Bull. Korean Math. Soc. **54** (2017), No. 6, pp. 1913–1925
https://doi.org/10.4134/BKMS.b160401
pISSN: 1015-8634 / eISSN: 2234-3016

RINGS WITH IDEAL-SYMMETRIC IDEALS

Juncheol Han, Yang Lee, and Sangwon Park

Abstract. Let R be a ring with identity. An ideal N of R is called *ideal-symmetric* (resp., *ideal-reversible*) if $ABC \subseteq N$ implies $ACB \subseteq N$ (resp., $AB \subseteq N$ implies $BA \subseteq N$) for any ideals A, B, C in R. A ring R is called *ideal-symmetric* if zero ideal of R is ideal-symmetric. Let $S(R)$ (called the *ideal-symmetric radical* of R) be the intersection of all ideal-symmetric ideals of R. In this paper, the following are investigated: (1) Some equivalent conditions on an ideal-symmetric ideal of a ring are obtained; (2) Ideal-symmetric property is Morita invariant; (3) For any ring R, we have $S(M_n(R)) = M_n(S(R))$ where $M_n(R)$ is the ring of all n by n matrices over R; (4) For a quasi-Baer ring R, R is semiprime if and only if R is ideal-symmetric if and only if R is ideal-reversible.

1. Introduction and basic definitions

Throughout this paper, all rings are associative with identity unless otherwise specified. Let R be a ring. Let $J(R)$ and $P(R)$ denote the Jacobson radical and the prime radical of R respectively. Denote the n by n full (resp., upper triangular) matrix ring over R by $M_n(R)$ (resp., $U_n(R)$). $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). $R[x]$ denotes the polynomial ring with an indeterminate x over R.

Lambek introduced the concept of a symmetric right ideal, unifying the sheaf representation of commutative rings and reduced rings in [10]. Lambek called a right ideal I of a ring R *symmetric* if $rst \in I$ implies $rts \in I$ for all $r, s, t \in R$. If zero ideal of R is symmetric, then R is called a *symmetric* ring; while Anderson and Camillo [1] used the term ZC_3 for this concept. It is proved by Lambek that an ideal I of a ring R is symmetric if and only if $r_1 r_2 \cdots r_n \in I$ implies $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} \in I$ for any permutation σ of the set $\{1, 2, \ldots, n\}$, where $n \geq 1$ and $r_i \in R$ for all i (see [10], Proposition 1).

As a generalizaton of symmetric rings, Kwak, at el. [3] extended the concept of symmetric rings to ideal-symmetric rings. A ring R is called *ideal-symmetric*

Received May 9, 2016; Accepted September 21, 2016.
2010 *Mathematics Subject Classification.* 16D25, 16S50.
Key words and phrases. symmetric ideal, ideal-symmetric ideal, ideal-reversible ideal, ideal-symmetric ring, ideal-reversible ring, Morita invariant, ideal-symmetric radical, Baer ring, quasi-Baer ring.
This work was supported by 2-year Research Grant of Pusan National University.

if $ABC = 0$ implies $ACB = 0$ for all ideals A, B, C of R. It is evident that symmetric rings are ideal-symmetric, but the converse need not hold by [3, Example 1.2].

In this note, we will extend the concepts of symmetric ideals of a ring to ideal-symmetric ideals. We will call an ideal N of a ring R *ideal-symmetric* if $ABC \subseteq N$ implies $ACB \subseteq N$ for any ideals A, B, C in R. Note that if the zero ideal of a ring R is ideal-symmetric, then R is ideal-symmetric ([3]).

It is obvious that every prime ideal of a ring R is ideal-symmetric. Moreover, observe that any semiprime ideal of a ring R is also ideal-symmetric. Indeed, let N be a semiprime ideal of R such that $ABC \subseteq N$ for any ideals A, B, C of R. Since N is semiprime and $(ACB)^2 = A(CBA)(CB) \subseteq ABC \subseteq N$, we have $ACB \subseteq N$, yielding that N is ideal-symmetric. However, the converse need not be true by the following examples:

Example 1.1. Let $n, k \geq 2$ and consider the ideal $I = n^k\mathbb{Z}$ of $\mathbb{Z}$. Then I is clearly an ideal-symmetric ideal of $\mathbb{Z}$, but I is not a semiprime ideal of $\mathbb{Z}$ since $\mathbb{Z}/n^k\mathbb{Z}$ is isomorphic to $\mathbb{Z}_{n^k}$.

Example 1.2. Let $\mathbb{H}$ be the Hamilton quaternions over the real numbers. Consider the subring

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in \mathbb{H} \right\}$$

of $U_3(\mathbb{H})$. Then R is a noncommutative local ring with $J^2 \neq 0 = J^3$, where $J = J(R) = \left\{ \left(\begin{smallmatrix} 0 & b & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{smallmatrix}\right) \mid b, c \in \mathbb{H} \right\}$. Note that $\{R, J, J^2, 0\}$ is the set of all ideals of R, and so all ideals of R are ideal-symmetric. But 0 and J^2 are not semiprime ideals of R.

According to Cohen [5], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for all $a, b \in R$. Anderson and Camillo [1] used the term ZC_2 for the reversible condition. It is evident that a symmetric ring is reversible. But the converse could not hold by [1, Example 1.5] or [11, Examples 5 and 7]. An ideal N of a ring R is called *reversible* if $ab \in N$ implies $ba \in N$ for all $a, b \in R$. In [12], this ideal N is called *completely reflexive*. We will also extend the concepts of reversible ideals to ideal-reversible ideals. We will call an ideal N of a ring R *ideal-reversible* if $AB \subseteq N$ implies $BA \subseteq N$ for any ideals A, B in R. In particular, if the zero ideal of a ring R is ideal-reversible, then R is usually called *ideal-reversible*. Anderson and Camillo demonstrated that there exists a reversible ring but not ideal-symmetric in [1, Example 1.5]. On the other hand, it is clear that any ideal-symmetric ideal of a ring is ideal-reversible. The following example tells us that there exists an ideal-reversible ideal in some ring but not ideal-symmetric:

Example 1.3. By [1, Example 1.5], there exists a reversible ring but not ideal-symmetric. Hence we can take a reversible ring R_1 which is not ideal-symmetric. Consider $R = R_1 \times R_2$ for some ring R_2, and let $N = \{0\} \times R_2$ be an ideal of R. Note that R/N is isomorphic to R_1. Since R_1 is reversible, R_1 is clearly ideal-reversible. Thus R/N is ideal-reversible, and so N is ideal-reversible by the below Theorem 2.8. On the other hand, since R_1 is not ideal-symmetric, R/N is not also ideal-symmetric, and then N is not ideal-symmetric by the below Theorem 2.8.

In Section 2, we will show that every symmetric ideal of a ring R is ideal-symmetric, but the converse does not hold. Some equivalent conditions that an ideal N of R is ideal-symmetric are investigated, for example, an ideal N of R is ideal-symmetric if and only if $I_1 I_2 \cdots I_n \subseteq N$ implies $I_{\sigma(1)} I_{\sigma(2)} \cdots I_{\sigma(n)} \subseteq N$ for any permutation σ of the set $\{1, 2, \ldots, n\}$ where $n \geq 3$ and I_i is an (right, left) ideal of R for all i. It is shown that an ideal N of R is ideal-symmetric if and only if R/N is an ideal-symmetric ring.

We call the intersection of all ideal-symmetric ideals of a ring R the *ideal-symmetric radical* of R and denote it by $S(R)$. It is evident that $S(R)$ is the smallest ideal-symmetric ideal of R. If R has no proper ideal-symmetric ideals, then $S(R) = R$. It is clear that $S(R) \subseteq P(R) \subseteq J(R)$ since every prime ideal of R is ideal-symmetric and every maximal ideal is prime. In Section 2, we will also show that the ideal-symmetric property is Morita invariant, and for any ring R, $S(M_n(R)) = M_n(S(R))$ and $S(R)[x] \subseteq S(R[x])$.

In [12], a right ideal I of a ring R is called *reflexive* if $aRb \subseteq I$ implies that $bRa \subseteq I$ for any $a, b \in R$. R is called *reflexive* if the zero ideal of R is a reflexive ideal (i.e., $aRb = 0$ implies that $bRa = 0$ for $a, b \in R$. It was shown in [9] that a ring R is ideal-reversible if and only if R is reflexive. In Section 3, we will show that any ideal N is ideal-reversible if and only if N is reflexive if and only if $IJ \subseteq N$ implies $JI \subseteq N$ for any right (or left) ideals I, J of R.

Kaplansky [8] introduced the concept of *Baer* rings as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. Clark [4] extended the concept of Baer rings to quasi-Baer rings. A ring R is called *quasi-Baer* if the right (left) annihilator of every nonzero ideal is generated by an idempotent (See [2], [7]). Note that the definition of Baer rings and quasi-Baer rings are left-right symmetric by [4] and [8]. Also note that in a reduced ring R (i.e., R has no nonzero nilpotent), R is Baer if and only if R is quasi-Baer. In Section 3, it was shown that for any ideal B of an ideal-reversible ring R, if R is quasi-Baer, then $ann(B) = Re$ for some central idempotent $e \in R$. In [3, Proposition 1.9], it was proved that for a Baer ring R, R is semiprime if and only if R is ideal-symmetric if and only if R is reflexive (equivalently, ideal-reversible). In this note, it was shown that for an ideal N of a ring R such that R/N is Baer, N is semiprime if and only if N is ideal-symmetric if and only if N is ideal-reversible.

2. Ideal-symmetric ideals of rings

In this section we study the structure of ideal-symmetric ideals.

Proposition 2.1. *Every symmetric (resp., reversible) ideal of a ring R is ideal-symmetric (resp., ideal-reversible).*

Proof. Let N be a symmetric ideal of a ring R. Suppose that $ABC \subseteq N$ for any ideals A, B, C in R, and let $\alpha \in ACB$ be arbitrary. Then $\alpha = \sum_{i=1}^{n} a_i q_i$ where $a_i \in A, q_i = \sum_{j=1}^{\ell_i} c_{ij} b_{ij} \in CB$ ($c_{ij} \in C, b_{ij} \in B$) for each $i = 1, \ldots, n$. So $\alpha = \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} a_i c_{ij} b_{ij}$. Note that each $a_i b_{ij} c_{ij} \in ABC \subseteq N$. Since N is symmetric, $a_i c_{ij} b_{ij} \in N$, and so $\alpha \in N$, yielding that N is ideal-symmetric. Similarly, we also show that every reversible ideal of a ring R is ideal-reversible. □

The converse of above Proposition 2.1 could not be true by the following example:

Example 2.2. Let $\mathbb{Z}_4$ be the rings of integers modulo 4 and $R = \mathrm{Mat}_2(\mathbb{Z}_4)$. Then $\{R, N = \mathrm{Mat}_2(2\mathbb{Z}_4), 0\}$ is the set of all ideals of R. Note that R is not reversible (hence R is not symmetric) because $ab = 0 \neq ba$ for some $a, b \in R$ where $a = \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right), b = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Note that the ideal N of R is not reversible (hence N is not symmetric) because $pq \in N, qp \notin N$ for some $p, q \in R$ where $p = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right), q = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right)$. On the other hand, we observe that all ideals of R are ideal-symmetric. Indeed, let A, B, C be ideals of R. First, suppose that $ABC = 0$. If one of A, B, C is zero, then clearly, $ACB = 0$. Let $A, B, C \neq 0$. Since $ABC = 0$, ABC is one of NNR, NRN, RNN, and so $ACB = 0$, yielding that 0 is ideal-symmetric. Next, suppose that $ABC \subseteq N$. If one of A, B, C is zero, then clearly, $ABC = ACB = 0 \subseteq N$. Let $A, B, C \neq 0$. Since $ABC \subseteq N$, $ABC = 0$ or $ABC = N$. If $ABC = 0$, then $ACB = 0$ as above argument. If $ABC = N$, then ABC is one of NRR, RRN, RNR, and so $ACB = N$, yielding that N is ideal-symmetric (hence N is ideal-reversible).

Proposition 2.3. *Let N be any ideal of a ring R. Then N is ideal-symmetric if and only if $(a)(b)(c) \subseteq N$ implies $(a)(c)(b) \subseteq N$ for all $a, b, c \in R$.*

Proof. Suppose that $(a)(b)(c) \subseteq N$ implies $(a)(c)(b) \subseteq N$ for all $a, b, c \in R$. Let $ABC \subseteq N$ for any ideals A, B, C in R, and let $\alpha \in ACB$ be arbitrary. Then $\alpha = \sum_{i=1}^{n} a_i q_i$ where $a_i \in A, q_i = \sum_{j=1}^{\ell_i} c_{ij} b_{ij} \in CB$ ($c_{ij} \in C, b_{ij} \in B$) for each $i = 1, \ldots, n$. So $\alpha = \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} a_i c_{ij} b_{ij}$. Since each $a_i b_{ij} c_{ij} \in (a_i)(b_{ij})(c_{ij}) \subseteq ABC \subseteq N$, we have $a_i c_{ij} b_{ij} \in (a_i)(c_{ij})(b_{ij}) \subseteq N$ by assumption, and so $\alpha \in N$. Thus $ACB \subseteq N$, yielding that N is an ideal-symmetric ideal of a ring R. The converse is clear. □

Lemma 2.4. *Let N be an ideal-symmetric ideal of a ring R and I_1, I_2, I_3 be ideals of R. Then $I_1 I_2 I_3 \subseteq N$ if and only if $I_{\sigma(1)} I_{\sigma(2)} I_{\sigma(3)} \subseteq N$ for any permutation σ of the set of $\{1, 2, 3\}$.*

Proof. Suppose that N is ideal-symmetric such that $I_1I_2I_3 \subseteq N$. Since N is ideal-symmetric, $I_1I_3I_2 \subseteq N$. Since N is ideal-symmetric and $RI_1(I_2I_3) \subseteq N$, we have $R(I_2I_3)I_1 \subseteq N$, and so $I_2I_3I_1 \subseteq N$, also $I_2I_1I_3 \subseteq N$. By applying the similar argument to $RI_1(I_3I_2) \subseteq N$, we also have that $I_3I_1I_2, I_3I_2I_1 \subseteq N$. The converse is clear. $\square$

Let S_n be the symmetric group on n letters for any positive integer.

Proposition 2.5. *For any ideal N of a ring R, the following conditions are equivalent*:

(1) *N is ideal-symmetric*;

(2) *$aRbRc \subseteq N$ implies $aRcRb \subseteq N$ for all $a, b, c \in R$*;

(3) *$I_1I_2 \cdots I_n \subseteq N$ implies $I_{\sigma(1)}I_{\sigma(2)} \cdots I_{\sigma(n)} \subseteq N$ for any $\sigma \in S_n$ where I_i is an ideal of R for $i = 1, 2, \ldots, n$ and $n \geq 3$ is any positive integer*;

(4) *$a_1Ra_2 \cdots Ra_n \subseteq N$ implies $a_{\sigma(1)}Ra_{\sigma(2)} \cdots Ra_{\sigma(n)} \subseteq N$ for any $\sigma \in S_n$ where I_i is an ideal of R for $i = 1, 2, \ldots, n$ and $n \geq 3$ is any positive integer*;

(5) *$I_1I_2 \cdots I_n \subseteq N$ implies $I_{\sigma(1)}I_{\sigma(2)} \cdots I_{\sigma(n)} \subseteq N$ for any $\sigma \in S_n$ where I_i is a right (left) ideal of R for $i = 1, 2, \ldots, n$ and $n \geq 3$ is any positive integer*;

(6) *$ABC \subseteq N$ implies $BAC \subseteq N$ for all ideals A, B, C of R*;

(7) *$aRbRc \subseteq N$ implies $bRaRc \subseteq N$ for all $a, b, c \in R$.*

Proof. (1) $\Rightarrow$ (2). Suppose that N is ideal-symmetric and $aRbRc \subseteq N$ for all $a, b, c \in R$. Since N is an ideal of R, we have that $(RaR)(RbR)(RcR) \subseteq N$. Since N is ideal-symmetric and RaR, RbR, RcR are ideals of R,

$$(RaR)(RcR)(RbR) \subseteq N$$

by assumption. Clearly, $aRcRb \subseteq (RaR)(RcR)(RbR) \subseteq N$ as desired.

(2) $\Rightarrow$ (1). Suppose that $aRbRc \subseteq N$ implies $aRcRb \subseteq N$ for all $a, b, c \in R$. Let $ABC \subseteq N$ for any ideals A, B, C in R, and let $\alpha \in ACB$ be arbitrary. Then $\alpha = \sum_{i=1}^{m} a_ic_ib_i$ where $a_i \in A, c_i \in C, b_i \in B$ for some positive integer m. Since each $a_ib_ic_i \in a_iRb_iRc_i \subseteq ABC \subseteq N$, we have $a_ic_ib_i \in a_iRc_iRb_i \subseteq N$ by assumption, and so $\alpha \in N$. Thus $ACB \subseteq N$, yielding that N is ideal-symmetric.

(1) $\Rightarrow$ (3). Suppose that N is ideal-symmetric and $I_1I_2 \cdots I_n \subseteq N$ $(n \geq 3)$. Since S_n is generated by the transpositions $\tau_0 = (1, 2), \tau_1 = (2, 3), \ldots, \tau_{n-2} = (n-1, n), \tau_{n-1} = (n, 1) \in S_n$, it is enough to show that $I_{\tau_k(1)}I_{\tau_k(2)} \cdots I_{\tau_k(n)} \subseteq N$ for all $k = 0, 1, \ldots, n-1$. Note that $I_2I_3 \cdots I_1 \subseteq N$, and so $I_3I_4 \cdots I_2 \subseteq N$, $\ldots, I_nI_1 \cdots I_{n-1} \subseteq N$, i.e.,

$$(*) \qquad I_{\mu_k}(1)I_{\mu_k}(2) \cdots I_{\mu_k}(n) \subseteq N,$$

where $\mu = (1, 2, \ldots, n) \in S_n$ and $\mu_k = \mu^k(= \mu \cdot \mu \cdots \mu)$ for any $k = 0, 1, \ldots, n-1$. By Lemma 2.4, we have $I_2I_1I_3 \cdots I_n \subseteq N$, i.e.,

$$(**) \qquad I_{\tau_0(1)}I_{\tau_0(2)} \cdots I_{\tau_0(n)} \subseteq N.$$

By $(*)$ and $(**)$, we have that

$$I_{\tau_k(1)}I_{\tau_k(2)} \cdots I_{\tau_k(n)} = I_{\mu_k\tau_0\mu_{n-k}(1)}I_{\mu_k\tau_0\mu_{n-k}(2)} \cdots I_{\mu_r\tau_0\mu_{n-r}(n)} \subseteq N$$

by observing that $\mu_k\tau_0\mu_{n-k} = \tau_k$ for all $k = 0, 1, \ldots, n-1$.

(3) $\Rightarrow$ (1). Clear.

(1) $\Leftrightarrow$ (6). It follows from Lemma 2.4.

(1) $\Leftrightarrow$ (7). It follows from (1) $\Leftrightarrow$ (6) and the similar arguments given in the proof of (1) $\Leftrightarrow$ (2).

(3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (3) are obvious.

(4) $\Rightarrow$ (3). Suppose that $a_1Ra_2\cdots Ra_n \subseteq N$ implies $a_{\sigma(1)}Ra_{\sigma(2)}\cdots Ra_{\sigma(n)} \subseteq N$ for any $\sigma \in S_n$. Let $I_1I_2\cdots I_n \subseteq N$ where I_i $(1 \le i \le n)$ is an ideal of R. Let $\beta \in I_{\sigma(1)}I_{\sigma(2)}\cdots I_{\sigma(n)}$ be arbitrary for any $\sigma \in S_n$. Then

$$\beta = \sum_{j=1}^{\ell} a_{\sigma(1)_j}a_{\sigma(2)_j}\cdots a_{\sigma(n)_j},$$

where $a_{\sigma(i)_j} \in I_{\sigma(i)}$ $(1 \le i \le n, 1 \le j \le \ell)$. Since each $a_{1_j}a_{2_j}\cdots a_{n_j} \in a_1Ra_2\cdots Ra_n \subseteq N$, $a_{\sigma(1)_j}a_{\sigma(2)_j}\cdots a_{\sigma(n)_j} \in a_{\sigma(1)}Ra_{\sigma(2)}\cdots Ra_{\sigma(n)} \subseteq N$ by assumption, and so $\beta \in N$, yielding that $I_{\sigma(1)}I_{\sigma(2)}\cdots I_{\sigma(n)} \subseteq N$ for any $\sigma \in S_n$.

(3) $\Rightarrow$ (5). Suppose that $I_1I_2\cdots I_n \subseteq N$ implies $I_{\sigma(1)}I_{\sigma(2)}\cdots I_{\sigma(n)} \subseteq N$ for any $\sigma \in S_n$ where I_i $(1 \le i \le n)$ is an ideal of R and n is any positive integer. Let $J_1J_2\cdots J_n \subseteq N$ where J_i $(1 \le i \le n)$ is a right ideal of R. Note that $(RJ_1R)(RJ_2R)\cdots(RJ_nR) \subseteq N$ where RJ_iR $(1 \le i \le n)$ is an ideal of R. Let $K_i = RJ_iR$ for each $i = 1, 2, \ldots, n$. By assumption, we have that $(RJ_{\sigma(1)}R)(RJ_{\sigma(2)}R)\cdots(RJ_{\sigma(n)}R) \subseteq N$ for any $\sigma \in S_n$, and so $J_{\sigma(1)}J_{\sigma(2)}\cdots J_{\sigma(n)} \subseteq (RJ_{\sigma(1)}R)(RJ_{\sigma(2)}R)\cdots(RJ_{\sigma(n)}R) \subseteq N$, as desired. The proof for the left ideal case is shown by the similar argument given in the right ideal case. □

Corollary 2.6. *For a ring R, the following conditions are equivalent*:

(1) *R is ideal-symmetric*;

(2) *$aRbRc = 0$ implies $aRcRb = 0$ for all $a, b, c \in R$*;

(3) *$I_1I_2\cdots I_n = 0$ implies $I_{\sigma(1)}I_{\sigma(2)}\cdots I_{\sigma(n)} = 0$ for any $\sigma \in S_n$ where I_i is an ideal of R for $i = 1, 2, \ldots, n$ and $n \ge 3$ is any positive integer*;

(4) *$a_1Ra_2\cdots Ra_n = 0$ implies $a_{\sigma(1)}Ra_{\sigma(2)}\cdots Ra_{\sigma(n)} = 0$ for any $\sigma \in S_n$ where I_i is an ideal of R for $i = 1, 2, \ldots, n$ and $n \ge 3$ is any positive integer*;

(5) *$I_1I_2\cdots I_n = 0$ implies $I_{\sigma(1)}I_{\sigma(2)}\cdots I_{\sigma(n)} = 0$ for any $\sigma \in S_n$ where I_i is a right (left) ideal of R for $i = 1, 2, \ldots, n$ and $n \ge 3$ is any positive integer*;

(6) *$ABC = 0$ implies $BAC = 0$ for all ideals A, B, C of R*;

(7) *$aRbRc = 0$ implies $bRaRc = 0$ for all $a, b, c \in R$.*

Proof. It follows from Proposition 2.5. □

The following theorem implies that the ideal-symmetric property of any ideal of a ring is Morita invariant.

Theorem 2.7. *Let R be a ring and N be an ideal of R. Then we have the following*:

(1) *If* N *is ideal-symmetric, then* eNe *is an ideal-symmetric ideal of* eRe *for each* $e^2 = e \in R$.

(2) N *is ideal-symmetric in* R *if and only if* $M_n(N)$ *is ideal-symmetric in* $M_n(R)$ *for all* $n \geq 1$.

Proof. (1) Suppose that N is ideal-symmetric. Let $a, b, c \in eRe$ such that $a(eRe)b(eRe)c \subseteq eNe$. Since $a(eRe)b(eRe)c = aeRebRec \subseteq eNe \subseteq N$ and N is ideal-symmetric, $aeRecReb \subseteq N$, and so $a(eRe)c(eRe)b = aeRecReb = e(aeRecReb)e \subseteq eNe$, yielding that the ideal eNe is ideal-symmetric.

(2) Suppose that N is ideal-symmetric. Let A, B, C be ideals of $M_n(R)$ such that $ABC \subseteq M_n(N)$. Note that there exist ideals I, J, K such that $A = M_n(I), B = M_n(J), C = M_n(K)$. Note that $ABC = M_n(I)M_n(J)M_n(K) = M_n(IJK)$ and then $IJK \subseteq N$. Since N is ideal-symmetric, $IKJ \subseteq N$, and so $ACB = M_n(IJK) \subseteq M_n(N)$. Thus $M_n(N)$ is ideal-symmetric.

Conversely, if $\mathrm{Mat}_n(N)$ is ideal-symmetric, then $N \cong e_{11}\mathrm{Mat}_n(N)e_{11}$ is ideal-symmetric by (1) where e_{11} is the matrix in $\mathrm{Mat}_n(N)$ with $(1,1)$-entry 1 and elsewhere 0. $\square$

We already knew that for an ideal N of a ring R, N is a prime (resp., semiprime) ideal if and only if R/N is prime (resp., semiprime). Here we also have the following:

Theorem 2.8. *For an ideal* N *of a ring* R, N *is ideal-symmetric* (*resp., ideal-reversible*) *if and only if* R/N *is an ideal-symmetric* (*resp., ideal-reversible*) *ring.*

Proof. Suppose that N is ideal-symmetric. Let A, B, C be ideals of R/N such that $ABC = N$, zero of R/N. Then there exist ideals $A_0, B_0, C_0 \supseteq N$ of R such that $A = A_0/N, B = B_0/N, C = C_0/N$. Since $ABC = (A_0/N)(B_0/N)(C_0/N) = (A_0B_0C_0)/N = N$, $A_0B_0C_0 = N$. Since N is ideal-symmetric, $A_0C_0B_0 \subseteq N$. Thus $ACB = (A_0C_0B_0)/N = N$, which yields that R/N is ideal-symmetric ring.

Suppose that R/N is an ideal-symmetric ring. Let A, B, C be ideals of N such that $ABC \subseteq N$. Then $ABC + N = N$. Note that $(A+N)(B+N)(C+N) \subseteq ABC + N = N$, and so $(A+N)(B+N)(C+N) = N$. Since R/N is an ideal-symmetric ring, $((A+N)/N)(C+N)/N)(B+N)/N) = N$, yielding that $(A+N)(C+N)(B+N) \subseteq N$, and so $ACB \subseteq (A+N)(C+N)(B+N) \subseteq N$, which means that N is ideal-symmetric.

Similarly, ideal-reversible case is also shown. $\square$

Corollary 2.9. *Let* I *be an ideal-symmetric ideal of a ring* R. *If* I *is semiprime* (*as a ring without identity*), *then* R *is ideal-symmetric.*

Proof. Since I is ideal-symmetric, R/I is ideal-symmetric by Theorem 2.8. Since I is semiprime (as a ring without identity), R is ideal-symmetric ring by [3, Proposition 2.11]. $\square$

Note that a subring of ideal-symmetric ring could not be ideal-symmetric by the following example:

Example 2.10. Let R be an ideal-symmetric ring and consider $U_2(R)$. By Theorem 2.8, $\mathrm{Mat}_2(R)$ is an ideal-symmetric ring. Let $A = \left(\begin{smallmatrix} R & R \\ 0 & 0 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} 0 & R \\ 0 & R \end{smallmatrix}\right)$ be two ideals of $U_2(R)$. Since $ABA = 0$ and $AAB \neq 0$, $U_2(R)$ is not ideal-symmetric. On the other hand, we note that A (or B) is an ideal-symmetric ideal of $U_2(R)$ and it is also ideal-symmetric as a subring of $U_2(R)$ because there is the unique nonzero proper ideal $\left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right)$ in A (or B) even though $U_2(R)$ is not an ideal-symmetric ring.

Lemma 2.11. (1) *The intersection of two ideal-symmetric ideals of a ring R is ideal-symmetric.*

(2) *The intersection of all ideal-symmetric ideals of a ring R is ideal-symmetric.*

Proof. Clear. □

Recall that the intersection of all ideal-symmetric ideals of a ring R introduced in Lemma 2.11 is called the *ideal-symmetric radical* of R and denoted $S(R)$. It is evident that $S(R)$ is the smallest ideal-symmetric ideal of R, and R is a ring such that $S(R) = 0$ if and only if R is ideal-symmetric.

Corollary 2.12. *For any ring R, we have $S(R/S(R)) = 0$.*

Proof. Since $S(R)$ is ideal-symmetric ideal of R by Lemma 2.11, $R/S(R)$ is an ideal-symmetric ring by Theorem 2.8, and so $S(R/S(R)) = 0$. □

Now we raise a question:

Question 1. For an ideal I of a ring R that is considered as ring, $S(I) = I \cap S(R)$?

The answer is negative by the following examples:

Example 2.13. Let $R = U_2(F)$ over a field F, and consider the ideal $I = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$ of R. By Example 2.8, $U_2(F)$ is not ideal-symmetric (i.e., the zero ideal of R is not ideal-symmetric). Since $R/I \cong F \times F$, which is ideal-symmetric, I is ideal-symmetric by Theorem 2.8, i.e., $S(I) = 0$. On the other hand, observe that all nonzero ideals of R are I, $I_1 = \left(\begin{smallmatrix} 0 & F \\ 0 & F \end{smallmatrix}\right)$ and $I_2 = \left(\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right)$ which are clearly ideal-symmetric. Hence $S(R) = I \cap I_1 \cap I_2 = I$, and so $I \cap S(R) = I \neq 0 = S(I)$. Note that even though all nonzero ideals of $U_2(F)$ are ideal-symmetric, $U_2(F)$ is not ideal-symmetric.

Example 2.14. Let R be not any ideal-symmetric ring with $I = J(R) \neq 0$. Then $S(I) = 0$ because I is a semiprime ideal of R (hence I is ideal-symmetric). Thus $I \cap S(R) = J(R) \cap S(R) = S(R) \neq 0$ because R is not ideal-symmetric, yielding that $S(I) = 0 \neq I \cap S(R)$.

Corollary 2.15. *If $S(R)$ is semiprime* (*as a ring without identity*) *for a ring R, then R is ideal-symmetric.*

Proof. Since $S(R)$ is an ideal-symmetric ideal of R, it follows from Corollary 2.9. $\square$

Theorem 2.16. *For any ring R, we have $S(M_n(R)) = M_n(S(R))$.*

Proof. Let $N = S(R)$. By Lemma 2.11, N is ideal-symmetric ideal of R, and so $M_n(N)$ is ideal-symmetric ideal of $M_n(R)$ by Theorem 2.7. Since $M_n(N)$ is ideal-symmetric ideal of $M_n(R)$, $S(M_n(R)) \subseteq M_n(N)$. Next, we will show that $M_n(N) \subseteq S(M_n(R))$. Let A be any ideal-symmetric ideal of $M_n(R)$. Then there exists an ideal A_0 of R such that $A = M_n(A_0)$. Since A is ideal-symmetric, A_0 is ideal-symmetric by Theorem 2.7, and so $N \subseteq A_0$. Thus $M_n(N) \subseteq M_n(A_0) = A$, yielding that $M_n(N) \subseteq S(M_n(R))$. Therefore, we have $S(M_n(R)) = M_n(S(R))$. $\square$

Proposition 2.17. *For any ring R, we have the following*:

(1) $S(R)[x] \subseteq S(R[x])$;

(2) *If $R[x]$ is ideal-symmetric, then R is ideal-symmetric.*

Proof. (1) Let $N = S(R)$. It is enough to show that $N[x] \subseteq A$ for any ideal-symmetric ideal A of $R[x]$. We note that $A \cap R$ is an ideal-symmetric ideal of R. Indeed, if $aRbRc \subseteq A \cap R$ for all $a, b, c \in R$, then $aR[x]bR[x]c = (aRbRc)[x] \subseteq A$, and then $aR[x]cR[x]b = (aRcRb)[x] \subseteq A$ by Proposition 2.5 because A is ideal-symmetric, and so $aRcRb \subseteq A \cap R$, yielding that $A \cap R$ is ideal symmetric. Since $A \cap R$ is ideal-symmetric, $N \subseteq A \cap R \subseteq A$. Therefore, $N[x] \subseteq A$, as desired.

(2) It follows from (1). $\square$

Note that the converse of (2) of Proposition 2.17 could not be true by [3, Example 2.4]. A ring R is *quasi-Armendariz* [6] provided that $a_iRb_j = 0$ for all i, j whenever $f = \sum_{i=0}^{m} a_i x^i, g = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $fR[x]g = 0$. In [3], V. Camilo et al. have shown that for a quasi-Armendariz ring R, R is ideal-symmetric if and only if $R[x]$ is ideal-symmetric.

Theorem 2.18. *Let R be a ring R. If $R/S(R)$ is quasi-Armendariz, then $S(R)[x] = S(R[x])$.*

Proof. By Proposition 2.17, we have $S(R)[x] \subseteq S(R[x])$. To show the reverse inclusion, let $N = S(R)$. Since N is ideal-symmetric, R/N is an ideal-symmetric ring by Theorem 2.8. Since R/N is quasi-Armendariz, $(R/N)[x]$ is ideal-symmetric by [3, Remark 2.5]. Since $(R/N)[x] \cong R[x]/N[x]$ is ideal-symmetric, $N[x]$ is an ideal-symmetric ideal of $R[x]$ by Theorem 2.8, and so $N[x] \supseteq S(R[x])$ as desired. $\square$

Now we raise the following open questions:

Question 2. (1) Is $S(R)[x] = S(R[x])$ for a quasi-Armendariz ring R?

(2) For a ring R, what is $S(S(R))$?

3. Ideal-reversible ideals of rings

Let B be a subset of a ring R and N be an ideal of R. The set $\{a \in R \,|\, aB \subseteq N\}$ is a left ideal of R, which is actually an ideal if B is a left ideal. The set $\{a \in R \,|\, aB \subseteq N\}$ is called the *left annihilator* of B in N and is denoted $ann_\ell(B; N)$. Similarly, the set

$$ann_r(B; N) = \{a \in R \,|\, Ba \subseteq N\}$$

is an ideal of R if B is a right ideal. The set $ann_r(B; N)$ is called the *right annihilator* of B in N. When $ann_\ell(B; N) = ann_r(B; N)$, it is denoted $ann(B; N)$, and called *annihilator* of B in N. In particular, if $N = 0$, then $ann_\ell(B; 0)$ (resp., $ann_r(B; 0)$) is called *left annihilator* of B (resp., *right annihilator* of B), and is simply denoted $ann_\ell(B)$ (resp., $ann_r(B)$). When $ann_\ell(B) = ann_r(B)$, it is denoted $ann(B)$.

Proposition 3.1. *For an ideal N of a ring R, the following are equivalent*:
(1) *N is ideal-reversible*;
(2) *N is reflexive*;
(3) *For each $a \in R$, $ann_\ell(Ra; N) = ann_r(aR; N)$*;
(4) *$ARB \subseteq N$ implies $BRA \subseteq N$ for any nonempty subsets A, B of R*;
(5) *For each ideal B of R, $ann_\ell(B; N) = ann_r(B; N)$*;
(6) *$IJ \subseteq N$ implies $JI \subseteq N$ for any right* (*or left*) *ideals I, J of R.*

Proof. (1) $\Rightarrow$ (2) Suppose that N is ideal-reversible. Let $aRb \subseteq N$ for $a, b \in R$. Then $(RaR)(RbR) \subseteq N$. Since N is ideal-reversible, we have that $bRa \subseteq (RbR)(RaR) \subseteq N$, and so N is reflexive.

(2) $\Rightarrow$ (1) Suppose that N is reflexive. Let $IJ \subseteq N$ for any ideals $I, J \in R$. Let $\alpha \in JI$ be arbitrary. Then $\alpha = \sum_{i=1}^{n} b_i a_i$ where $a_i \in I, b_i \in J$ for some positive integer n. Since each $a_i b_i \in a_i R b_i \subseteq IJ \subseteq N$, we have $b_i a_i \in b_i R a_i \subseteq N$ by assumption, yielding $JI \subseteq N$. Hence N is ideal-reversible.

(2) $\Rightarrow$ (3) Suppose that N is reflexive. Let $b \in ann_r(aR; N)$ for each $a \in R$ be arbitrary. Then $aRb \subseteq N$. Since N is reflexive, we have that $bRa \subseteq N$, yielding that $b \in ann_\ell(Ra; N)$, and so $ann_r(aR; N) \subseteq ann_\ell(Ra; N)$. Similarly, we also have that $ann_\ell(Ra; N) \subseteq ann_r(aR; N)$.

(3) $\Rightarrow$ (2) Suppose that For each $a \in R$, $ann_\ell(Ra; N) = ann_r(aR; N)$. Let $aRb \subseteq N$ for $a, b \in R$. Then $b \in ann_r(aR; N) = ann_\ell(Ra; N)$ by assumption, yielding that $bRa \subseteq N$, and so N is reflexive.

(1) $\Rightarrow$ (5) Suppose that N is ideal-reversible. Let $A = ann_\ell(B; N)$ and $A_0 = ann_r(B; N)$. Then A and A_0 are ideals of R. Since $AB \subseteq N$ and N is ideal-reversible, we have that $BA \subseteq N$, yielding $A \subseteq A_0$. Similarly, we also have that $A_0 \subseteq A$. Hence $ann_\ell(B; N) = ann_r(B; N)$ as desired.

(5) $\Rightarrow$ (1) Suppose that (5) holds. Let $AB \subseteq N$ for any ideals I, J of R. Then $A \subseteq ann_\ell(B; N) = ann_r(B; N)$ by assumption, yielding that $BA \subseteq N$, and so N is ideal-reversible.

(6) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) are clear.

(4) $\Rightarrow$ (6) is clear.

(2) $\Rightarrow$ (4) Suppose that N is reflexive. Let A, B be two nonempty subsets of R with $ARB \subseteq N$. Then $aRb \subseteq N$ for any $a \in A$ and $b \in B$, and so $bRa \subseteq N$ by assumption. Thus $BRA = \sum_{a\in A, b\in B} bRa \subseteq N$. □

Corollary 3.2. *For a ring R, the following are equivalent*:

(1) *R is ideal-reversible*;
(2) *R is reflexive*;
(3) *For each $a \in R$, $ann_\ell(Ra) = ann_r(aR)$*;
(4) *$ARB = 0$ implies $BRA = 0$ for any nonempty subsets A, B of R*;
(5) *For each ideal B of R, $ann_\ell(B) = ann_r(B)$*;
(6) *$IJ = 0$ implies $JI = 0$ for any right (or left) ideals I, J of R.*

Proof. It follows from Proposition 3.1. □

We also have that the ideal-reversible property of any ideal of a ring is Morita invariant.

Proposition 3.3. *Let R be a ring and N be an ideal of R. Then we have the following*:

(1) *If N is ideal-reversible, then eNe is an ideal-symmetric ideal of eRe for each $e^2 = e \in R$.*

(2) *N is ideal-reversible in R if and only if $M_n(N)$ is ideal-reversible in $M_n(R)$ for all $n \geq 1$.*

Proof. (1) Suppose that N is ideal-reversible. Let $a, b \in eRe$ such that $a(eRe)b \subseteq eNe$. Since $(eae)R(ebe) = a(eRe)b \subseteq N$ and N is reflexive by Proposition 3.1, $b(eRe)a \subseteq N$, and then $b(eRe)a = (ebe)R(eae) \subseteq eNe$, yielding that the ideal eNe is ideal-reversible.

(2) It follows from the similar proof given in Theorem 2.7. □

Proposition 3.4. *Let B be an ideal of an ideal-reversible ring R. If R is quasi-Baer, then $ann(B) = Re$ for some central idempotent $e \in R$.*

Proof. Since R is quasi-Baer, $ann_\ell(B) = Re$ for some idempotent $e \in R$. Similarly, $ann_r(B) = fR$ for some idempotent $f \in R$. Since R is an ideal-reversible, $ann_\ell(B) = ann_r(B)$ by Corollary 3.2, and so $Re = fR$. Observe that $e = f$. Indeed, since $Re = fR$, $e = fa$ for some $a \in R$, and then $fe = fa = e$. Also $f = be$ for some $b \in R$, and then $fe = be = f$. Thus $e = f$. Let $r \in R$ be arbitrary. Since $Re = eR$, $re = ex$ for some $x \in R$, and so $ere = ex = re$. Similarly, $er = ye$ for some $y \in R$, and so $ere = ye = er$. Thus we have that for all $r \in R$, $ere = re = er$, yielding that e is central. □

Theorem 3.5. (1) *Let N be an ideal of a ring R. If R/N is Baer, then the following conditions are equivalent*:

(1) *N is semiprime*;
(2) *N is ideal-symmetric*;
(3) *N is ideal-reversible.*

Proof. It suffices to show that (3) $\Rightarrow$ (1). Suppose that $aRa \subseteq N$ for $a \in R$. Let $\overline{R} = R/N$ and $\overline{x} = x + \overline{R}$ for all $x \in R$. Then $\overline{R}$ is ideal-reversible by Theorem 2.8. Since $\overline{R}$ is Baer, there exists $\overline{e}^2 = \overline{e} \in \overline{R}$ with $ann_r(\overline{a}\overline{R}) = \overline{e}\overline{R}$. Then $\overline{a} = \overline{e} \cdot \overline{a}$ since $\overline{a} \in ann_r(\overline{a}\overline{R}) = \overline{e}\overline{R}$, and so $a - ea \in N$. Note that both $\overline{e}\overline{R}$ and $\overline{a}\overline{R}$ are right ideals of $\overline{R}$ and $(\overline{a}\overline{R})(\overline{e}\overline{R}) = \overline{0}$. Since $\overline{R}$ is ideal-reversible, $(\overline{a}\overline{R})(\overline{e}\overline{R}) = \overline{0}$ implies that $(\overline{e}\overline{R})(\overline{a}\overline{R}) = \overline{0}$ by Corollary 3.2, entailing $ea \in N$. Hence we have $a = (a - ea) + ea \in N$, which implies that N is semiprime. $\square$

Corollary 3.6. (1) *Let R be a Baer ring. Then the following conditions are equivalent*:

(1) *R is semiprime*;
(2) *R is ideal-symmetric*;
(3) *R is ideal-reversible.*

Proof. It follows from Theorem 3.5. $\square$

Acknowledgements. The authors thank the referee for his/her very careful reading of the manuscript and valuable suggestions that improved the paper.

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Juncheol Han
Department of Mathematics Education
Pusan National University
Pusan 46241, Korea
E-mail address: jchan@pusan.ac.kr

Yang Lee
Institute of Basic Science
Daejin University
Pocheen 11159, Korea
E-mail address: iissylee@hanmail.net

Sangwon Park
Department of Mathematics
Dong-A University
Pusan 49315, Korea
E-mail address: swpark@donga.ac.kr