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DING PROJECTIVE DIMENSION OF GORENSTEIN FLAT MODULES

Junpeng Wang

Abstract. Let R be a Ding-Chen ring. Yang [24] and Zhang [25] asked whether or not every R-module has finite Ding projective or Ding injective dimension. In this paper, we give a new characterization of that all modules have finite Ding projective and Ding injective dimension in terms of the relationship between Ding projective and Gorenstein flat modules. We also give an example to obtain negative answer to the above question.

1. Introduction and preliminaries

Unless stated otherwise, throughout this paper R denotes an associative ring with identity and every R-module is a unitary left R-module. All classes are closed under isomorphisms.

Enochs and co-authors [13, 15] introduced Gorenstein projective, injective and flat modules and then established Gorenstein homological algebra. Gorenstein homological algebra has been developed rapidly during the past several years. The theory of modules of finite Gorenstein dimensions has some interesting applications in representation theory; these include the structure of the stable category of Cohen-Macaulay modules, the Auslander-Reiten theory, the existence of Serre duality at the level of perfect complexes and the theory of singularities. In addition, the formal properties of Gorenstein projective dimension may proved to be useful to study certain group-theoretical problems (cf. [11]). Also, the theory of modules of finite Gorenstein dimensions is closed related to Auslander categories (cf. [8, 17]) and Tate (co)homology theory (cf. [25]).

Ding projective and Ding injective modules, as a special case of Gorenstein projective and Gorenstein injective modules respectively, were initially called strongly Gorenstein flat and Gorenstein FP-injective modules respectively by Ding and co-authors in [10, 21], and was renamed by Gillespie in [18]. Dimensions by Ding modules has been studied in [10, 18, 20, 24, 25]. Let R be a Ding-Chen ring. Yang [24, Remark 2.9] and Zhang [25, p. 821] asked whether

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or not every R-module has finite Ding projective or Ding injective dimension. This question is also concerned by Becerril and co-authors in [2, remarks on Example 4.1.6(2)]. The goal of this paper is to obtain a negative answer to Yang and Zhang's question by investigating Ding projective dimension of Gorenstein flat modules. The paper is organized as follows.

In Section 2, we investigate the relationship of modules with finite Ding projective dimension and finite Gorenstein flat dimension. For any left GF-closed ring R, it is proved that $\overline{\mathcal{DP}}(R) = \overline{\mathcal{GF}}(R)$ if and only if $\mathcal{DP}(R) \subseteq \mathcal{GF}(R)$ and $\mathcal{GF}(R) \subseteq \overline{\mathcal{DP}}(R)$. If R is a right coherent ring, then we show that $\overline{\mathcal{DP}}(R) = \overline{\mathcal{GF}}(R)$ if and only if $\mathcal{GF}(R) \subseteq \overline{\mathcal{DP}}(R)$, note that this result has parallel to the fact of that $\overline{\mathcal{P}}(R) = \overline{\mathcal{F}}(R)$ if and only if $\mathcal{F}(R) \subseteq \overline{\mathcal{P}}(R)$ and extends the corresponding result in [17].

Section 3 is devoted to giving some further applications and examples. Some new characterization of the finiteness of Ding projective dimension of Ding-Chen rings are established and some examples are given to obtain a negative answer to Yang and Zhang's question. Also, a partial answer to Bennis' question [4, Question B] is given and the (pre)covering property of the class $\mathcal{GP}$ is studied.

Next we explain some notations which we have used above, and recall some notions which we need in the later sections.

Notation. Let R be a ring. We denote by $\mathcal{P}(R)$, $\mathcal{F}(R)$, $\overline{\mathcal{P}}(R)$, $\overline{\mathcal{F}}(R)$, $\mathcal{GP}(R)$, $\mathcal{GF}(R)$, $\mathcal{DP}(R)$, $\overline{\mathcal{DP}}(R)$ and $\overline{\mathcal{GF}}(R)$ the classes of all projective and flat R-modules, the classes of all R-modules with finite projective dimension and finite flat dimension, the classes of all Gorenstein projective, Gorenstein flat and Ding projective R-modules and the classes of all modules with finite Ding projective dimension and finite Gorenstein flat dimension, respectively.

Let R be a ring and M an R-module. As usual, we denote $\mathrm{pd}_R(M)$, $\mathrm{fd}_R(M)$ by projective and flat dimension of M respectively. M is said to be *Gorenstein projective* [13] (resp. *Ding projective* [10,18]) if there exists a $\mathrm{Hom}_R(-, \mathcal{P}(R))$-exact (resp. $\mathrm{Hom}_R(-, \mathcal{F}(R))$-exact) exact sequence of projective R-modules $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ such that

$$M \cong \mathrm{Im}(P_0 \to P^0).$$

The *Gorenstein projective dimension* (resp. *Ding projective dimension*) of an R-module N is denoted by $\mathrm{Gpd}_R(N)$ (resp. $\mathrm{Dpd}_R(N)$). M is said to be *Gorenstein flat* [15] if there exists an exact sequence of flat R-modules $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ such that

$$M \cong \mathrm{Im}(F_0 \to F^0)$$

and that remains exact whenever the functor $I \otimes_R -$ is applied for any injective right R-module I. The *Gorenstein flat dimension* of an R-module N is denoted by $\mathrm{Gfd}_R(N)$. A ring R is said to be *left GF-closed* [3] if the class $\mathcal{GF}(R)$ is closed under extensions. It follows from [3, Example 3.6] that left GF-closed rings includes strictly the one of right coherent rings and the one of rings of finite weak global dimension.

2. Modules with finite Ding projective dimension versus modules with finite Gorenstein flat dimension

In this section we will investigate the relationship between modules with finite Ding projective dimension and finite Gorenstein flat dimension over left GF-closed rings and right coherent rings respectively.

2.1. When $\overline{\mathcal{DP}} = \overline{\mathcal{GF}}$ over left GF-closed rings

The main result in this subsection is Proposition 2.3, which is obtained immediately from the following two lemmas.

Lemma 2.1. *Let R be a ring. Consider the following three conditions*:
(1) $\overline{\mathcal{DP}}(R) \subseteq \overline{\mathcal{GF}}(R)$.
(2) $\mathcal{DP}(R) \subseteq \overline{\mathcal{GF}}(R)$.
(3) $\mathcal{DP}(R) \subseteq \mathcal{GF}(R)$.
Then (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1). *If R is left GF-closed, then also* (2) $\Rightarrow$ (3) *and hence all three conditions are equivalent.*

Proof. Note that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2) are trivial.

(2) $\Rightarrow$ (1). Let $M \in \overline{\mathcal{DP}}(R)$ with $\mathrm{Dpd}_R(M) = n < \infty$. We will prove $M \in \overline{\mathcal{GF}}(R)$ by induction on n. If $n = 0$, then it follows from (2). Let $n > 0$. Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective. Then $\mathrm{Dpd}_R(K) = n - 1$ by [20, Proposition 2.2] and hence $K \in \overline{\mathcal{GF}}(R)$ by induction. Whence there exists a nonnegative integer m and an exact sequence

$$0 \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow K \longrightarrow 0$$

with all $F_i \in \mathcal{GF}(R)$, which yields an exact sequence

$$0 \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow P \longrightarrow M \longrightarrow 0$$

with all $F_i \in \mathcal{GF}(R)$ and $P \in \mathcal{P}(R)$. Thus $M \in \overline{\mathcal{GF}}(R)$.

(2) $\Rightarrow$ (3). Let D be a Ding projective R-module. By (2), there exists a nonnegative integer n such that $\mathrm{Gfd}_R(D) = n < \infty$. If $n = 0$, then there is nothing to prove. Let $n > 0$. Consider the exact sequence

$$(\dagger) \qquad 0 \longrightarrow G \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow D \longrightarrow 0$$

with all P_i projective. Then G is Gorenstein flat by [3, Theorem 2.8] since R is left GF-closed. We decompose the above exact sequence ($\dagger$) into short exact sequences $0 \to K_{j+1} \to P_j \to K_j \to 0$ for $j = 0, 1, \ldots, n-1$, where $K_0 = D$ and $K_n = G$. Then each $K_j \in \mathcal{DP}(R)$ by [20, Theorem 2.1]. Let us first consider the short exact sequence $0 \to G \to P_{n-1} \to K_{n-1} \to 0$. Since G is Gorenstein flat, there is a short exact sequence $0 \to G \to F \to C \to 0$ with

F flat and C Gorenstein flat by the definition. Now we consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & G & \longrightarrow & P_{n-1} & \longrightarrow & K_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & K_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & C & = & C & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By the middle column of the diagram we see that Q is Gorenstein flat since both P_{n-1} and C are so. On the other hand, $\mathrm{Ext}^1_R(K_{n-1}, F) = 0$ by [20, Proposition 2.1], it follows that the short exact sequence $0 \to F \to Q \to K_{n-1} \to 0$ splits and hence K_{n-1} is Gorenstein flat by [3, Corollary 2.6]. If continue the above procedure to the remaining short exact sequences, $0 \to K_{j+1} \to P_j \to K_j \to 0$ for $j = n-2, n-3, \ldots, 0$, then one can obtain that D is Gorenstein flat. □

Lemma 2.2. *Let R be a ring. Consider the following conditions*:

(1) $\overline{\mathcal{GF}}(R) \subseteq \overline{\mathcal{DP}}(R)$.

(2) $\mathcal{GF}(R) \subseteq \overline{\mathcal{DP}}(R)$.

Then (1) $\Rightarrow$ (2). *If R is left GF-closed, then also* (2) $\Rightarrow$ (1) *and hence* (1) *and* (2) *are equivalent.*

Proof. Note that (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Let $M \in \overline{\mathcal{GF}}(R)$ with $\mathrm{Gfd}_R(M) = n < \infty$. We will show $M \in \overline{\mathcal{DP}}(R)$ by induction on n. If $n = 0$, then it follows from (2). Let $n > 0$. Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective. Then $\mathrm{Gfd}_R(K) = n-1$ by [3, part 2 of Theorem 2.11] since R is left GF-closed, which implies that $K \in \overline{\mathcal{DP}}(R)$ by induction. Thus M is in $\overline{\mathcal{DP}}(R)$ by [20, Proposition 2.3] since K and P are so. □

By the above lemmas, we have the following result.

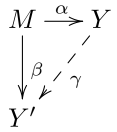
Proposition 2.3. *Let R be a left GF-closed ring. Then the following are equivalent*:

(1) $\overline{\mathcal{DP}}(R) = \overline{\mathcal{GF}}(R)$.

(2) $\mathcal{DP}(R) \subseteq \mathcal{GF}(R)$ *and* $\mathcal{GF}(R) \subseteq \overline{\mathcal{DP}}(R)$.

2.2. When $\overline{\mathcal{DP}} = \overline{\mathcal{GF}}$ over right coherent rings

Given an R-module M and a class $\mathcal{Y}$ of R-modules, recall that a *$\mathcal{Y}$-preenvelope* of M is a homomorphism $\alpha : M \to Y$ with $Y \in \mathcal{Y}$ such that for any homomorphism $\beta : M \to Y'$ with $Y' \in \mathcal{Y}$, there is a homomorphism $\gamma : Y \to Y'$ with the following commutative diagram (more details see Section 3.3):



A $\mathcal{Y}$-preenvelope is called a *$\mathcal{Y}$-envelope* if γ must be an isomorphism whenever $\beta = \alpha$ in the above diagram.

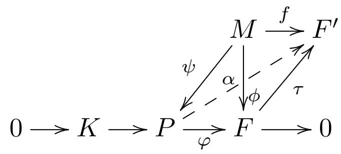
In this subsection, we will characterize when every Gorenstein flat module has finite Ding projective dimension over right coherent rings. We start it with the following lemma, which gives a method to construct a *right projective resolution* (in the sense of [19]) for some modules.

Lemma 2.4. *Let R be a right coherent ring and M an R-module with $Gfd_R(M) < \infty$ such that $Ext^i_R(M, F) = 0$ for all $F \in \mathcal{F}(R)$ and all $i > 0$. Then M admits a flat preenvelope $\psi : M \longrightarrow P$ in which P is projective. Moreover, ψ is monic and also an $\overline{\mathcal{F}}(R)$-preenvelope.*

Proof. Since R is right coherent, M has a flat preenvelope: $\alpha : M \longrightarrow F$. Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow P \xrightarrow{\varphi} F \longrightarrow 0$$

with P projective. Let us explain how M admits a flat preenvelope $\psi : M \longrightarrow P$ in which P is projective by the following commutative diagram (each triangle is commutative):



The existence of ψ is followed from the long exact sequence $\mathrm{Hom}_R(M, P) \to \mathrm{Hom}_R(M, F) \to \mathrm{Ext}^1_R(M, K) = 0$ since K is flat. In addition, the left-handed triangle commutes, i.e., $\alpha = \varphi\psi$. For any homomorphism $f : M \to F'$ with F' flat, note that α is a flat preenvelope of M, there is a homomorphism $\tau : F \to F'$ such that $f = \tau\alpha$ and so the right-handed triangle is commutative. Set $\phi = \tau\varphi$. It follows that $f = \tau\alpha = \tau\varphi\psi = \phi\psi$ and so the top triangle commutes as well, this shows that $\psi : M \to P$ is a flat preenvelope of M.

Before proving that ψ is monic, we show that $\mathrm{Ext}^i_R(M, L) = 0$ for all $L \in \overline{\mathcal{F}}(R)$ and all $i > 0$ by induction on $\mathrm{fd}_R(L) = m < \infty$. If $m = 0$, then it is nothing to prove. Let $m > 0$. Consider the short exact sequence

$$0 \longrightarrow L' \longrightarrow Q \longrightarrow L \longrightarrow 0$$

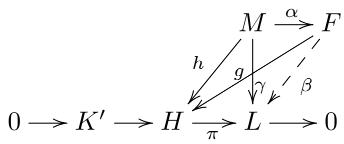
with Q projective. Then $\mathrm{fd}_R(L') = m - 1$ and so $\mathrm{Ext}^i_R(M, L') = 0$ for all $i > 0$ by induction. Also, $\mathrm{Ext}^i_R(M, Q) = 0$ for all $i > 0$ since Q is flat. Hence, we get that $\mathrm{Ext}^i_R(M, L) = 0$ for all $i > 0$ by the long exact sequence $0 = \mathrm{Ext}^i_R(M, Q) \to \mathrm{Ext}^i_R(M, L) \to \mathrm{Ext}^{i+1}_R(M, L') = 0$.

Having shown that $\mathrm{Ext}^i_R(M, L) = 0$ for all $L \in \overline{\mathcal{F}}(R)$ and all $i > 0$, we see that ψ is monic. It suffices to show that α is so since $\varphi\psi = \alpha$. In view of that there is a short exact sequence

$$0 \longrightarrow M \stackrel{\iota}{\longrightarrow} N \longrightarrow C \longrightarrow 0$$

with $\mathrm{fd}_R(N) = \mathrm{Gfd}_R(M) < \infty$ by [8, Lemma 2.19], it is sufficient to show that α is an $\overline{\mathcal{F}}(R)$-preenvelope. Indeed, this will obtain a homomorphism $\theta : F \longrightarrow N$ such that $\iota = \theta\alpha$, which yields that α is monic since ι is so. On the other hand, by the proof in the first paragraph, we can see that ψ is also an $\overline{\mathcal{F}}(R)$-preenvelope. Thus, the proof will be finished.

Now, we need only to explain how α is an $\overline{\mathcal{F}}(R)$-preenvelope of M by the following commutative diagram (each triangle is commutative):



For any homomorphism $g : M \to L$ with $L \in \overline{\mathcal{F}}(R)$, consider the short exact sequence

$$0 \longrightarrow K' \longrightarrow H \longrightarrow L \longrightarrow 0$$

with H flat. Then $K' \in \overline{\mathcal{F}}(R)$ since $L \in \overline{\mathcal{F}}(R)$. Hence $\mathrm{Ext}^1_R(M, K') = 0$ by what we have proved in the second paragraph, which yields an exact sequence $\mathrm{Hom}_R(M, H) \to \mathrm{Hom}_R(M, L) \to 0$. It follows that the homomorphism h exists and that the left-handed triangle commutes, i.e., $\pi h = g$. Also, the homomorphism $\gamma : M \to L$ exists since α is a flat preenvelope of M, in addition, the bottom triangle commutes, i.e., $h = \gamma\alpha$. Set $\beta = \pi\gamma$. Then $g = \pi h = \pi\gamma\alpha = \beta\alpha$ and so the right-handed triangle commutes as well, we conclude that α is an $\overline{\mathcal{F}}(R)$-preenvelope of M, as required. □

Proposition 2.5. *Let R be a right coherent ring, n a nonnegative integer and M an R-module. Then the following are equivalent*:

(1) *$\mathrm{Dpd}_R(M) \leqslant n$.*
(2) *$\mathrm{Gfd}_R(M) \leqslant n$ and $\mathrm{Ext}^i_R(M, F) = 0$ for all $F \in \mathcal{F}(R)$ and all $i > n$.*
(3) *$\mathrm{Gfd}_R(M) \leqslant n$ and $\mathrm{Ext}^i_R(M, L) = 0$ for all $L \in \overline{\mathcal{F}}(R)$ all $i > n$.*
(4) *$\mathrm{Gfd}_R(M) < \infty$ and $\mathrm{Ext}^i_R(M, F) = 0$ for all $F \in \mathcal{F}(R)$ and all $i > n$.*
(5) *$\mathrm{Gfd}_R(M) < \infty$ and $\mathrm{Ext}^i_R(M, L) = 0$ for all $L \in \overline{\mathcal{F}}(R)$ all $i > n$.*

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) $\Rightarrow$ (4) are trivial, (2) $\Rightarrow$ (3) holds by the second paragraph of the proof in Lemma 2.4. Next we show that (4) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) and so we are done.

(1) $\Rightarrow$ (2). Note that every Ding projective R-module is Gorenstein flat by [10, Proposition 2.3] since R is right coherent. Then the result follows easily from [20, Theorem 2.4] and [19, Theorem 3.14].

(4) $\Rightarrow$ (1). Suppose that (4) holds. Consider the exact sequence

$$(\ddagger) \qquad 0 \longrightarrow D \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all P_i projective. It suffices to show that D is Ding projective by [20, Theorem 2.4]. By [20, Proposition 2.1], we need only to construct a $\mathrm{Hom}_R(-, \mathcal{F}(R))$-exact exact sequence

$$0 \longrightarrow D \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with all P^i projective, and to show that $\mathrm{Ext}_R^i(D, F) = 0$ for all $F \in \mathcal{F}(R)$ and all $i > 0$. Let F be any flat R-module. Since $\mathrm{Ext}_R^i(M, F) = 0$ for all $i > n$, it follows that $\mathrm{Ext}_R^i(D, F) \cong \mathrm{Ext}_R^{i+n}(M, F) = 0$ for all $i > 0$ by dimension shifting.

Now, we construct the $\mathrm{Hom}_R(-, \mathcal{F}(R))$-exact exact sequence

$$0 \longrightarrow D \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with all P^i projective and this will finish the proof. Applying [19, Theorem 3.14] into the exact sequence ($\ddagger$) in which $\mathrm{Gfd}_R(M) < \infty$ we have $\mathrm{Gfd}_R(D) < \infty$. Hence D has a monic flat preenvelope $\alpha^0 : D \longrightarrow P^0$ with P^0 projective by Lemma 2.4. Suppose that F is any flat R-module. Consider the short exact sequence

$$0 \longrightarrow D \longrightarrow P^0 \longrightarrow C \longrightarrow 0,$$

where $C = \mathrm{Coker}\alpha^0$. We claim that $\mathrm{Hom}_R(-, F)$ leaves the exactness of the short exact sequence $0 \to D \to P^0 \to C \to 0$, and that C possesses the same properties as D, that is, $\mathrm{Gfd}_R(C) < \infty$ and $\mathrm{Ext}_R^i(C, F) = 0$ for all $i > 0$. The first assertion is followed since $\alpha^0 : D \to P^0$ is a flat preenvelope. The second assertion will be proved by three steps. First, if $i > 1$, then $\mathrm{Ext}_R^i(C, F) = \mathrm{Ext}_R^{i-1}(D, F) = 0$ by dimension shifting. Secondly, consider the case of that $i = 1$. By the first assertion, we have an exact sequence $\mathrm{Hom}_R(C, F) \to \mathrm{Hom}_R(P^0, F) \to \mathrm{Hom}_R(D, F) \to 0$, which implies that $\mathrm{Ext}_R^1(C, F) = 0$ by 5-lemma. Finally, [19, Proposition 3.12] yields that $\mathrm{Gfd}_R(C) = \mathrm{Gfd}_R(D) + 1 < \infty$ since we have proved that $\mathrm{Gfd}_R(D) < \infty$. Thus, the claim holds, as desired. Therefore, if continuing the above process, we can establish the required $\mathrm{Hom}_R(-, \mathcal{F}(R))$-exact exact sequence

$$0 \longrightarrow D \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with all P^i projective. □

Remark 2.6. (1) Let $n = 0$. Then the equivalence from (1) to (5) in Proposition 2.5 gives some new characterizations of Ding projective modules over right coherent rings.

(2) It is easy to verify that $\mathcal{DP}(R) = \mathcal{GP}(R)$ whenever $\overline{\mathcal{P}}(R) = \overline{\mathcal{F}}(R)$ by [19, Proposition 2.3] and the counterpart for Ding projective modules. Let R be a commutative Noetherian ring with finite Krull dimension. Then $\overline{\mathcal{P}}(R) = \overline{\mathcal{F}}(R)$ and hence $\mathcal{DP}(R) = \mathcal{GP}(R)$. Thus, [17, Theorem 3.2] is an immediate consequence of Proposition 2.5.

We set $\mathrm{l.FPD}(R) = \sup\{\mathrm{pd}_R(F) \mid F \text{ is any flat } R\text{-module}\}$. Similarly, we set $\mathrm{r.FPD}(R) = \sup\{\mathrm{pd}_R(F) \mid F \text{ is any flat right } R\text{-module}\}$. A ring R is called to be *left perfect* (resp., *right perfect*) if $\mathrm{l.FPD}(R) = 0$ (resp., $\mathrm{r.FPD}(R) = 0$) and R is said to be *perfect* if it is left and right perfect. Recall that an R-module C is *cotorsion* if $\mathrm{Ext}^1_R(F, C) = 0$ for each flat R-module F. We are now in a position to give the main result of the paper which has many applications.

Theorem 2.7. *Let R be a right coherent ring and n a nonnegative integer. Then the following are equivalent*:
(1) *$\mathrm{Dpd}_R(M) \leqslant n$ for all Gorenstein flat modules M.*
(2) *$\mathrm{Dpd}_R(M) \leqslant n$ for all flat modules M.*
(3) *$\mathrm{l.FPD}(R) \leqslant n$.*

Proof. Note that (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). Let F be a flat R-module. Then by (2), $\mathrm{Dpd}_R(F) \leqslant n$. But then [20, Proposition 2.5] gives that $\mathrm{pd}_R(F) = \mathrm{Dpd}_R(F) \leqslant n$. Thus, $\mathrm{l.FPD}(R) \leqslant n$.

(3) $\Rightarrow$ (1). Let $M \in \mathcal{GF}(R)$ and $F \in \mathcal{F}(R)$. If R is a left perfect ring, then F is cotorsion by [23, Proposition 3.3.1]. Hence [19, Proposition 3.22] implies that $\mathrm{Ext}^i_R(M, F) = 0$ for all $i > 0$. So M is Ding projective by Proposition 2.5. Suppose that $\mathrm{l.FPD}(R) \leqslant n$ with $n > 0$. Consider the minimal pure injective resolution of F (see [23, p. 39 to p. 92]):

$$0 \longrightarrow F \longrightarrow PE^0(F) \longrightarrow PE^1(F) \longrightarrow \cdots \longrightarrow PE^n(F) \longrightarrow \cdots .$$

Note that each $PE^i(F)$ is both flat and cotorsion by [23, Lemma 3.16] and [14, Lemma 5.3.23]. Also $PE^i(F) = 0$ for all $i > n$ by [23, Remark 3.4.9]. Thus $\mathrm{Ext}^i_R(M, F) = \mathrm{Ext}^{i-n}_R(\mathrm{Coker}(\mathrm{PE}^n(F) \to \mathrm{PE}^{n+1}(F))) = 0$ for all $i > n$ by dimension shifting. Therefore, $\mathrm{Dpd}_R(M) \leqslant n$ by Proposition 2.5. □

Remark 2.8. Note that the equivalence between (2) and (3) holds for any ring R. Indeed, suppose that $\mathrm{Dpd}_R(M) \leqslant n$ for all flat modules F. Then it follows from [20, Proposition 2.5] that $\mathrm{pd}_R(F) = \mathrm{Dpd}_R(F) \leqslant n$. That is, $\mathrm{l.FPD}(R) \leqslant n$. Conversely, let $\mathrm{l.FPD}(R) \leqslant n$. Then for any flat R-module F, $\mathrm{Dpd}_R(F) \leqslant \mathrm{pd}_R(F) \leqslant n$.

Corollary 2.9. *Let R be a right coherent. Then the following are equivalent*:
(1) *$\overline{\mathcal{DP}}(R) = \overline{\mathcal{GF}}(R)$.*

(2) l.FPD(R) $< \infty$, i.e., every flat R-module has finite projective dimension. If the above equivalent conditions are satisfied, then the following are equivalent for any R-module M:

(i) $Gpd_R(M) < \infty$.

(ii) $Gfd_R(M) < \infty$.

In particular, if R is a right coherent ring with finite left finitistic projective dimension d, that is, $\sup\{pd_R(M) \mid M$ *has finite projective dimension*$\} = d < \infty$*, then for any R-module M,* $Gpd_R(M) < \infty$ *if and only if* $Gfd_R(M) < \infty$*, and in this case,* $Gpd_R(M) \leqslant d$.

Proof. (1) $\Leftrightarrow$ (2) follows from Proposition 2.3 and Theorem 2.7, and so (i) $\Leftrightarrow$ (ii) holds since $\mathcal{DP}(R) = \mathcal{GP}(R)$ by Remark 2.6(2). Note that $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$ whenever $\mathrm{pd}_R(M) < \infty$ by [19, Proposition 2.27], and that l.FPD(R) $\leqslant d$ whenever $\sup\{pd_R(M) \mid M$ has finite projective dimension$\}$ is at most d by the proof of [19, Proposition 3.4]. Thus the last assertion follows. □

Remark 2.10. It is well-known that every commutative Noetherian ring with finite Krull dimension admits finite (left) finitistic projective dimension, so [17, Theorem 3.4] is an immediate consequence of Corollary 2.9.

3. Applications

This section is divided into three subsections, by which applications of Theorem 2.7 are given.

3.1. Ding projective dimension of Ding-Chen rings

In this subsection, we investigate Ding projective dimension of Ding-Chen rings. *Ding-Chen ring* was initially called *n-FC ring* by Ding and Chen in [9] and renamed by Gillespie in [18], which is defined as two-sided coherent rings with finite self-FP-injective dimension at most n on both sides for some nonnegative integer n. We set l.gl.DP.dim$(R) = \sup\{Dpd_R(M) \mid M$ is any R-module$\}$.

As we all known, Gorenstein injective and Ding injective modules are dualities of Gorenstein projective and Ding projective modules respectively. We denote by $\mathrm{Gid}_R(M)$ and $\mathrm{Did}_R(M)$ the Gorenstein injective and Ding injective dimensions of an R-module M respectively; and we set l.gl.DI.dim$(R) = \sup\{Did_R(M) \mid M$ is any R-module$\}$. According to [6, Theorem 1.1], Bennis and Mahdou showed that the equality $\sup\{Gpd_R(M) \mid M$ is any R-module$\} = \sup\{Gid_R(M) \mid M$ is any R-module$\}$ holds for any ring R, and called the common value the *left Gorenstein global dimension* of R which is denoted by l.G-gl.dim (R) (note that the result was first proved in commutative setting in [5] and in some particular case in [12]). Also, it follows from [20, Theorem 3.2] that l.gl.DP.dim(R) = l.G-gl.dim (R) for any ring R. On the other hand, it was shown in [24, Theorem 3.6] that l.gl.DP.dim(R) = l.gl.DI.dim(R) whenever R is a Ding-Chen ring. Thus for any Ding-Chen ring R, we get that l.gl.DP.dim(R) = l.gl.DI.dim(R) = l.G-gl.dim (R), but we don't know whether or not the equality of these dimensions of rings holds for any ring. Similarly, the *left weak Gorenstein global dimension* of a ring R is denoted by l.G-wgl.dim (R) and defined as l.G-wgl.dim $(R) = \sup\{Gfd_R(M) \mid M$ is any R-module$\}$.

Let R be a Ding-Chen ring. It was proved in [10, Corollary 3.5] that l.gl.DP.dim$(R) < \infty$ if and only if l.FID$(R) < \infty$, that is, l.gl.DP.dim$(R) < \infty$ if and only if $\sup\{id_R(F) \mid F$ is any flat R-module$\} < \infty$. Next we give some new characterization of the finiteness of l.gl.DP.dim(R).

Theorem 3.1. (1) *Let R be a right coherent ring. Then the following are equivalent*:

(i) *l.gl.DP.dim*$(R) < \infty$.

(ii) *l.G-wgl.dim*$(R) < \infty$ *and l.FPD*$(R) < \infty$.

(2) *If R is an n-FC ring with l.FPD$(R) = m$ for some integers $m, n \geqslant 0$, then we get that l.gl.DP.dim $(R) \leqslant n + m$.*

(3) *If R is a Ding-Chen ring, then the following are equivalent*:

(iii) *l.gl.DP.dim*$(R) < \infty$.

(iv) $Dpd_R(M) < \infty$ *for each Gorenstein flat module M.*

(v) $Dpd_R(M) < \infty$ *for each flat module F.*

(vi) *l.FPD*$(R) < \infty$.

Proof. (i) ⇒ (ii). The first assertion follows from [11, Remark 5.4(2)] and [20, Theorem 3.2]. In order to see the second assertion, let l.gl.DP.dim$(R) < \infty$ and F be a flat R-module. It follows from [20, Proposition 2.5] that $\mathrm{pd}_R(F) = \mathrm{Dpd}_R(F) < \infty$, i.e., l.FPD$(R) < \infty$.

(ii) ⇒ (i). Suppose that R admits l.FPD$(R) = m < \infty$ and admits l.G-wgl.dim$(R) = n < \infty$ for some integers $m, n \geqslant 0$. We claim that $\mathrm{Dpd}_R(M) \leqslant \mathrm{Gfd}_R(M) + m$ for all R-modules M and this will finish the proof since $\mathrm{Gfd}_R(M) \leqslant n$ is guaranteed by l.G-wgl.dim$(R) = n < \infty$. We will prove the claim by induction on $\mathrm{Gfd}_R(M)$. If M is Gorenstein flat, then we are done by Theorem 2.7. Let $\mathrm{Gfd}_R(M) > 0$. Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective. Then $\mathrm{Gfd}_R(K) = \mathrm{Gfd}_R(M) - 1$ by [19, Proposition 3.12], which implies that $\mathrm{Dpd}_R(K) \leqslant \mathrm{Gfd}_R(K) + m$ by induction. It follows that $\mathrm{Dpd}_R(M) = \mathrm{Dpd}_R(K) + 1 \leqslant \mathrm{Gfd}_R(K) + 1 + m = \mathrm{Gfd}_R(M) + m$ by [19, Proposition 3.12] and [20, Proposition 2.2]. Thus the claim holds, as desired.

(2) is followed by the proof of (ii) ⇒ (i) and since l.G-wgl.dim$(R) = n$ for any n-FC ring R by [16, Theorem 2.3.3]. Finally, we see (3).

(iii) ⇒ (iv) is trivial and (iv) ⇔ (v) ⇔ (vi) comes from Theorem 2.7.

(vi) ⇒ (iii). Since R is Ding-Chen, it follows from [16, Theorem 2.3.3] that l.G-wgl.dim$(R) = n < \infty$ for some nonnegative n. Let l.FPD$(R) = m < \infty$. Then we are done by (2). □

Recall that a ring R is Gorenstein if R is an *n-Gorenstein ring* for some nonnegative integer n, i.e., R is a two-sided Noetherian ring with self-injective dimension at most n on both sides for some integer n.

Remark 3.2. Let R be any ring. Then we get that $\mathrm{l.FID}(R) < \infty$ provided that $\mathrm{l.gl.DP.dim}\ (R) < \infty$ by [10, Proposition 3.2]. We also get that $\mathrm{l.FPD}(R) < \infty$ provided that $\mathrm{l.gl.DP.dim}\ (R) < \infty$ by the proof of (1) $\Rightarrow$ (2) in Theorem 3.1. It is known that any commutative coherent ring R with $\mathrm{l.FID}(R) < \infty$ is a Ding-Chen ring. Hence for any commutative coherent ring R, the condition $\mathrm{l.FID}(R) < \infty$ can guarantee that $\mathrm{l.gl.DP.dim}(R) < \infty$ by [10, Corollary 3.5]. However, a commutative coherent ring R admitting $\mathrm{l.FPD}(R) < \infty$ can not assure that $\mathrm{l.gl.DP.dim}(R) < \infty$ in general. For example, let R be a commutative Noetherian ring with finite Krull dimension which is not Gorenstein. It is clear that R is a commutative coherent ring with $\mathrm{l.FPD}(R) < \infty$. But note that $\mathrm{l.gl.DP.dim}(R) < \infty$ if and only if $\mathrm{l.FID}(R) < \infty$ by what we have noted, equivalently, R is Gorenstein by [14, Theorem 9.1]. Thus R satisfies $\mathrm{l.gl.DP.dim}(R) = \infty$.

It was shown in [10, Theorem 3.6] that all finitely presented left and right modules over a Ding-Chen ring admit finite Ding projective dimension. The following example shows that there exists a commutative Ding-Chen ring R with $\mathrm{gl.DP.dim}(R) = \infty$, i.e., neither all left R-modules nor all right R modules admit finite Ding projective dimension, which gives a negative answer to Yang and Zhang's question as mentioned in the introduction.

Example 3.3. Let $R = F_\alpha$ be the free Boolean ring on $\aleph_\alpha$ generators with α an infinite cardinality (more details see [22]). Then by [22, Corollary 5.2], R admits infinite global dimension. Hence, R is a Ding-Chen ring admits neither $\mathrm{l.FPD}(R) < \infty$ nor $\mathrm{r.FPD}(R) < \infty$ since every Boolean ring is commutative and von Neumann regular and hence is automatically commutative Ding-Chen, and satisfies that all left and right R-modules are flat. Thus, by Theorem 3.1, R satisfies $\mathrm{l.gl.DP.dim}(R) = \mathrm{r.gl.DP.dim}(R) = \infty$, which, in addition, shows that R is a Ding-Chen ring that admits infinite left and right Gorenstein global dimension by [20, Theorem 3.2].

It is known that every Gorenstein ring is always a Ding-Chen ring and there exist Ding-Chen rings which are not Gorenstein. The following example shows that there exists a Ding-Chen ring R with $\mathrm{l.gl.DP.dim}(R) < \infty$ and with $\mathrm{r.gl.DP.dim}(R) < \infty$ that is not Gorenstein.

Example 3.4. As we all known, there exists a left and right hereditary ring which is neither left nor right Noetherian. For instance, let

$$R = \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{pmatrix}.$$

Then R is a (left and right) hereditary and (left and right) perfect ring which is neither left nor right Noetherian by [1, Exercise 10 (3), p. 215 and Example

28.12] and the fact of that a left (resp. right) Noetherian ring R is left (resp. right) artinian if and only if R is left (resp. right) perfect (see [1, Theorem 15.20, Theorem 28.4 and Corollary 28.8]).

Since every hereditary ring S is always a 1-FC ring with $\mathrm{l.gl.DP.dim}(S) = \mathrm{r.gl.DP.dim}(S) \leqslant 1$, the above example shows that R is a Ding-Chen ring with $\mathrm{l.gl.DP.dim}(R) = \mathrm{r.gl.DP.dim}\ (R) < \infty$ which is not Gorenstein. In addition, it follows that R is a 1-FC and perfect ring that is not 1-Gorenstein, although it is known that 0-FC and perfect rings are 0-Gorenstein rings (recall that 0-FC rings are just FC rings and 0-Gorenstein rings just QF rings).

3.2. Bennis' question

Bennis [4, Question B] asked when Gorenstein flat module is Gorenstein projective? Bennis' question is the converse of Holm's question (see [10, Remark 4.5(4)], also see [4, Question A]). These questions are still open and subtle. Up to now, it is known that Holm's question holds true provided that the ring R is a right coherent ring with $\mathrm{l.FPD}(R) < \infty$ (see the proof of [19, Proposition 3.4]) or R satisfies that all injective right R-modules admit finite flat dimension (see [4, Proposition 3.2]). It was shown in [4, Theorem 3.3] that, for any (in)finitely presented module M over any ring, M is Gorenstein projective if and only if it is Gorenstein flat, note that this result give a partial answer to both Holm's question and Bennis' question.

In this subsection, we will give another partial answer to Bennis' question, as an application of Theorem 2.7.

Proposition 3.5. *Let R be a right coherent ring. Then the following are equivalent*:

(1) $\mathcal{GF}(R) \subseteq \mathcal{DP}(R)$.
(2) $\mathcal{GF}(R) = \mathcal{DP}(R)$.
(3) *R is left perfect.*

If one of the above equivalent conditions is satisfied, then $\mathcal{GF}(R) \subseteq \mathcal{GP}(R)$ (more precisely, $\mathcal{GF}(R) = \mathcal{GP}(R)$).

Proof. Note that (2) ⇒ (1) is trivial, (1) ⇒ (2) holds by [10, Proposition 2.3] and (2) ⇔ (3) follows easily from Theorem 2.7. □

3.3. Gorenstein projective (pre)cover

Let us recall some notions. Let R be a ring and $\mathcal{X}, \mathcal{Y}$ be classes of R-modules. A *cotorsion pair* (*cotorsion theory*) is a pair $(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{X}^{\perp} = \mathcal{Y}$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$. Here $\mathcal{X}^{\perp} = \{A \in \mathcal{A} \,|\, \mathrm{Ext}^1_{\mathcal{A}}(X, A) = 0, \forall X \in \mathcal{X}\}$, and similarly we can define ${}^{\perp}\mathcal{Y}$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be *complete* if it has enough projectives and injectives, i.e., for any R-module A, there are exact sequences $0 \to Y \to X \to A \to 0$ and $0 \to A \to Y' \to X' \to 0$ with $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$. Given an R-module M, an $\mathcal{X}$-*precover* of M is defined as a homomorphism $\alpha : X \longrightarrow M$ with $X \in \mathcal{X}$ such that the sequence, $\mathrm{Hom}_R(X', X) \longrightarrow \mathrm{Hom}_R(X', M) \longrightarrow 0$ is exact for all $X' \in \mathcal{X}$. An $\mathcal{X}$-precover

$\alpha : X \longrightarrow M$ is said to be an $\mathcal{X}$-*cover* if every endomorphism $f : X \longrightarrow X$ satisfying that $\alpha f = \alpha$ is an isomorphism. An $\mathcal{X}$-precover $\alpha : X \to A$ of A is called *special* if α is an epimorphism and $\mathrm{Ker}\alpha \in \mathcal{X}^{\perp}$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be *perfect* if every module has an $\mathcal{X}$-cover and a $\mathcal{Y}$-envelope. A class $\mathcal{X}$ is called to be (*pre*)*covering* (resp., *special precovering*, (*pre*)*enveloping*) if every object has an $\mathcal{X}$-(pre)cover (resp. a special $\mathcal{X}$-precover, an $\mathcal{X}$-(pre)envelope, a special $\mathcal{X}$-preenvelope).

It was shown in [7, Theorem 8.5 and Remarks in p. 30] that the class $\mathcal{GP}(R)$ is special precovering provided that R is a right coherent ring with $\mathrm{l.FPD}(R) < \infty$. The proof of this result involves the fact that the class $\mathcal{GP}(R)^{\perp}$ is *thick* (in the sense of [7, p. 4]. In this subsection, we give another proof of the above result.

Proposition 3.6. *Let R be a right coherent ring with l.FPD$(R) < \infty$. Then the class $\mathcal{GP}(R)$ is special precovering. In particular, the pair $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ is a complete cotorsion pair.*

Proof. Let M be any R-module. Then by [16, Theorem 3.19], there is a short exact sequence

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0$$

with $F \in \mathcal{GF}(R)$ and $L \in \mathcal{GF}(R)^{\perp}$. Since $\mathcal{GP}(R) \subseteq \mathcal{GF}(R)$ by the proof of [19, Proposition 3.4], it follows that $L \in \mathcal{GP}(R)^{\perp}$. Note that $\mathrm{Gpd}_R(F) < \infty$ by Corollary 2.9, we have a short exact sequence

$$0 \longrightarrow K \longrightarrow G \longrightarrow F \longrightarrow 0$$

with $G \in \mathcal{GP}(R)$ and $K \in \overline{\mathcal{P}}(R) \subseteq \mathcal{GP}(R)^{\perp}$ by [19, Theorem 2.10]. Now we get the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Q & \longrightarrow & G & \xrightarrow{\alpha} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then by the left column of the diagram we have Q is in $\mathcal{GP}(R)^{\perp}$ since both K and L are so. Thus, the middle raw

$$0 \longrightarrow Q \longrightarrow G \xrightarrow{\alpha} M \longrightarrow 0$$

of the diagram gives a special Gorenstein projective precover $\alpha : G \longrightarrow M$.

By [14, Proposition 7.1.7], to see the last assertion, it suffices to prove that the pair $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ is a cotorsion pair. We need only to show that

${}^{\perp}(\mathcal{GP}(R)^{\perp}) = \mathcal{GP}(R)$. It is trivial that $\mathcal{GP}(R) \subseteq {}^{\perp}(\mathcal{GP}(R)^{\perp})$. Conversely, let $X \in {}^{\perp}(\mathcal{GP}(R)^{\perp})$. By what we have proved above, there is a short exact sequence

$$0 \longrightarrow K \longrightarrow G \longrightarrow X \longrightarrow 0$$

with $G \in \mathcal{GP}(R)$ and $K \in \mathcal{GP}(R)^{\perp}$. Hence $\mathrm{Ext}_R^1(X, K) = 0$ by the assumption. It follows that the short exact sequence $0 \to K \to G \to X \to 0$ splits and hence X is Gorenstein projective by [19, Theorem 2.5]. Therefore, ${}^{\perp}(\mathcal{GP}(R)^{\perp}) = \mathcal{GP}(R)$, as required. □

Let R is a right coherent ring with l.FPD$(R) < \infty$. It follows from Proposition 3.6 that the class $\mathcal{GP}(R)$ is special precovering. We will end the paper by the following result which discuss when the class $\mathcal{GP}(R)$ is covering.

Proposition 3.7. *Let $n \geqslant 1$ be an integer and R a right coherent ring with l.FPD$(R) = n < \infty$. Then the following are equivalent*:

(1) *The class $\mathcal{GP}(R)$ is covering.*

(2) *The class $\mathcal{GP}(R)$ is closed under direct limits.*

(3) *The pair $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ is a perfect cotorsion pair.*

(4) *Every Gorenstein flat R-module M has a Gorenstein projective cover.*

(5) *Every flat R-module M has a Gorenstein projective cover.*

(6) *R is left perfect.*

(7) *The classes $\mathcal{GP}(R)$ and $\mathcal{GF}(R)$ coincide.*

(8) *The class $\mathcal{GP}(R)$ is closed under pure submodules and pure-epimorphic images.*

Proof. (2) ⇒ (3) follows from Proposition 3.6 and [14, Theorem 7.2.6], (3) ⇒ (1) ⇒ (4) ⇒ (5) are trivial.

(5) ⇒ (6). Let F be a flat R-module and $\alpha : P \to F$ a Gorenstein projective cover of F. So α must epic since the class $\mathcal{GP}(R)$ contains all projective modules. Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow P \xrightarrow{\alpha} F \longrightarrow 0$$

with $K = \mathrm{Ker}\alpha$. Then K belongs to $\mathcal{GP}(R)^{\perp}$ by Wakamutsu lemma [23, Lemma 2.1.1] since the class $\mathcal{GP}(R)$ is closed under extensions (see [19, Theorem 2.5]), which implies that $P \in \mathcal{GP}(R)^{\perp}$ since $F \in \overline{\mathcal{P}}(R) \subseteq \mathcal{GP}(R)^{\perp}$. Note that there is a short exact sequence

$$0 \longrightarrow P \longrightarrow Q \longrightarrow G \longrightarrow 0$$

with $Q \in \mathcal{P}(R)$ and $G \in \mathcal{GP}(R)$ by the definition. We conclude that $\mathrm{Ext}_R^1(G, P) = 0$, which implies that $P \in \mathcal{P}(R)$. Thus $\alpha : P \to F$ is also a projective cover of F. Therefore, R is left perfect by the proof of (3) ⇒ (1) in [14, Theorem 5.3.2].

(6) ⇒ (7) holds by Proposition 3.5, (7) ⇒ (8) comes from [21, Lemma 5.1] and its proof.

(8) $\Rightarrow$ (2). Note that any direct limit of a family of modules is always a pure-epimorphic image of direct sums of such modules, and that the class $\mathcal{GP}(R)$ is closed under arbitrary direct sums by [19, Theorem 2.5]. Thus $\mathcal{GP}(R)$ is closed under direct limits by (8). $\square$

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Junpeng Wang
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P. R. China
E-mail address: `wangjunpeng1218@163.com`