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ENUMERATION OF GRAPHS WITH GIVEN WEIGHTED NUMBER OF CONNECTED COMPONENTS

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Abstract. We give a generating function for the number of graphs with given numerical properties and prescribed weighted number of connected components. As an application, we give a generating function for the number of q-partite graphs of given order, size and number of connected components.

1. Introduction

In this paper, we consider the generating function for the number of graphs with prescribed numerical properties and the *weighted number* of connected components: Given a weight vector $\omega = (\omega_1, \omega_2, \dots) \in \mathbb{Q}^\infty$, the ω*-weighted number of connected components* of a graph G with connected components $G_1, \dots, G_s$ is defined to be

$$h_0^\omega(G) := \sum_{i=1}^{\infty} \omega_i \mathrm{Card}\{j \mid i = |G_j|\}.$$

Here, $|G_j|$ means the order of the component G_j and $\mathrm{Card}\{j \mid i = |G_j|\}$ is the number of connected components of order i, so $h_0^\omega(G)$ counts the number of connected components weighted by ω_i. For instance, when ω is the uniform trivial weight $(1, 1, \dots,)$, $h_0^\omega(G)$ is just the number of connected components. The notion of weighted number of components frequently arises in many areas of Mathematics, such as groups or bundle discounts. More complex and sophisticated applications may be found in network analysis.

The goal of this paper is to give a generating function for the number of graphs of given order, size, and h_0^ω. In fact, order, size can be replaced by any homogeneous properties (Definition 2). Our method is a slight modification of the exponential formula in [12, Section 5.1]. We first introduce an auxiliary

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multivariate exponential generating function (Section 3, Equation (1)) which enumerates the number of graphs with prescribed number of connected components of given order. Next, we use a ring homomorphism τ_ω to find the desired generating function (Definition 3).

Let f_i be additive functions on graphs (Definition 1) and $\mathcal{P}$ be a collection of homogeneous properties. Our main theorem, which will be proved in Section 3, can be stated as following:

Theorem. *The number of graphs with properties $\mathcal{P}$, given f_i values and h_0^ω is generated by*

$$\tau_\omega\left(\exp\left(\sum_{n,k_i} g_{n,k_1,\dots,k_s}\frac{1}{n!}x^n\prod_{i=1}^{s} y_i^{k_i} z_n\right)\right),$$

where $g_{n,k_1,\dots,k_s}$ is the number of connected graphs with properties $\mathcal{P}$ and given numerical values of f_i, $i = 1,\dots,s$, and τ_ω is the ring homomorphism determined by mapping $\prod z_i^{\alpha_i}$ to $z^{\sum \omega_i\alpha_i}$.

As an application, we shall enumerate the number $c^q_{n,k,\nu}$ (resp. $c'^q_{n,k,\nu}$) of q-colored (resp. q-partite) graphs of order n, size k with ν connected component (Proposition 1), and give a numerical table of bipartite graphs in the Appendix A computed by using Mathematica. In this numerical table, we specifically choose the uniform trivial weight $\omega = (1, 1, 1, \dots)$ since it will be used in our forthcoming paper [10]. We remark that, although enumeration of bipartite graphs have been studied by many authors ([2–5, 12, 13]), our search did not turn up a table of $c'^2_{n,k,\nu}$.

Throughout this paper, graphs are assumed to be labeled.

2. The exponential formula

In this section, we quickly review the exponential formula for generating functions [12, Section 5.1] and give a few variant forms of it that will be useful for our purpose here and in [10].

Definition 1. We say that an integer valued function f on the set of graphs is *additive* if $f(G + G') = f(G) + f(G')$ for any two graphs G, G' with disjoint vertex sets.

For example, order, size and rank (of the incidence matrix) are all additive.

Definition 2. We say that a graph property P is *homogeneous* if G satisfies P then all of its components satisfy P.

Let $f_1, \dots, f_s$ be additive functions on graphs. Say one is interested in enumerating graphs with given order and given values of f_i, $i = 1, \dots, s$. Then the exponential formula allows one to relate the total number of graphs with the desired numerical properties and the number of connected graphs with the same properties. Let $[n] = \{1, 2, \dots, n\}$.

Theorem 1 ([7, Theorem 1], [12, Section 5.1]). *Let P be a homogeneous graph property. Let $g_{n,k_1,\dots,k_s}$ (resp. $\bar{g}_{n,k_1,\dots,k_s}$) be the number of graphs (resp. connected graphs) G on $[n]$ with property P and $f_i(G) = k_i$, for all i. Then the generating function for $g_{n,k_1,\dots,k_s}$ is given by*

$$g(x, y_1, \dots, y_s) = \exp\left(\sum \bar{g}_{n,k_1,\dots,k_s} \frac{1}{n!} x^n y_1^{k_1} \cdots y_s^{k_s}\right)$$

or equivalently, the generating function for $\bar{g}_{n,k_1,\dots,k_s}$ is given by the formal logarithm

$$\overline{g}(x, y_1, \dots, y_s) = \log\left(\sum g_{n,k_1,\dots,k_s} \frac{1}{n!} x^n y_1^{k_1} \cdots y_s^{k_s}\right).$$

By using this theorem, one can sort out unwanted components when enumerating graphs. Suppose that the generating function $g(x, y_1, \dots, y_s)$ is known and we want to enumerate the graphs with the same numerical features but without certain components.

Corollary 1. *Let $\mathcal{I} \subset \mathbb{Z}^{s+1}$. The number of graphs of given order and given values of f_i without a connected component of order n^* and f_i value of k_i^*, $\forall i \in \mathcal{I}$, is generated by*

$$\exp\left(\log\left(g(x, y_1, \dots, y_s) - \sum_{(n^*, k_i^*) \in \mathcal{I}} \overline{g}_{n^*, k_1^*, \dots, k_s^*} \frac{1}{n^*!} x^{n^*} y_1^{k_1^*} \cdots y_s^{k_s^*}\right)\right).$$

Proof. By construction, the $x^n \prod y_i^{k_i}$-coefficient comes from the partitions of n and k_i that do not involve n^* and k_i^*. The assertion follows immediately. $\square$

In particular, the number of graphs with desired numerical properties but without an isolated vertex is generated by

$$\exp\left(\log\left(g(x, y_1, \dots, y_s) - \left(\left.\frac{\partial g}{\partial x}\right|_{x=0}\right) x\right)\right).$$

3. Graphs with a given number of weighted connected components

We shall use the multi-vector notation and simply write $g(x, y)$ for $g(x, y_1, \dots, y_s)$, y^k for $\prod_{i=1}^s y_i^{k_i}$ and so on. Suppose we have the generating function

$$g(x, y) = \sum_{n \geq 1, k_i} g_{n,k_1,\dots,k_s} \frac{1}{n!} x^n \prod_{i=1}^s y_i^{k_i}$$

for the number of connected graphs with certain homogeneous properties $\mathcal{P}$ and given numerical values of f_i, $i = 1, \dots, s$.

To control the weighted number of connected components, we define an auxiliary generating function

(1)

$$\exp\ \tilde{g}(x,y,z) = \exp \sum_{n,k_i} g_{n,k_1,\ldots,k_s} \frac{1}{n!} x^n \prod_{i=1}^{s} y_i^{k_i} z_n \in \mathbb{Q}[x, y_1, \ldots, y_s, z_1, z_2, z_3 \ldots],$$

which involves infinitely many variables z_i, $i \in \mathbb{N}$. The upshot is that the auxiliary z_i's allow one to keep track of the orders of connected components.

Definition 3. Let $\omega = (\omega_1, \omega_2, \omega_3, \ldots) \in \mathbb{Q}^\infty$ be a finite weight vector, i.e., $\omega_i = 0$ except for finitely many i. Let $\tau_\omega : \mathbb{Q}[z_1, z_2, z_3, \ldots] \to \mathbb{Q}(z)$ be the ring homomorphism determined by

$$z^\alpha := \prod z_i^{\alpha_i} \mapsto z^{\omega.\alpha} = z^{\sum \omega_i \alpha_i}.$$

Since $z^\alpha z^\beta = z^{\alpha+\beta}$, it is easy to see that this is a well defined ring homomorphism.

Theorem 2. *The number of graphs with properties $\mathcal{P}$, given f_i values and h_0^ω is generated by $\tau_\omega(\exp \tilde{g})$ where $\exp \tilde{g}(x,y,z)$ is obtained from $g(x,y)$ by using Equation* (1) *above. That is, if*

$$\tau_\omega(\exp \tilde{g}) = \sum T_{n,k,\nu} \frac{1}{n!} x^n y^k z^\nu,$$

then $T_{n,k,\nu}$ is precisely the number of graphs with properties $\mathcal{P}$, f_i-value k_i and $h_0^\omega = \nu$.

Proof. We shall give a proof for the case $s = 1$ for the sake of simplicity. The general case can be treated in a similar manner, but it is just more cumbersome to write. Fix ν and choose a set of partitions

(2)

$$n = \textstyle\sum_{i=1}^{\ell} (\sum_j m_{ij}) n_i, \quad k = \sum_{i,j} m_{ij} k_{ij}, \quad \text{such that } \nu = \sum_i (\omega_i \sum_j m_{ij}),$$
$$n_1 < n_2 < \cdots < n_\ell \text{ and } k_{i1} < k_{i2} < \cdots,\ \forall i.$$

We first count the number of graphs with the properties $\mathcal{P}$ which has precisely m_{ij} connected components of order n_i and numerical value k_{ij}. We first choose $m_i = \sum_j m_{ij}$ many sets V_{ijr} of vertices of cardinality n_i out of $[n]$. Note that we have triply indexed the vertex sets since for each fixed i and j, we need to choose m_{ij} many sets of vertices, i.e., $V_{ij1}, V_{ij2}, \ldots, V_{ijm_{ij}}$. There are $\binom{n}{\underbrace{n_1,\ldots,n_1}_{m_1},\ldots,\underbrace{n_\ell,\ldots,n_\ell}_{m_\ell}}$ ways to do this, where $\binom{n}{k_1,k_2,\ldots,k_m} = \frac{n!}{k_1!\cdots k_m!}$ is the multinomial coefficient defined by

$$(x_1 + \cdots + x_m)^n = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{i=1}^{m} x_i^{k_i}.$$

Once we have chosen V_{ijr}, on each V_{ijr}, there are $g_{n_i,k_{ij}}$ many ways to draw the graphs (connected components) on V_{ijr} with the desired properties. If the last index of $V_{ij1}, \ldots, V_{ijm_{ij}}$ mattered, then we would simply have $g_{n_i,k_{ij}}^{m_{ij}}$ many graphs on V_{ijr}. It does not, so we divide out by $m_{ij}!$ to remove the repetition. All in all, there are

$$\sum \binom{n}{\underbrace{n_1, \ldots, n_1}_{m_1}, \ldots, \underbrace{n_\ell, \ldots, n_\ell}_{m_\ell}} \frac{1}{\prod_{i,j} m_{ij}!} \prod g_{n_i,k_{ij}}^{m_{ij}} \tag{3}$$

many graphs with ℓ components, where the sum runs over all partitions of n and k as in Equation (2) (note that ℓ is fixed here).

On the other hand, consider $\exp \tilde{g} = \prod \exp\left(g_{n,k} \frac{1}{n!} z_n y^k x^n\right)$. We shall compute its $\prod(z_{n_i}^{m_i}) y^k x^n$-coefficient. By writing each exponential factor in $\exp \tilde{g}$ as a power series

$$\prod \left(1 + (g_{n',k'} \frac{1}{n'!} y^{k'} x^{n'}) + \frac{1}{2!}(g_{n',k'} \frac{1}{n'!} y^{k'} x^{n'})^2 + \frac{1}{3!}(g_{n',k'} \frac{1}{n'!} y^{k'} x^{n'})^3 + \cdots\right)$$

and expanding the product, we see that a term $\prod(z_{n_i}^{m_i}) y^k x^n$ appears when some of the factors of the product contribute $(z_{n_i} y^{k_{ij}} x^{n_i})^{m_{ij}}$ such that $n = \sum_{i,j} n_i m_{ij}$, $k = \sum_{i,j} k_{ij} m_{ij}$ and all other factors contribute 1.

Hence the $\prod(z_{n_i} y^{k_{ij}} x^{n_i})^{m_{ij}}$-coefficient of $\exp \tilde{g}$ is simply the sum of the product of the coefficients of $z_{n_i} y^{k_{ij}} x^{n_i}$

$$\sum \prod \frac{1}{m_{ij}!} \left(\frac{1}{(n_i)!} g_{n_i,k_{ij}}\right)^{m_{ij}} = \sum \frac{1}{(n_1!)^{m_1} \cdots (n_\ell!)^{m_\ell}} \prod \frac{1}{m_{ij}!} \prod g_{n_i,k_{ij}}^{m_{ij}}, \tag{4}$$

where the sum runs over all partitions $n = \sum_{i,j} n_i m_{ij}, k = \sum_{i,j} m_{ij} k_{ij}$ as in the first two equations of Equation (2).

Let τ_ω be as in Definition 3 but extended to the rings with $\mathbb{Q}$ replaced by $\mathbb{Q}[x, y_1, \ldots, y_s]$ in the obvious manner. We have

$$\begin{aligned} \tau_\omega \left(\prod (z_{n_i} y^{k_{ij}} x^{n_i})^{m_{ij}}\right) &= z^{\sum_i (\omega_i \sum_j m_{ij})} \prod y^{\sum m_{ij} k_{ij}} x^{\sum m_{ij} n_i} \\ &= z^{\sum_i (\omega_i \sum_j m_{ij})} y^k x^n. \end{aligned}$$

Hence the $x^n y^k z^\nu$-coefficient of $\tau_\omega(\exp \tilde{g})$ is

$$\sum \left(\prod (z_{n_i} y^{k_{ij}} x^{n_i})^{m_{ij}} \text{ coefficient of } \exp \tilde{g}\right)$$

where the sum runs over all partitions as in Equation (2). Substituting Equation (4), we obtain

$$T_{n,k,\nu} = \sum \frac{n!}{(n_1!)^{m_1} \cdots (n_\ell!)^{m_\ell}} \prod \frac{1}{m_{ij}!} \prod g_{n_i,k_{ij}}^{m_{ij}}$$

which precisely equals Equation (3). $\square$

4. Applications: q-colored and q-partite graphs of given order, size and rank

Our motivation for studying the enumerative combinatorics of bipartite graphs comes from certain hyperplane arrangements [9, 11], but they are certainly interesting on their own [2–5, 8, 13] due to the intricacy of the enumeration and the beautiful theory of generating functions exploited to overcome the difficulty. There are some interesting applications as well: bipartite blocks in which the color classes are of equal size appear in the study of a modified penetrable sphere model of liquid-vapor equilibrium [6]. It is then further related to statistical mechanics as investigated in [1].

We shall give a generalization of some of these earlier works and enumerate q-colored graphs (as well as q-partite graphs) of given order, size and rank. As a special case, we shall obtain a method for enumerating bipartite graphs of given order, size and rank.

Definition 4. (1) A graph is *q-colored* if each of its vertices is assigned precisely one of the colors $c_1, \dots, c_q$ so that no two adjacent vertices have the same color.
(2) A graph is *q-partite* if it admits a q-coloring or equivalently, its vertices can be partitioned into q different independent sets.

Proposition 1. *The number of connected q-colored graphs of given order and size is generated by the formal logarithm*

$$C_q(x, y) = \log\left(1 + \sum_{n\geq 1, k\geq 0}\left(\sum_{i_1+\cdots+i_q=n}\binom{n}{i_1, i_2, \dots, i_q}\binom{\sum_{1\leq i_s<i_t\leq q} i_s i_t}{k}\frac{1}{n!}x^n y^k\right)\right)$$

where the second sum runs over all partitions $i_1+i_2+\cdots+i_q = n$. The number of connected q-partite graphs of given order and size is generated by $\frac{1}{q!}C_q$.

Proof. We shall first enumerate the number of q-colored graphs of order n and size k. We start with the set V of n labeled vertices. To construct a q-colored graph on V of size k, we first decompose V into q blocks and assign q colors: There are $\binom{n}{i_1, i_2, \dots, i_q}$ ways to decompose V such that $i_1, \dots, i_q$ are the numbers of vertices which have colors $c_1, \dots, c_q$, respectively. Now we draw k edges. By definition, there are no edges between two vertices of the same color. So we need to consider the number of possible edges between blocks of different colors, say c_1 and c_2: We can draw any many as $i_1 i_2$ edges. In total, there are $\sum_{1\leq s<t\leq q} i_s i_t$ possible edges, and we choose k edges among them. Hence the total number of q-colored graphs of size k which has i_s vertices of color c_s is

$$\binom{n}{i_1,\, i_2,\, \dots,\, i_q}\binom{\sum_{1\leq s<t\leq q} i_s i_t}{k}.$$

The first assertion of the proposition now follows from the exponential formula (Theorem 1).

Since a connected q-partite graph admits precisely $q!$ many q-colorings, the number of connected q-partite graphs of given order and size is exactly $\frac{1}{q!}$ of the number of connected q-colorable graphs of the same order and size. $\square$

By applying Theorem 2 to $C_q(x, y)$, we obtain:

Corollary 2. *The number of q-colored graphs of given order, size and number of connected components is given by*

$$\tau_\omega(\exp \widetilde{C_q}(x, y, z))$$

with $\omega = (1, 1, \dots)$. That is, if

$$\tau_\omega(\exp \widetilde{C_q}) = \sum c^q_{n,k,\nu} \frac{1}{n!} x^n y^k z^\nu,$$

then $c^q_{n,k,\nu}$ is precisely the number of q-colored graphs of order n, size k with ν connected components. Similarly, if

$$\tau_\omega(\exp \frac{1}{q!} \widetilde{C_q}) = \sum c'^q_{n,k,\nu} \frac{1}{n!} x^n y^k z^\nu,$$

with $\omega = (1, 1, \dots)$, then $c'^q_{n,k,\nu}$ is precisely the number of q-partite graphs of order n, size k with ν connected components.

This generalizes the main theorem in [8] which enumerates the q-colored graphs on n nodes (with any number of edges and components).

Taking $q = 2$, we obtain the number of connected bi-colored graphs of given order and size. It is generated by the formal logarithm

$$\log \left(1 + \sum_{n \geq 1, k \geq 0} \left(\sum_i \binom{n}{i} \binom{i(n-i)}{k} \frac{1}{n!} x^n y^k \right) \right).$$

Let $\bar{b}_{n,k}$ denote the number of connected bipartite graphs of order n and size k. We have

$$\begin{aligned} \mathcal{B}(x, y) &:= \sum_{n \geq 0, k \geq 0} \bar{b}_{n,k} \frac{1}{n!} x^n y^k \\ &= \frac{1}{2} \log \left(1 + \sum_{n \geq 1, k \geq 0} \left(\sum_i \binom{n}{i} \binom{i(n-i)}{k} \frac{1}{n!} x^n y^k \right) \right) \end{aligned}$$

and from Corollary 2, we know that if

$$\tau_\omega(\exp \tilde{\mathcal{B}}) = \sum b_{n,k,\nu} \frac{1}{n!} x^n y^k z^\nu,$$

then $b_{n,k,\nu}$ is precisely the number of bipartite graphs of order n, size k with ν connected components. The upshot is that the generating function $\tau_\omega(\exp \tilde{B})$ can be easily computed by computer algebra systems such as Mathematica. In the subsequent section, we shall list the first 100 or so terms of the generating function (order of the graph up to 10).

Appendices

A. The generating function

The generating function for the number of bipartite graphs of given order, size and number of connected components as computed by Mathematica.

$$
\begin{aligned}
&\tau_\omega(\exp\tilde{\mathcal{B}})\\
&= 1 + \tfrac{1}{2}x^2yz + \tfrac{1}{2}x^3y^2z + x^4\Big(\tfrac{y^2z^2}{8} + \tfrac{2y^3z}{3} + \tfrac{y^4z}{8}\Big) + x^5\Big(\tfrac{y^3z^2}{4} + \tfrac{25y^4z}{24} + \tfrac{y^5z}{2}\\
&\quad + \tfrac{y^6z}{12}\Big) + x^6\Big(\tfrac{y^3z^3}{48} + \tfrac{11y^4z^2}{24} + y^5\Big(\tfrac{z^2}{16} + \tfrac{9z}{5}\Big) + \tfrac{19y^6z}{12} + \tfrac{2y^7z}{3} + \tfrac{7y^8z}{48} + \tfrac{y^9z}{72}\Big)\\
&\quad + x^7\Big(\tfrac{y^4z^3}{16} + \tfrac{41y^5z^2}{48} + y^6\Big(\tfrac{5z^2}{16} + \tfrac{2401z}{720}\Big) + y^7\Big(\tfrac{z^2}{24} + \tfrac{55z}{12}\Big) + \tfrac{10y^8z}{3} + \tfrac{37y^9z}{24}\\
&\quad + \tfrac{37y^{10}z}{80}\Big) + x^8\Big(\tfrac{zy^{16}}{1152} + \tfrac{11zy^{15}}{720} + \tfrac{zy^{14}}{8} + \tfrac{91zy^{13}}{144} + \tfrac{1583zy^{12}}{720} + \tfrac{1327zy^{11}}{240}\\
&\quad + \Big(\tfrac{z^2}{144} + \tfrac{491z}{48}\Big)y^{10} + \Big(\tfrac{7z^2}{96} + \tfrac{124z}{9}\Big)y^9 + \Big(\tfrac{49z^2}{128} + \tfrac{611z}{48}\Big)y^8 + \Big(\tfrac{9z^2}{8} + \tfrac{2048z}{315}\Big)y^7\\
&\quad + \Big(\tfrac{z^3}{64} + \tfrac{1183z^2}{720}\Big)y^6 + \tfrac{7z^3y^5}{48} + \tfrac{z^4y^4}{384}\Big) + x^9\Big(\tfrac{zy^{20}}{2880} + \tfrac{zy^{19}}{144} + \tfrac{143zy^{18}}{2160} + \tfrac{2zy^{17}}{5}\\
&\quad + \tfrac{2471zy^{16}}{1440} + \tfrac{133zy^{15}}{24} + \tfrac{140281zy^{14}}{10080} + \tfrac{1}{9}\Big(\tfrac{z^2}{32} + \tfrac{9947z}{40}\Big)y^{13} + \tfrac{1}{9}\Big(\tfrac{3z^2}{8} + \tfrac{31357z}{80}\Big)y^{12}\\
&\quad + \tfrac{1}{9}\Big(\tfrac{343z^2}{160} + \tfrac{7779z}{16}\Big)y^{11} + \tfrac{1}{9}\Big(\tfrac{123z^2}{16} + \tfrac{14763z}{32}\Big)y^{10} + \tfrac{1}{9}\Big(\tfrac{305z^2}{16} + \tfrac{24903z}{80}\Big)y^9\\
&\quad + \tfrac{(140z^3+47670z^2+177147z)y^8}{13440} + \tfrac{(135z^3+4697z^2)y^7}{1440} + \tfrac{61z^3y^6}{192} + \tfrac{z^4y^5}{96}\Big)\\
&\quad + x^{10}\Big(\tfrac{1771zy^{20}}{720} + \tfrac{4129zy^{19}}{480} + \tfrac{2927zy^{18}}{120} + \tfrac{1}{10}\Big(\tfrac{5z^2}{1152} + \tfrac{164105z}{288}\Big)y^{17}\\
&\quad + \tfrac{1}{10}\Big(\tfrac{11z^2}{144} + \tfrac{2976223z}{2688}\Big)y^{16} + \tfrac{1}{10}\Big(\tfrac{5z^2}{8} + \tfrac{339671z}{189}\Big)y^{15} + \tfrac{1}{10}\Big(\tfrac{115z^2}{36} + \tfrac{38943z}{16}\Big)y^{14}\\
&\quad + \tfrac{1}{10}\Big(\tfrac{1097z^2}{96} + \tfrac{98255z}{36}\Big)y^{13} + \tfrac{1}{10}\Big(\tfrac{52303z^2}{1728} + \tfrac{89665z}{36}\Big)y^{12} + \tfrac{1}{10}\Big(\tfrac{z^3}{96} + \tfrac{3661z^2}{72}\\
&\quad + \tfrac{1}{8}\Big(\tfrac{z^2}{18} + \tfrac{491z}{6}\Big)z + \tfrac{16070z}{9}\Big)y^{11} + \tfrac{1}{10}\Big(\tfrac{7z^3}{64} + \tfrac{5783z^2}{72} + \tfrac{1}{8}\Big(\tfrac{7z^2}{12} + \tfrac{992z}{9}\Big)z\\
&\quad + \tfrac{16709z}{18}\Big)y^{10} + \tfrac{1}{10}\Big(\tfrac{17z^3}{32} + \tfrac{5743z^2}{72} + \tfrac{1}{8}\Big(\tfrac{49z^2}{16} + \tfrac{611z}{6}\Big)z + \tfrac{156250z}{567}\\
&\quad + \tfrac{5}{48}\big(z^3+50z^2\big) + \tfrac{1}{16}\big(z^3+110z^2\big)\Big)y^9 + \tfrac{(15330z^3+268559z^2)y^8}{40320} + \tfrac{(15z^4+3916z^3)y^7}{5760}\\
&\quad + \tfrac{17z^4y^6}{576} + \tfrac{z^5y^5}{3840}\Big) + \cdots .
\end{aligned}
$$

B. Number of bipartite graphs of given (order, size, number of components)

A missing triple (n, k, ν) (such as $(3, 1, 1)$) means there are no bipartite graphs of order n, size k with ν connected components.

2,1,1	3,2,1	4,3,1	4,4,1	4,2,2	5,4,1	5,5,1
1	3	16	3	3	125	60
5,6,1	5,3,2	6,5,1	6,6,1	6,7,1	6,8,1	6,9,1
10	30	1296	1140	480	105	10
6,10,1	6,4,2	6,5,2	6,3,3	7,6,1	7,7,1	7,8,1
0	330	45	15	16,807	23,100	16,800
7,9,1	7,10,1	7,5,2	7,6,2	7,7,2	7,4,3	8,7,1
7,770	2,331	4,305	1,575	210	315	262,144
8,8,1	8,9,1	8,10,1	8,6,2	8,7,2	8,8,2	8,9,2
513,240	555,520	412,440	66,248	45,360	15,435	2,940
8,10,2	8,5,3	8,6,3	8,4,4	9,8,1	9,9,1	9,10,1
280	5,880	630	105	4,782,969	12,551,112	18,601,380
9,7,2	9,8,2	9,9,2	9,10,2	9,6,3	9,7,3	9,8,3
1,183,644	1,287,090	768,600	309,960	115,290	34,020	3,780
9,5,4	10,9,1	10,10,1	10,8,2	10,9,2	10,10,2	10,7,3
3,780	100,000,000	336,853,440	24,170,310	37,948,680	34,146,000	2,467,080
10,8,3	10,9,3	10,10,3	10,6,4	10,7,4	10,5,5	
1,379,700	392,175	66,150	107,100	9,450	945	

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