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ENUMERATION OF OPTIMALLY LABELLED GRAPHS OF BANDWIDTH 2

Gab-Byung Chae, MinSeok Cheong, and Sang-Mok Kim

Abstract. An optimally labelled graph of bandwidth 2 is an ordered pair (G, f) where G is a simple graph with $bw(G) = 2$ and $f : V(G) \to [n]$ is a bijection such that $bw(G, f) = 2$. In this paper, the number of optimally labelled graphs of bandwidth two of order n is enumerated by counting linear forests.

1. Introduction

Let $G = (V, E)$ be a graph of order n and $f : V(G) \to [n]$ be a bijection (labelling). The *bandwidth* of f, denoted by $bw(f, G)$, is the maximum of $|f(u) - f(v)|$ for $uv \in E(G)$. The bandwidth of G, written $bw(G)$, is the minimum bandwidth over all bijections f. That is,

$$bw(G) = \min_f bw(f, G). \tag{1}$$

A bijection satisfying

$$bw(G) = bw(f, G) \tag{2}$$

is called an *optimal labelling* of G, and the pair (G, f) an *optimally labelled graph.* Hence an optimally labelled graph of bandwidth 2 is an ordered pair (G, f) where G is a simple graph with $bw(G) = 2$ and $f : V(G) \to [n]$ is a bijection such that $bw(G, f) = 2$.

We follow the usual convention of regarding two labelled graphs as identical if and only if their adjacency matrices are the same. For the rest of the paper we work directly with adjacency matrices. Let $A_f = [a_{i,j}]$ be an adjacency matrix of a simple labelled graph G of order n with a bijection f. If $bw(A_f)$

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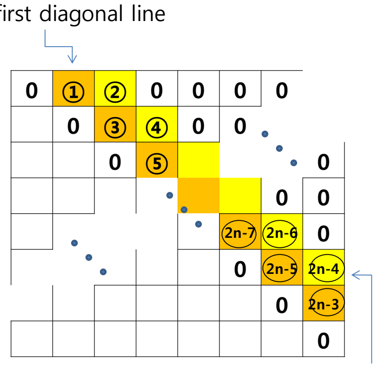

FIGURE 1. Asset positions

denotes $\max\{j-i : a_{ij} = 1,\ j \geq i\}$, then $bw(A_f) = bw(f, G)$, since $bw(f, G) = \max\{|f(u) - f(v)| : uv \in E(G)\}$. Hence we have

$$bw(G) = \min_f \{bw(A_f)\}.$$

The A_f satisfying $bw(G) = bw(A_f)$ is called an *optimally labelled matrix*. Hence we now count the number of optimally labelled graphs of bandwidth 2 and of order n by counting the corresponding optimally labelled matrices, although the characterization of 2-bandwidth graphs has not been accomplished yet [1]. Since adjacency matrix is symmetric, it is enough to consider the matrices which have all zero entries except the first two upper diagonal lines, say the first diagonal line and the second diagonal line of the matrix above the main diagonal line as in Figure 1 when we consider the adjacency matrices of graphs of order n and of bandwidth less than or equal to 2. We call such matrix in Figure 1 *two-diagonal matrix* from now on for convenience. Thus each of two-diagonal matrix represents an ordered pair (G, f) where G is a graph and $f : V(G) \to [n]$ is a bijection such that $bw(G) \leq 2$ and that $bw(G, f) \leq 2$. Thus optimally labelled matrix A_f of bandwidth 2 is a two-diagonal matrix representing an ordered pair (G, f) where G is a graph and $f : V(G) \to [n]$ is a bijection such that $bw(G) = 2$ and that $bw(G, f) = 2$. There are $(n-1) + (n-2) = 2n-3$ available entries at the first two diagonal lines above the main diagonal line in the two-diagonal matrices A_f of order n depending on given bijection f. Hence these $2n-3$ entries shall be filled up by 0 or 1. We call each of these $2n-3$ entries an *asset position* of the two-diagonal matrix.

Hence the total number of two-diagonal matrices is 2^{2n-3}, and clearly the total number of two-diagonal matrices of bandwidth one is 2^{n-1} [2]. There are some two-diagonal matrices which are not labelled optimally, that is, even though there is at least one 1 at the second diagonal line of the matrix, they could be relabelled to be matrices of bandwidth one. They are called *non-optimal matrices*. Hence a non-optimal matrix is an adjacency matrix of graph G with an bijection f such that an ordered pair (G, f) of a graph G and bijection $f : V(G) \to [n]$ satisfies $bw(G) = 1$ but $bw(G, f) = 2$. Hence this whole counting is completed by excluding non-optimal matrices and the two-diagonal matrices of graphs of bandwidth one, from all two-diagonal matrices.

2. Counting optimally labelled graph of bandwidth 2

The following definitions are needed for characterizing the non-optimal matrix. If $bw(G \setminus \{v\}) \leq bw(G) - 1$ for any $v \in V$, then G is called an *irreducible graph*. If G is irreducible and its bandwidth is k, then G is called k*-irreducible*. It is easy to obtain the following results.

Theorem 1 ([3])**.** *A graph G is 1-irreducible if and only if G is K_2.*

The proof is trivial.

Theorem 2 ([3])**.** *A graph G is 2-irreducible if and only if G is $K_{1,3}$ or C_n, $n \geq 3$.*

Proof. If G is $K_{1,3}$ or C_n, $n \geq 3$, then clearly G is irreducible and $bw(G) = 2$. Conversely, suppose there is an irreducible graph $G = (V, E)$ which is neither $K_{1,3}$, nor C_n, $n \geq 3$. Since G is irreducible, G does not contain $K_{1,3}$, and C_n as its induced subgraph. Since $K_{1,3}$ is not an induced subgraph of G, it is clear that $\deg(v) \leq 2$ for $v \in G$. Note that G cannot be a path since bandwidth of a path is 1. If there is a vertex of degree 1, then there is a vertex of degree 3 since G is not a path. This implies that G contains $K_{1,3}$ as its induced subgraph, i.e., a contradiction. Hence, $\deg(v) = 2$ for all $v \in V$. This implies G is C_n. Therefore, G is one of $K_{1,3}$, and C_n. □

Theorem 3 ([3])**.** *G is a graph with $bw(G) = 1$ if and only if G has K_2 as a subgraph but no 2-irreducible graph as a subgraph.*

Proof. Suppose that G is a graph with $bw(G) = 1$. If G have any 2-irreducible graphs as a subgraph, then $bw(G) \geq 2$. This is a contradiction. Hence G does not have any 2-irreducible graphs. And clearly G contains K_2. Conversely, Suppose that G has K_2 as a subgraph but no 2-irreducible graph as a subgraph. Then we have $1 = bw(K_2) \leq ld(G)$. Since G has no 2-irreducible graph as a subgraph, we have $bw(G) < 2$. Hence, $bw(G) = 1$. □

If a graph G has K_2 as a subgraph, then its two-diagonal matrix has at least 1 on the first diagonal line. If G has a $K_{1,3}$ as a subgraph, then its two-diagonal matrix has a sequence which represents a vertex of degree greater than or equal to three (see Figure 2). And if G has any C_n, $n \geq 3$ as a subgraph, then each of its two-diagonal matrix has a sequence which represents any cycle in the two-diagonal matrix (see Figure 2 also).

So we have the following lemma by Theorem 3.

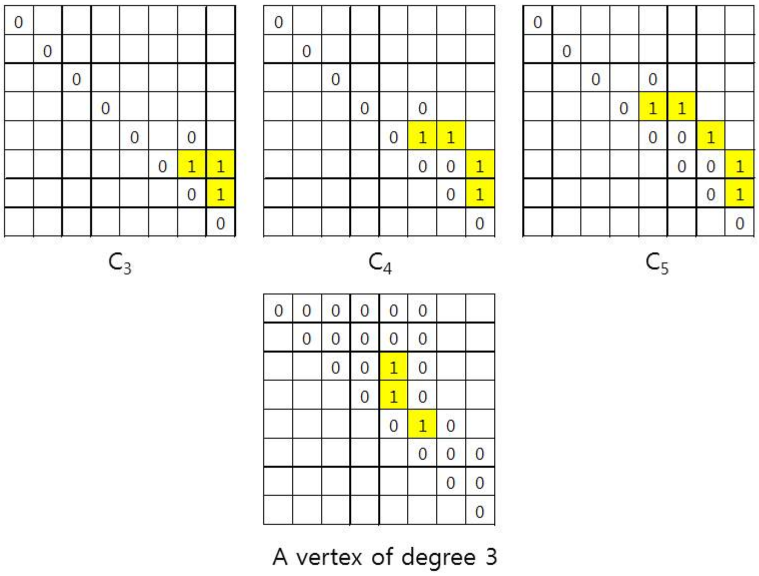

FIGURE 2. The two-diagonal matrices that have sequences which represent the cycles or vertices of degree three

Lemma 4. *A two-diagonal matrix of order n is non-optimal if and only if there is at least one 1 at the second diagonal line and there is no sequence which represents any cycle or vertex of degree greater than or equal to three.*

Proof. Suppose a two-diagonal matrix of order n is non-optimal. It means that it represents a ordered pair (G, f) of a graph G and bijection $f : V(G) \to [n]$ such that $bw(G) = 1$ but $bw(G, f) = 2$. By Theorem 3, the graph G with a proper f has K_2 as a subgraph but no 2-irreducible graph as a subgraph. That means there is no sequence which represents a cycle, C_n, $n \geq 3$ or vertex of degree greater than or equal to three which is $K_{1,3}$ in the two-diagonal matrix. Since $bw(G, f) = 2$, it also has at least one 1 at the second diagonal line of the two-diagonal matrix. Conversely, suppose there is at least one 1 at the second diagonal line and there is no sequence which represents a cycle or vertex of degree greater than or equal to three in the two-diagonal matrix A_f. The existence of at least one 1 at the second diagonal line implies that $bw(G, f) = 2$ with the given labelling f of graph G where A_f is the adjacency matrix of G. Since there is no sequence which represents a cycle or vertex of degree greater than or equal to three in the two-diagonal matrix A_f, G does not have $K_{1,3}$ or C_n, $n \geq 3$ as a subgraph. That means G has no 2-irreducible graph as a subgraph by Theorem 2. Hence $bw(G) = 1$ by Theorem 3. So this two-diagonal matrix is an adjacency matrix of graph G with an bijection f such

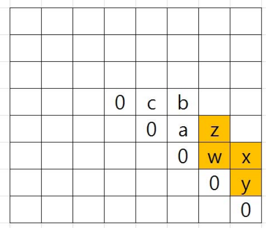

FIGURE 3. The positions of x, y, z, w, a, b, and c for b_n^{ij}

that an ordered pair (G, f) of a graph G and bijection $f : V(G) \to [n]$ satisfies $bw(G) = 1$ but $bw(G, f) = 2$. □

Here we count the number of graphs of bandwidth 0 or 1 and the graphs of which the adjacency matrix are non-optimal. By Theorem 3 and Lemma 4, the graphs of bandwidth 0 or 1 and the graphs of which the adjacency matrix are non-optimal of order n must be a linear forest, that is, it must be acyclic and have maximum degree less than 3. Let a_n be the number of graphs of labelled linear forests of order n. And let b_n^{ij} be the number of graphs of order n in which the vertex $n-1$ has degree 0 or 1 if $i = 1$ and degree 2 if $i = 2$, and in which the vertex n has degree 0 or 1 if $j = 1$ and degree 2 if $j = 2$. Let $A(x) = \sum a_n x^n$, $B_{11}(x) = \sum b_n^{11} x^n$, $B_{12}(x) = \sum b_n^{12} x^n$, $B_{21}(x) = \sum b_n^{21} x^n$, and $B_{22}(x) = \sum b_n^{22} x^n$. Note that

$$A(x) = 1 + x + B_{11} + B_{12} + B_{21} + B_{22} \tag{3}$$

with $a_0 = 1$ and $a_1 = 1$ for convenience.

Note that we deal with the two-diagonal matrices of bandwidth less than or equal to two, hence the 3-irreducible graphs are not necessary for our consideration. Suppose $M = [m_{i,j}]$ is the adjacency matrix of a graph of order n counted by b_n^{11}. Let $x = m_{n-2,n}$, $y = m_{n-1,n}$, $z = m_{n-3,n-1}$, $w = m_{n-2,n-1}$, $a = m_{n-3,n-2}$, $b = m_{n-4,n-2}$, and $c = m_{n-4,n-3}$ in the matrix M (See Figure 3).

Here to obtain the adjacency matrix of graph of order n counted by b_n^{11} one may append the $(n-1)$-th and n-th column and row to the adjacency matrix of graph of order $n-2$ so as to maintain the acyclic and minimum degree 2. The graphs that are counted by b_n^{11} are obtained from the graphs of order $n-2$ or $n-3$ by the following several subcases.

1) Some graphs in which the vertex $n-1$ and the vertex n have degree 0 are obtained from those counted by a_{n-2}. That is $x = 0$, $y = 0$, $z = 0$, and $w = 0$.
2) Some graphs in which the vertex $n-1$ has degree 0 and the vertex n is adjacent to vertex $n-1$ only are obtained from those counted by a_{n-2}. That is $x = 0$, $y = 1$, $z = 0$, and $w = 0$.

3) Some graphs in which the vertex $n-1$ and the vertex n are adjacent to vertex $n-2$ are obtained from those counted by a_{n-3}. That is $x=1$, $y=0$, $z=0$, and $w=1$.
4) Some graphs in which the vertex $n-1$ is adjacent to vertex $n-2$ only and the vertex n have degree 0 are obtained from those counted by b^{11}_{n-2} and b^{21}_{n-2}. That is $x=0$, $y=0$, $z=0$, and $w=1$.
5) Some graphs in which the vertex $n-1$ is adjacent to vertex $n-3$ only and the vertex n have degree 0 are obtained from those counted by b^{11}_{n-2} and b^{12}_{n-2}. That is $x=0$, $y=0$, $z=1$, and $w=0$.
6) Some graphs in which the vertex $n-1$ have degree 0 and the vertex n is adjacent to vertex $n-2$ only are obtained from those counted by b^{11}_{n-2} and b^{21}_{n-2}. That is $x=1$, $y=0$, $z=0$, and $w=0$.
7) Some graphs in which the vertex $n-1$ is adjacent to vertex $n-3$ only and the vertex n is adjacent to vertex $n-2$ only are obtained from those counted by b^{11}_{n-2}. That is $x=1$, $y=0$, $z=1$, and $w=0$.

Since a linear forest should be acyclic and have maximum degree less than 3, the asset position a, b, and c cannot have some set of values. Hence we have

$$b^{11}_n = 2a_{n-2} + a_{n-3} + 4b^{11}_{n-2} + 2b^{21}_{n-2} + b^{12}_{n-2}, \quad n \geq 4. \tag{4}$$

Equivalently,

$$B_{11} = (2x^2 + x^3)A + 4x^2 B_{11} + 2x^2 B_{21} + x^2 B_{12} + 2x^3 \tag{5}$$

with the boundary condition $b^{11}_3 = 5$.

Similarly, the graphs that are counted by b^{12}_n are obtained from the graphs of order $n-2$ by appending the $(n-1)$-th and n-th columns and rows to the adjacency matrix of graph of order $n-2$ with the values $x=1$, $y=1$, $z=0$, and $w=0$ so as to maintain the acyclic and minimum degree 2. The graphs counted by b^{11}_{n-2} and b^{21}_{n-2} contribute to the number b^{12}_n. But b^{12}_{n-2} causes graphs in which a vertex of degree 3 arise. Therefore we have

$$b^{12}_n = b^{21}_{n-2} + b^{11}_{n-2}, \quad n \geq 4. \tag{6}$$

Equivalently,

$$B_{12} = x^2 B_{21} + x^2 B_{11} + x^3 \tag{7}$$

with the boundary condition $b^{12}_3 = 1$.

Here to obtain the adjacency matrix of graph of order n counted by b^{21}_n one may append the n-th column and row to the adjacency matrix of graph of order $n-1$ so as to maintain the acyclic and minimum degree 2. There are 4 subcases to consider which are b^{22}_{n-1}, b^{21}_{n-1}, b^{12}_{n-1}, and b^{11}_{n-1}.

Subcase b^{22}_{n-1}: When we attach n-th column as $(x,y)=(1,1)$, $(x,y)=(1,0)$, and $(x,y)=(0,1)$ cannot generate the matrix counted by b^{21}_n. But $(x,y)=(0,0)$ generates matrices counted by b^{21}_n.

Subcase b^{21}_{n-1}: When we attach n-th column as $(x,y)=(1,1)$, $(x,y)=(1,0)$, and $(x,y)=(0,0)$ cannot generate the matrix counted by b^{21}_n. But $(x,y)=$

$(0,1)$ generates matrices counted by b_n^{21} except the matrices which have all zero $(n-1)$-th column.

Subcase b_{n-1}^{12}: When we attach n-th column as $(x,y)=(1,1)$, and $(x,y)=(0,1)$ cannot generate the matrix counted by b_n^{21}. But $(x,y)=(1,0)$ and $(x,y)=(0,0)$ generate matrices counted by b_n^{21}.

Subcase b_{n-1}^{11}: When we attach n-th column as $(x,y)=(1,1)$, $(x,y)=(1,0)$, and $(x,y)=(0,0)$ cannot generate the matrix counted by b_n^{21}. But $(x,y)=(0,1)$ generates matrices counted by b_n^{21} except the matrices which have all zero $(n-1)$-th column.

In all cases above, the matrices which have all zero $(n-1)$-th column are counted by a_{n-2} so we have

$$b_n^{21}=b_{n-1}^{21}+b_{n-1}^{22}+2b_{n-1}^{12}+b_{n-1}^{11}-a_{n-2},\quad n\geq 3. \tag{8}$$

Equivalently,

$$B_{21}=xB_{21}+xB_{22}+2xB_{12}+xB_{11}-x^2A+x^2 \tag{9}$$

with the boundary condition $b_2^{21}=0$.

Here to obtain the adjacency matrix of graph of order n counted by b_n^{22} one may append the $(n-1)$-th and n-th column and row to the adjacency matrix counted by b_{n-2}^{11} of order $n-2$ with the values $x=1$, $y=1$, $z=1$, and $w=0$ so as to maintain the acyclic and minimum degree 2. But there are matrices which has a sequence denoting cycle on it and they are counted by

$$\sum_{i=0}^{n-4}a_i.$$

Hence we have

$$b_n^{22}=b_{n-2}^{11}-\sum_{i=0}^{n-4}a_i. \tag{10}$$

Equivalently,

$$B_{22}=x^2B_{11}-\frac{x^4A}{1-x}. \tag{11}$$

Solving the system of Equations (3), (5), (7), (9), and (11) gives us

$$\begin{aligned}&(-1+2x+4x^2-4x^3+4x^4-6x^5+4x^6)A(x)\\ =\;&-1+x+4x^2-3x^3-3x^5+2x^6.\end{aligned} \tag{12}$$

Or

$$\begin{aligned}&(-1+4x-4x^2+4x^3-4x^4+2x^5)A(x)\\ =\;&-1+3x-2x^2+x^3-2x^4+x^5.\end{aligned} \tag{13}$$

Then we can have the recurrence relation for a_n by extracting the coefficient of x^n as follows.

Theorem 5. *The recurrence relation for the number of bandwidth* 0 *and* 1 *graphs and the graphs of which the adjacency matrix are non-optimal of order* n, a_n, *is*

$$a_n = 4a_{n-1} - 4a_{n-2} + 4a_{n-3} - 4a_{n-4} + 2a_{n-5} \tag{14}$$

with the boundary conditions $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 7$, $a_4 = 22$, *and* $a_5 = 65$.

Theorem 6. *Let* $Bw_2(n)$ *be the number of optimally labelled graphs of order* n *and bandwidth* 2. *Then we have*

$$Bw_2(n) = 2^{2n-3} - a_n. \tag{15}$$

The corresponding numbers are given in Table 1.

The partial fraction method provides an asymptotic expression for the number a_n. From Equation (13), we have

$$\begin{aligned} A(x) &= \frac{-1+3x-2x^2+x^3-2x^4+x^5}{-1+4x-4x^2+4x^3-4x^4+2x^5} \\ &= \frac{1}{2} - \frac{-1/2+x-x^3}{-1+4x-4x^2+4x^3-4x^4+2x^5} \\ &= \frac{1}{2} - \frac{0.0924327...}{-0.335209...+x} \\ &\quad + \frac{0.126302... - 0.366435...x}{2.04547... - 3.03653...x + 1.41421...x^2} \\ &\quad + \frac{0.237007... + 0.497155...x}{1.45845... + 0.682159...x + 1.41421...x^2}. \end{aligned} \tag{16}$$

One can find that

$$a_n = Cr^n(1 + O(\rho^n)), \tag{17}$$

where $C = 0.275746...$ and $r = 2.98321...$ by the term

$$\frac{-0.0924327...}{-0.335209...+x}, \tag{18}$$

and

$$\rho = r/0.984717558361022906 = 0.330086260284945543888 \text{ and } 0.984717558361022906$$

is the growth rate of the next fastest growing term, and other terms are slower growing as n tends to infinity. Here r is the only real root of characteristic equation of Equation (14). Therefore by Equation (15) we have

$$Bw_2(n) = (2^{2n-3} - Cr^n)(1 + O((0.984717558361022906/4)^n)). \tag{19}$$

Acknowledgement. We thank the anonymous referee for remarks that showed us the alternative method to find the numbers and the asymptotic result.

Table 1. Number of optimally labelled graphs of bandwidth 2 and a_n

n	$Bw_2(n)$	a_n
3	1	7
4	10	22
5	63	65
6	318	194
7	1468	580
8	6462	1730
9	27608	5160
10	115678	15394
11	478364	45924
12	1960152	137000
13	7979908	408700
14	32335192	1219240
15	130580476	3637252
16	526020216	10850696
17	2115113712	32369936
18	8493368184	96566408
19	34071660224	288078144
20	136579555064	859398408

References

[1] A. Caprara, F. Malucelli, and D. Pretolani, *On bandwidth-2 graphs*, Discrete Appl. Math. **17** (2002), no. 1-3, 1–13.

[2] G.-B. Chae, M. Cheong, and S.-M. Kim, *Irreducible posets of linear discrepancy* 1 *and* 2, Far East J. Math. Sci. (JMS) **22** (2006), no. 2, 217–226.

[3] ———, *Graph bandwidth with irreducibility*, Preprint.

Gab-Byung Chae
Division of Mathematics and Informational Statistics
Wonkwang University
Iksan 54538, Korea
E-mail address: rivendell@wonkwang.ac.kr

MinSeok Cheong
College of Informatics, Information Security Convergence
Korea University
Seoul 02841, Korea
E-mail address: toset@hanmail.net

Sang-Mok Kim
Department of Mathematics
Kwangwoon University
Seoul 01897, Korea
E-mail address: smkim@kw.ac.kr