^{-\beta}(-t)^{-\mu}\\
&\quad\times \sum_{m,n=0}^{\infty} \frac{\Gamma(\beta+m)\Gamma(\mu+n)\Gamma(\alpha-(\beta+m))\Gamma(\lambda-(\mu+n))(-(-z)^{-m})(-(-t)^{-n})}{\Gamma(\gamma-(\beta+m))\Gamma(\nu-(\mu+n))(a-(\beta+m)-(\mu+n))^s m!n!}
\end{aligned} \tag{40}$$

$(\min\{\Re(a),\ \Re(s)\} > 0,\ \max\{|\arg(-z)|,\ |\arg(-t)|\} < \pi;\ |x| \to \infty, |t| \to \infty)$.

Furthermore, by letting $s \to 0$ in (40), and using the formula:

$$(\alpha)_{-n} = \frac{(-1)^n}{(1-\alpha)_n} \qquad (\alpha \in \mathbb{C} \setminus \mathbb{N}; n \in \mathbb{N}_0),$$

we obtain the following interesting continuation formula for the product of two Gauss hypergeometric functions:

$$\begin{aligned} &{}_2F_1(\alpha,\beta;\gamma;z)\ {}_2F_1(\lambda,\mu;\nu;t) \\ (41)\quad &= \frac{\Gamma(\gamma)\Gamma(\nu)\Gamma(\beta-\alpha)\Gamma(\mu-\lambda)}{\Gamma(\beta)\Gamma(\mu)\Gamma(\gamma-\alpha)\Gamma(\nu-\lambda)}(-z)^{-\alpha}(-t)^{-\lambda} \\ &\times {}_2F_1\left(\alpha, 1+\alpha-\gamma; 1+\alpha-\beta; \frac{1}{z}\right)\ {}_2F_1\left(\lambda, 1+\lambda-\nu; 1+\lambda-\mu; \frac{1}{t}\right) \\ &+ \frac{\Gamma(\gamma)\Gamma(\nu)\Gamma(\alpha-\beta)\Gamma(\lambda-\mu)}{\Gamma(\beta)\Gamma(\mu)\Gamma(\gamma-\beta)\Gamma(\nu-\mu)}(-z)^{-\beta}(-t)^{-\mu} \\ &\times {}_2F_1\left(\beta, 1+\beta-\gamma; 1+\beta-\alpha; \frac{1}{z}\right)\ {}_2F_1\left(\mu, 1+\mu-\nu; 1+\mu-\lambda; \frac{1}{t}\right) \\ &(\max\{|\arg(-z)|, |\arg(-t)|\} < \pi;\ \min\{|z|, |t|\} > 1 \text{ when} \\ &\qquad \min\{\Re(\gamma-\alpha-\beta), \Re(\nu-\lambda-\mu)\} > 0). \end{aligned}$$

6. Fractional derivatives for $\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}(z,t,s,a)$

Recall the *Riemann-Liouville fractional derivative operator* $\mathcal{D}_z^\mu$ defined by (see, *e.g.* [10] and [17, p. 70 *et seq.*]):

(42)

$$\mathcal{D}_z^\mu \{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-t)^{-\mu-1}\ f(t)\, dt & \big(\Re(\mu) < 0\big), \\ \dfrac{d^m}{dz^m} \Big\{\mathcal{D}_z^{\mu-m}\{f(z)\}\Big\} & \big(m-1 \leqq \Re(\mu) < m \ \ (m \in \mathbb{N})\big). \end{cases}$$

It is easy to find the following formula:

$$\mathcal{D}_z^\mu \left\{z^\lambda\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\ z^{\lambda-\mu} \qquad \big(\Re(\lambda) > -1\big). \tag{43}$$

Theorem 6.1. *The following fractional derivative formula for* $\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}$ (z,s,a) *in* (4) *holds true*:

$$\begin{aligned} &\mathcal{D}_z^{\gamma-\xi}\mathcal{D}_t^{\nu-\zeta}\left\{z^{\gamma-1}t^{\nu-1}\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}(z^\tau, t^\kappa, s, a)\right\} \\ (44)\quad &= \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\xi)\Gamma(\zeta)} z^{\xi-1}t^{\zeta-1}\Phi_{\alpha,\beta;\xi:\lambda,\mu;\zeta}^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}(z^\tau, t^\kappa, s, a) \\ &\qquad (\min\{\Re(\gamma), \Re(\nu)\} > 0;\ \min\{\tau, \kappa\} > 0). \end{aligned}$$

Proof. By virtue of the definition (4) of $\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}(z,s,a)$ and the formula (43), the assertion (44) follows easily. □

Corollary 6.2. *Each of the following fractional derivative formulas holds true*:

$$\text{(45)}\qquad \mathcal{D}_z^{\gamma-\xi}\mathcal{D}_t^{\nu-\zeta}\left\{z^{\gamma-1}t^{\nu-1}\Phi_{\beta;\gamma:\mu;\nu}^{(\eta,\tau:\sigma,\kappa)}(z^\tau,t^\kappa,s,a)\right\}$$
$$=\frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\xi)\Gamma(\zeta)}z^{\xi-1}t^{\zeta-1}\Phi_{\beta;\xi:\mu;\zeta}^{(\eta,\tau:\sigma,\kappa)}(z^\tau,t^\kappa,s,a)$$
$$(\min\{\Re(\gamma),\,\Re(\nu)\}>0;\;\;\min\{\tau,\,\kappa\}>0);$$

$$\text{(46)}\qquad \mathcal{D}_z^{\gamma-\xi}\mathcal{D}_t^{\nu-\zeta}\left\{z^{\gamma-1}t^{\nu-1}\Phi_{\beta;\gamma:\mu;\nu}(z,t,s,a)\right\}$$
$$=\frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\xi)\Gamma(\zeta)}z^{\xi-1}t^{\zeta-1}\Phi_{\beta;\xi:\mu;\zeta}(z,t,s,a)$$
$$(\min\{\Re(\gamma),\,\Re(\nu)\}>0).$$

Theorem 6.3. *The following fractional derivative formula holds true*:

$$\text{(47)}\qquad \mathcal{D}_z^{\alpha-\gamma}\,\mathcal{D}_t^{\lambda-\nu}\{z^{\alpha-1}t^{\lambda-1}\,\Phi_{\beta:\mu}^*(z,t,s,a)\}$$
$$=\frac{\Gamma(\alpha)\Gamma(\lambda)}{\Gamma(\gamma)\Gamma(\nu)}\,z^{\gamma-1}\,t^{\nu-1}\,\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}(z,t,s,a)$$
$$(\Re(\nu)>\Re(\mu)>0).$$

Proof. Using the result (43) and (6), we can easily deduce fractional derivative formulas for (7) as in (47). □

7. Multi-parametric extension of $\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}(z,t,s,a)$

Here we present a multi-parametric extension of the $\Phi_{\alpha,\beta;\gamma:\lambda,\mu;\nu}^{(\delta,\eta,\tau:\rho,\sigma,\kappa)}(z,t,s,a)$ by introducing arbitrary number of numerator and denominator parameters as follows:

$$\text{(48)}\qquad \Phi_{\lambda_1,\ldots,\lambda_p;\gamma_1,\ldots,\gamma_q:\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_w}^{(\rho_1,\ldots,\rho_p,\tau_1,\ldots,\tau_q:\sigma_1,\ldots,\sigma_r,\kappa_1,\ldots,\kappa_w)}(z,t,s,a)$$
$$:=\sum_{m,n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{m\rho_j}\;\prod_{j=1}^{r}(\mu_j)_{n\sigma_j}}{\prod_{j=1}^{q}(\gamma_j)_{m\tau_j}\;\prod_{j=1}^{w}(\nu_j)_{n\kappa_j}}\frac{z^m t^n}{m!n!(m+n+a)^s}$$

$\big(\lambda_i,\ \mu_j\in\mathbb{C}\ (i=1,\ldots,p;\;\; j=1,\ldots,r);$
$a,\,\gamma_i,\,\nu_j\in\mathbb{C}\setminus\mathbb{Z}_0^-\,(i=1,\ldots,q;\;\; j=1,\ldots,w);$
$\rho_i,\,\tau_j,\,\sigma_k,\,\kappa_l\in\mathbb{R}^+\ (i=1,\ldots,p;\;\; j=1,\ldots,q;\;\; k=1,\ldots,r;\;\; l=1,\ldots,w);$
$\Delta>-1,\,\delta>-1$ when $s,\,z,\,t\in\mathbb{C}$;
$\Delta=-1,\,\delta=-1$ and $s\in\mathbb{C}$ when $|z|<\Delta^*,\,|t|<\delta^*$;
$\Delta=-1$ and $\Re(\Xi)>0$ when $|z|=\Delta^*,\,|t|=\delta^*$ where

$$\Delta^* := \left(\prod_{j=0}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=0}^{q} \tau_j^{\tau_j}\right), \ \Delta = \sum_{j=0}^{q} \tau_j - \sum_{j=0}^{p} \rho_j \ \text{ and}$$

$$\delta^* := \left(\prod_{j=0}^{r} \sigma_j^{-\sigma_j}\right) \cdot \left(\prod_{j=0}^{w} \kappa_j^{\kappa_j}\right), \ \delta = \sum_{j=0}^{w} \kappa_j - \sum_{j=0}^{r} \sigma_j \ \text{ and}$$

$$\Xi = s + \sum_{j=0}^{q} \gamma_j + \sum_{j=0}^{w} \nu_j - \sum_{j=0}^{p} \lambda_j - \sum_{j=0}^{r} \mu_j).$$

Theorem 7.1. *The following integral representation in* (48) *holds true*:

$$\begin{aligned} &\Phi^{(\rho_1,\ldots,\rho_p,\tau_1,\ldots,\tau_q:\sigma_1,\ldots,\sigma_r,\kappa_1,\ldots,\kappa_w)}_{\lambda_1,\ldots,\lambda_p;\gamma_1,\ldots,\gamma_q:\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_w}(z,t,s,a) \\ &:= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \, {}_p\Psi_q^* \left[\begin{matrix} (\lambda_1,\rho_1),\ldots,(\lambda_p,\rho_p); \\ (\gamma_1,\tau_1),\ldots,(\gamma_p,\tau_p); \end{matrix} \, ze^{-x} \right] \\ &\qquad \times {}_r\Psi_w^* \left[\begin{matrix} (\mu_1,\sigma_1),\ldots,(\mu_r,\sigma_r); \\ (\nu_1,\kappa_1),\ldots,(\nu_w,\kappa_w); \end{matrix} \, te^{-x} \right] \mathrm{d}x \end{aligned} \tag{49}$$

$$\big(\rho_j,\, \tau_k,\, \sigma_l,\, \kappa_m \in \mathbb{R}^+ (j=1,\ldots,p; \;\; k=1,\ldots,q; \;\; l=1,\ldots,r; \;\; m=1,\ldots,w)$$
$$\textit{and} \quad \min\{\Re(s),\, \Re(a)\} > 0; \; |z| < \Delta^*,\, |t| < \delta^*\big).$$

Theorem 7.2. *The following Mellin-Barnes contour integral representation in* (48) *holds true*:

$$\begin{aligned} &\Phi^{(\rho_1,\ldots,\rho_p,\tau_1,\ldots,\tau_q:\sigma_1,\ldots,\sigma_r,\kappa_1,\ldots,\kappa_w)}_{\lambda_1,\ldots,\lambda_p;\gamma_1,\ldots,\gamma_q:\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_w}(z,t,s,a) \\ &= \frac{\prod_{j=1}^{q} \Gamma(\gamma_j) \prod_{j=1}^{w} \Gamma(\nu_j)}{\prod_{j=1}^{p} \Gamma(\lambda_j) \prod_{j=1}^{r} \Gamma(\mu_j)} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \Gamma(-\xi)\Gamma(-\zeta) \\ &\qquad \times \frac{\prod_{j=1}^{p} \Gamma(\lambda_j + \rho_j\xi) \prod_{j=1}^{r} \Gamma(\mu_j + \sigma_j\zeta)(-z)^{\xi}(-t)^{\zeta}}{(\xi+\zeta+a)^s \prod_{j=1}^{q} \Gamma(\gamma_j + \tau_j\xi) \prod_{j=1}^{w} \Gamma(\nu_j + \kappa_j\zeta)} \, \mathrm{d}\xi \, \mathrm{d}\zeta \end{aligned} \tag{50}$$

$$(\max\{|\arg(-z)|,\, |\arg(-t)|\} < \pi).$$

Theorem 7.3. *The following fractional derivative formula in* (48) *holds true*:

$$\begin{aligned} &\mathcal{D}_z^{\gamma-\xi} \mathcal{D}_t^{\nu-\zeta} \left\{ z^{\gamma-1} t^{\nu-1} \Phi^{(\rho_1,\ldots,\rho_p,\tau_1,\ldots,\tau_q:\sigma_1,\ldots,\sigma_r,\kappa_1,\ldots,\kappa_w)}_{\lambda_1,\ldots,\lambda_p;\gamma_1,\ldots,\gamma_q:\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_w}(z^\tau, t^\kappa, s, a) \right\} \\ &= \frac{\Gamma(\gamma)\Gamma(\nu)}{\Gamma(\xi)\Gamma(\zeta)} z^{\xi-1} t^{\zeta-1} \Phi^{(\rho_1,\ldots,\rho_p,\tau,\tau_1,\ldots,\tau_q,\tau:\sigma_1,\ldots,\sigma_r,\kappa,\kappa_1,\ldots,\kappa_w,\kappa)}_{\lambda_1,\ldots,\lambda_p,\xi;\gamma_1,\ldots,\gamma_q,\gamma:\mu_1,\ldots,\mu_r,\zeta;\nu_1,\ldots,\nu_w,\nu}(z^\tau, t^\kappa, s, a) \end{aligned} \tag{51}$$

$$(\min\{\Re(\gamma),\, \Re(\nu)\} > 0; \; \min\{\tau,\, \kappa\} > 0).$$

Acknowledgments. The authors would like to express their deep-felt thanks for the reviewer's detailed and helpful comments.

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