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Effect of Three-dimensional Warping on Stiffness Constants of Closed Section Composite Beams

Manoj Kumar Dhadwal* and Sung Nam Jung**

Konkuk University, Seoul 05029, Republic of Korea

Abstract

This paper focuses on the investigation of three-dimensional (3D) warping effect on the stiffness constants of composite beams with closed section profiles. A finite element (FE) cross-sectional analysis is developed based on the Reissner's multifield variational principle. The 3D in-plane and out-of-plane warping displacements, and sectional stresses are approximated as linear functions of generalized sectional stress resultants at the global level and as FE shape functions at the local sectional level. The classical elastic couplings are taken into account which include transverse shear and Poisson deformation effects. A generalized Timoshenko level 6×6 stiffness matrix is computed for closed section composite beams with and without warping. The effect of neglecting the 3D warping on stiffness constants is shown to be significant indicating large errors as high as 93.3%.

Key words: Beam, Cross-section, Finite element model, Multifield principle

1. Introduction

In recent decades, a significant amount of research effort has been devoted to the accurate and efficient analysis of composite beams with various levels of refinements. Beams can be simply described as slender structures with one dimension much larger than the other two. In helicopters or wind turbines, they are primarily used to model and analyze rotor blades which are typically made of composite materials. These blades may exhibit various passive elastic couplings which can be favorable for the desired global behavior. A full 3D analysis of rotor blades requires extensive modeling and heavy computation. The alternative approach which is usually adopted is to decompose 3D beam analysis into a local two-dimensional (2D) sectional analysis and a global one-dimensional (1D) beam analysis [1]. The local 2D sectional analysis is a crucial step which involves treatment of classical elastic couplings, and nonclassical effects due to 3D warping displacements and boundary restraints. These effects are essential for accurate computation of sectional elastic properties which can be used to perform 1D beam static or dynamic analyses [2]. The proper recovery of 3D displacements, strains, and stresses is strongly dependent on the 2D sectional and 1D global beam analyses.

Several experimental as well as computational studies on the composite beams and blades can be found in the literature. The computational works can primarily be classified into analytical or FE based methods. Chandra and Chopra [3] studied experimentally and analytically the behavior of composite beams and blades. Berdichevsky et al. [4] proposed a variational asymptotic approach to refine the displacement approximations using an analytical based method. Jung et al. [2] developed an analytical shell-wall approach based on mixed force-displacement method including the effects of transverse shear and torsional restraints. Although the analytical shell-wall based approaches provide simple closed-form solutions, they are limited to thin- or thick-walled beams. A FE based analysis is applicable to arbitrary geometries and therefore more suited to generic beam sections. Giavotto et al. [5] proposed a FE analysis based on Saint-Venant (SV) central solutions without end effects along with extremity solutions using eigenmodes for end effects. Cesnik and Hodges [6] formulated a FE cross-sectional analysis called the variational asymptotic

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* Postdoctoral Researcher, Artificial Muscle Research Center
** Professor, Corresponding author: snjung@konkuk.ac.kr

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beam sectional analysis (VABS) based on the variational asymptotic method of Berdichevsky [7]. Kim and Kim [8] proposed a cross-sectional formulation based on a mixed variational principle with an asymptotic treatment of the stress field. They derived a Timoshenko-Vlasov level stiffness model considering transverse shear deformations and torsional restraint effects. Recently, a generalized refined displacement-based sectional analysis was proposed by Dhadwal and Jung [9] taking into account the 3D warping displacements and the boundary restraint effects due to nonuniform shear and torsional warping.

The present formulation is developed following the Reissner's multifield variational principle [10]. The proposed formulation is implemented into a FE program which is applicable for nonhomogeneous anisotropic beams with arbitrary geometries and material distributions. The unique features of the present formulation are: (a) the 3D warping displacements and transverse sectional stresses are modeled as unknown field variables. The other stress components are computed using Hooke's law and represented in terms of displacement derivatives in the kinematic relations; and (b) the 3D warping displacements and transverse sectional stresses are approximated as linear functions of generalized stress resultants (due to extension, shear, torsion, and bending) at the global beam level, and using FE shape functions at the local sectional level. This results in a generic nonlinear distribution of both 3D warping and reactive stresses over the beam cross-section. The present formulation incorporates the classical elastic couplings as well as nonclassical couplings due to transverse shear and Poisson's deformations. A 6×6 generalized stiffness matrix is subsequently obtained which incorporates Timoshenko-like model for transverse shear. The elastic constants are then computed considering the cases with and without the 3D warping. Note that the present multifield cross-sectional formulation is independent of the 1D beam analysis in that only stiffness constants are required to be provided to the latter which is similar to the displacement-based formulations. The importance of warping deformations for correct prediction of stiffness constants is investigated for elastically coupled composite beams with closed section profiles.

2. Multifield Beam Formulation

A brief description of the present formulation is discussed first. The beam is considered to be straight and prismatic. The volume of the beam can be described by extruding the cross-section along the reference line, which is an arbitrary curve in space. The beam reference line is aligned along ξ_1 axis. The schematic of the beam section is shown in Fig. 1.

2.1 Semi-inverted material constitutive relations

The constitutive relations for a linear elastic material in the material coordinate system can be defined using generalized Hooke's law as

$$\sigma_m = \mathbf{C}_m \varepsilon_m, \tag{1}$$

where σ_m is the stress vector, ε_m is the strain vector, and $\mathbf{C}_m$ is the material elastic constants matrix which may be fully populated for anisotropic materials. The constitutive relations in the beam coordinate system can be obtained by transforming the material coordinate system aligned along the fiber direction (θ_3, see Fig. 1) and the fiber plane coordinate system based from the fiber orientation angle (θ_1, see Fig. 1).

For the application of Reissner's multifield variational principle, the stresses and strains are expressed in a semi-inverted form through decomposition into sectional stresses acting on the beam section (normal and transverse shear stresses) and stresses acting on the planes normal to the beam section (normal and in-plane shear stresses), which gives

$$\begin{Bmatrix} \varepsilon_s^r \\ \sigma_n^a \end{Bmatrix} = \begin{bmatrix} \bar{\mathbf{C}}_{ss} & \bar{\mathbf{C}}_{sn} \\ -\bar{\mathbf{C}}_{sn}^T & \bar{\mathbf{C}}_{nn} \end{bmatrix} \begin{Bmatrix} \sigma_s^r \\ \varepsilon_n^a \end{Bmatrix}, \tag{2}$$

where the subscript s indicates the sectional stresses, subscript n represents the stresses on the planes normal to the beam section, the superscript a indicates the active components computed directly using Hooke's law, the superscript r indicates the reactive components, and $\bar{\mathbf{C}}_{ss}, \bar{\mathbf{C}}_{sn}, \bar{\mathbf{C}}_{nn}$ are the modified constitutive matrices.

2.2 Kinematics

The displacements $\mathbf{u}$ of an arbitrary material point on the beam section are defined as the sum of rotational and translational displacements $\mathbf{u}_b$ and the sectional warping

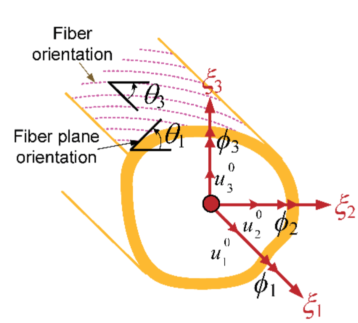

Fig. 1. Schematic of beam section

displacements Ψ, given as

$$\mathbf{u} = \mathbf{u}_b + \Psi, \tag{3}$$

with

$$\mathbf{u}_b = \mathbf{B}\mathbf{q}, \quad \mathbf{q} = \lfloor \mathbf{u}_0^T \quad \Phi^T \rfloor^T$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & \xi_3 & -\xi_2 \\ 0 & 1 & 0 & -\xi_3 & 0 & 0 \\ 0 & 0 & 1 & \xi_2 & 0 & 0 \end{bmatrix}, \tag{4}$$

where $\mathbf{q}$ represents the generalized beam displacements consisting of translations vector $\mathbf{u}^0$ and rotations vector Φ of the beam section.

The warping field is six times redundant which can be treated applying constraints defined in [9] as

$$\int_A \mathcal{D}_w \Psi dA = 0, \tag{5}$$

where A is the cross-sectional area, and operator matrix D_w is given by

$$\mathcal{D}_w = \begin{bmatrix} 1 & 0 & 0 & 0 & \partial/\partial\xi_3 & -\partial/\partial\xi_2 \\ 0 & 1 & 0 & -\partial/\partial\xi_3 & 0 & \partial/\partial\xi_1 \\ 0 & 0 & 1 & \partial/\partial\xi_2 & -\partial/\partial\xi_1 & 0 \end{bmatrix}^T. \tag{6}$$

Assuming small strains and small local rotations, the linear strain-displacement relations can be obtained as

$$\varepsilon_s^a = \mathbf{B}\Gamma + \mathcal{L}_s\Psi + \Psi', \quad \varepsilon_n^a = \mathcal{L}_n\Psi, \tag{7}$$

where $\Gamma = \mathcal{L}_q\mathbf{q} + \mathbf{q}'$ represents generalized strain measures, ()' denotes derivative with respect to axial coordinate ξ_1, and matrices $\mathcal{L}_s, \mathcal{L}_n, \mathcal{L}_q$ are respectively given as

$$\mathcal{L}_s = \begin{bmatrix} 0 & 0 & 0 \\ \partial/\partial\xi_2 & 0 & 0 \\ \partial/\partial\xi_3 & 0 & 0 \end{bmatrix}, \mathcal{L}_n = \begin{bmatrix} 0 & \partial/\partial\xi_2 & 0 \\ 0 & 0 & \partial/\partial\xi_3 \\ 0 & \partial/\partial\xi_3 & \partial/\partial\xi_2 \end{bmatrix}$$

$$\mathcal{L}_q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

The warping and reactive stress fields can be discretized using isoparametric shape functions as

$$\begin{aligned} \Psi(\xi_1, \xi_2, \xi_3) &= \mathbf{N}_\psi(\xi_2, \xi_3)\Lambda(\xi_1) \\ \sigma_s^r(\xi_1, \xi_2, \xi_3) &= \mathbf{N}_\sigma(\xi_2, \xi_3)\Upsilon(\xi_1), \end{aligned} \tag{9}$$

where $\mathbf{N}_\psi$ and $\mathbf{N}_\sigma$ represent FE shape function matrices for warping and reactive stress fields, respectively, with Λ and Υ as corresponding nodal values of warping and reactive stress fields. The warping constraints in Eq. (5) can be written in discretized form as

$$\left(\int_A \mathcal{D}_w \mathbf{N}_\psi dA\right)\Lambda = \mathbf{D}_\psi\Lambda = 0, \tag{10}$$

where $\mathbf{D}_\psi$ is the warping constraints matrix.

2.3 Governing equations

The present work considers sectional stresses to be unknowns in addition to displacements and employs variational principle to derive governing equations which leads to *a multifield variational formulation*. The variation of total energy per unit length $\delta\Pi_R$ of the beam can be stated as

$$\delta\Pi_R = \delta U_s - \delta W_s = 0, \tag{11}$$

where δU_s is the variation of sectional strain energy and δU_s is the variation of external work on the beam section due to applied loads. The sectional strain energy U_s can be defined in terms of Reissner's semi-complimentary energy functional Φ_R [10] as

$$U_s = \int_A [\Phi_R + (\sigma_s^r)^T \varepsilon_s^r]\, dA, \tag{12}$$

where Φ_R is given as

$$\Phi_R = \tfrac{1}{2}[(\varepsilon_n^a)^T \sigma_n^a - (\sigma_s^r)^T \varepsilon_s^r]. \tag{13}$$

The reactive strains ε_s^r obtained from semi-inverted material constitutive relations and active strains ε_s^a computed from kinematical relations should be compatible, which implies $\varepsilon_s^r=\varepsilon_s^a$. The first variation of sectional strain energy can then be obtained from Eq. (12) as

$$\delta U_s = \int_A [(\delta\varepsilon_n^a)^T \sigma_n^a + (\delta\varepsilon_s^a)^T \sigma_s^r + (\delta\sigma_s^r)^T(\varepsilon_s^a - \varepsilon_s^r)]\, dA. \tag{14}$$

The last term is a byproduct of the present multifield formulation which corresponds to the strain compatibility condition acting $\delta\sigma_s^r$ as Lagrange multipliers. The sectional stress resultants can be defined using tractions over the beam section σ_s as given by

$$\mathbf{F} = \int_A \mathbf{B}^T \sigma_s\, dA, \tag{15}$$

where $F=[F_1\ F_2\ F_3\ M_1\ M_2\ M_3]^T$ with F_1 as the extensional force, F_2 and F_3 as the transverse shear forces, M_1 as the torsional moment, and M_2 and M_3 as the bending moments. Neglecting the surface and body forces, the external work per unit length W_s is given as

$$W_s = \int_A (\mathbf{u}^T \sigma_s)'\, dA. \tag{16}$$

Using Eqs. (3), (4) and (15), and the definition of generalized strain measures Γ, the variation of external work W_s is obtained as

$$\begin{aligned} \delta W_s &= \int_A [(\delta\Psi')^T \sigma_s + \delta\Psi^T (\sigma_s)']\, dA \\ &\quad + \left[\delta\Gamma^T \mathbf{F} + \delta\mathbf{q}^T\left(\mathbf{F}' - \mathcal{L}_q^T \mathbf{F}\right)\right]. \end{aligned} \tag{17}$$

The warping and reactive stress fields are expressed as linear functions of generalized stress resultants, given as

$$\begin{aligned} \Lambda &= \tilde{\Lambda}\mathbf{F}, \Upsilon = \tilde{\Upsilon}\mathbf{F}, \Gamma = \tilde{\Gamma}\mathbf{F} \\ \Lambda' &= \tilde{\Lambda}_p\mathbf{F}, \Upsilon' = \tilde{\Upsilon}_p\mathbf{F}, \Gamma' = \tilde{\Gamma}_p\mathbf{F}. \end{aligned} \tag{18}$$

where $\tilde{\Lambda}$ and $\tilde{\Upsilon}$ are the nodal values of warping and reactive stress coefficients with nonuniform distribution over the section, and $\tilde{\Gamma}$ represents the strain measure coefficients which are constant over the section.

Substituting Eqs. (14) and (17) in Eq. (11), and using expressions from Eqs. (2), (7), (9), (10) and (18), the sets of equilibrium equations can be derived as:

$$\begin{bmatrix} -\mathbf{H} & \mathbf{G}^T & \mathbf{R} & \mathbf{0} \\ \mathbf{G} & \mathbf{E} & \mathbf{0} & \mathbf{D}_\psi^T \\ \mathbf{R}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_\psi & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\Upsilon}_p \\ \tilde{\Lambda}_p \\ \tilde{\Gamma}_p \\ \Theta_p \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathcal{L}_q^T \\ \mathbf{0} \end{bmatrix}, \tag{19a}$$

$$\begin{bmatrix} -\mathbf{H} & \mathbf{G}^T & \mathbf{R} & \mathbf{0} \\ \mathbf{G} & \mathbf{E} & \mathbf{0} & \mathbf{D}_\psi^T \\ \mathbf{R}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_\psi & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\Upsilon} \\ \tilde{\Lambda} \\ \tilde{\Gamma} \\ \Theta \end{bmatrix} = \begin{bmatrix} 0 & -\mathbf{A}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\Upsilon}_p \\ \tilde{\Lambda}_p \\ \tilde{\Gamma}_p \\ \Theta_p \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix}, \tag{19b}$$

where

$$\mathbf{A} = \int_A \mathbf{N}_\psi^T \mathbf{N}_\sigma \, dA, \;\; \mathbf{E} = \int_A \left(\mathcal{L}_s^n \mathbf{N}_\psi\right)^T \bar{\mathbf{C}}_{nn} \left(\mathcal{L}_s^n \mathbf{N}_\psi\right) dA$$

$$\mathbf{G} = \int_A \left(\mathcal{L}_s^n \mathbf{N}_\psi - \bar{\mathbf{C}}_{sn} \mathcal{L}_s^n \mathbf{N}_\psi\right)^T \mathbf{N}_\sigma \, dA \tag{20}$$

$$\mathbf{H} = \int_A \mathbf{N}_\sigma^T \bar{\mathbf{C}}_{ss} \mathbf{N}_\sigma \, dA, \;\; \mathbf{R} = \int_A \mathbf{N}_\sigma^T \mathbf{B} \, dA$$

The above submatrices **A**, **E**, **G**, **H**, and **R** incorporate the effects of geometric and material couplings. Once the warping and reactive stress coefficients are solved, the generalized Timoshenko-like 6×6 stiffness matrix **K** can be determined as

$$\mathbf{K} = \left(\begin{bmatrix} \tilde{\Lambda}_p \\ \tilde{\Upsilon} \\ \tilde{\Lambda} \\ \tilde{\Gamma} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}^T & -\mathbf{H} & \mathbf{G}^T & \mathbf{R} \\ \mathbf{0} & \mathbf{G} & \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\Lambda}_p \\ \tilde{\Upsilon} \\ \tilde{\Lambda} \\ \tilde{\Gamma} \end{bmatrix} \right)^{-1} \tag{21}$$

The stiffness matrix appeared in Eq. (21) takes into account the effects of elastic couplings, transverse shear, and Poisson deformations.

3. Results and Discussion

The present multifield formulation is implemented into a FE program called multifield variational sectional analysis code. The effect of 3D warping on stiffness constants is demonstrated for elastically coupled closed section beams.

3.1 Multilayer elliptical pipe

First example is a multilayer elliptical composite pipe studied in [11] which exhibits extension-torsion and shear-bending couplings. The schematic of the section is shown in Fig. 2. The material properties can be found in [11]. For the present analysis, the section is modeled using 2,800 eight-node quadrilateral elements. Fig. 3 illustrates the predicted 3D warping modes of the multilayered elliptical pipe. The extensional mode is influenced by the extension-torsion coupling as can be seen in Fig. 3(a). Note that the extensional mode would exhibit only in-plane contraction of the section when no couplings are present. The shear mode F_3 shown in Fig. 3(c) indicates in-plane deformation due to coupling with the bending mode M_2. It should be remarked that the shear mode without elastic coupling exhibits only out-of-plane deformation as that depicted by shear mode F_2 in Fig. 3(b). The uncoupled bending mode M_3 in Fig. 3(f) manifests only in-plane displacement due to Poisson effect.

The effect of 3D warping on stiffness constants is investigated next. It is noted that without cross-sectional warping deformation, the stiffness constants are considered to represent generalized Euler-Bernoulli beam theory. Table 1 shows the comparison of stiffness constants with and without considering 3D warping. The values obtained using a displacement-based approach [9] and those using variational asymptotic beam sectional analysis (VABS)

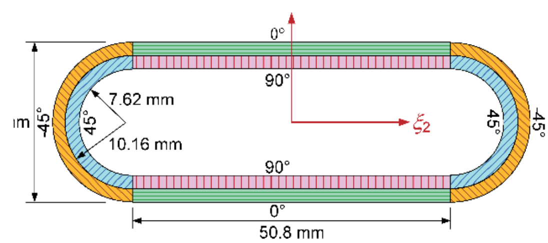

Fig. 2. Schematic of multilayer elliptical pipe

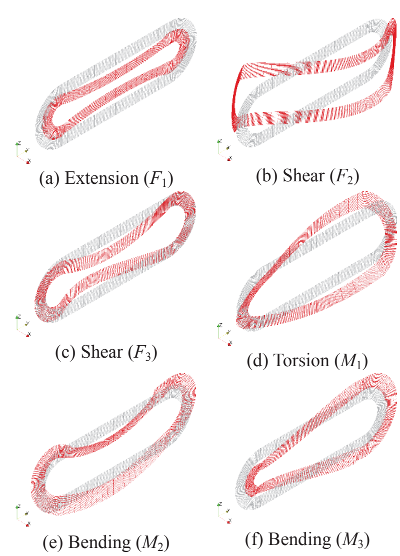

Fig. 3. Warping deformation modes of multilayer elliptical pipe (exaggerated)

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Table 1. Effect of 3D warping on stiffness constants of multilayer elliptical pipe

Stiffness Constants	VABS [11]	Ref. [8]	Present Multifield	
			No warping	3D warping (%)[a]
K_{11} (N)	4.621E+7	4.621E+7	5.755E+7	4.617E+7 (–19.9)
K_{14} (N m)	1.111E+4	1.112E+4	5.026E+4	1.120E+4 (–77.7)
K_{22} (N)	3.489E+6	3.493E+6	9.972E+6	3.493E+6 (–65.0)
K_{25} (N m)	–9.251E+2	–9.287E+2	–1.399E+4	–9.420E+2 (–93.3)
K_{33} (N)	1.463E+6	1.464E+6	9.972E+6	1.459E+6 (–85.4)
K_{36} (N m)	–5.859E+3	–5.878E+3	–3.627E+4	–5.896E+3 (–83.7)
K_{44} (N m^2)	1.971E+3	1.972E+3	1.068E+4	1.971E+3 (–81.6)
K_{55} (N m^2)	5.402E+3	5.402E+3	6.154E+3	5.397E+3 (–12.3)
K_{66} (N m^2)	1.547E+4	1.547E+4	2.563E+4	1.543E+4 (–39.8)

1–extension; 2,3–shear; 4–torsion; 5,6–bending

[a] Percentage difference with respect to the values computed without warping

[11] are also presented for comparison. The formulations presented in [9, 11] consider only displacements as primary variables whereas the present multifield formulation considers both displacements and sectional stresses as the unknowns. The present stiffness constants computed with 3D warping match well with those of VABS [11] and Ref. [9] with the maximum difference to be less than 2%. Neglecting 3D warping results in fairly large over-predictions in the stiffness constants with maximum errors as high as 93.3% for shear-bending coupling stiffness K_{25}. The minimum difference is 12.3% corresponding to bending stiffness K_{55}. The extension-torsion coupling stiffness K_{14} computed without warping is 77.7% higher than that with 3D warping. Because of extension-torsion coupling, the extensional stiffness K_{11} also indicates a large deviation of 19.8% when the warping is neglected. The torsional (K_{44}) and shear (K_{33}) stiffnesses show differences of larger than 80% without 3D warping. Since the stiffness constants are used to compute global beam behavior, the large errors in stiffness constants can result in highly inaccurate predictions of beam response.

3.2 Composite box beam

Next example is a composite box beam with symmetric layup [3]. The beam exhibits bending-torsion and extension-shear couplings. The geometry and layup of the section is presented in Fig. 4. The material properties can be found in [3]. For the present analysis, the section is discretized with 360 eight-node quadrilateral elements giving a total of 1,200 nodes. The 3D warping deformations computed from the present analysis are shown in Fig. 5. The extensional mode F_1 shown in Fig. 5(a) indicates out-of-plane deformation due to coupling with the shear mode F_2 in addition to in-plane displacement. The bending mode M_2 is affected by the coupling with the torsional mode M_1 as indicated in Fig. 5(e) resulting in out-of-plane deformation. The uncoupled shear mode F_3 in Fig. 5(c) and bending mode M_3 in Fig. 5(f) exhibit out-of-plane and in-plane deformations, respectively.

The influence of 3D warping on stiffness constants of

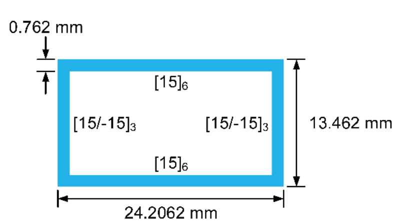

Fig. 4. Layup geometry of composite box section

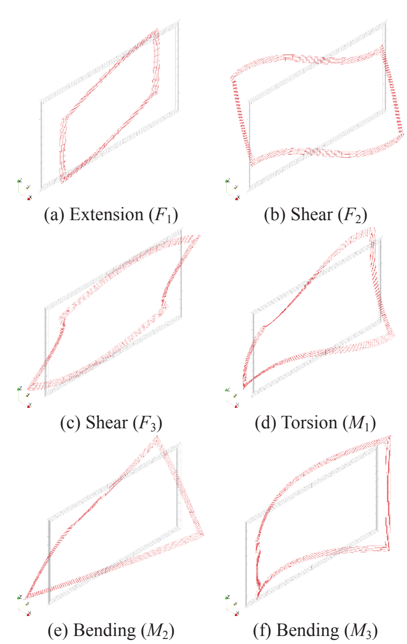

Fig. 5. Warping deformation modes of composite box beam (exaggerated)

Table 2. Effect of 3D warping on stiffness constants of composite box beam

Stiffness Constants	Ref. [8]	Present Multifield	
		No warping	3D warping (%)[a]
K_{11} (N)	6.127E+6	7.279E+6	5.999E+6 (–17.6)
K_{12} (N m)	8.160E+5	1.075E+6	8.084E+5 (–24.8)
K_{22} (N)	3.959E+5	6.083E+5	3.894E+5 (–36.0)
K_{33} (N)	1.774E+5	4.850E+5	1.719E+5 (–64.6)
K_{44} (N m^2)	5.010E+1	6.892E+1	4.949E+1 (–28.2)
K_{45} (N m^2)	–5.155E+1	–4.469E+1	–5.102E+1 (14.1)
K_{55} (N m^2)	1.752E+2	2.311E+2	1.701E+2 (–26.4)
K_{66} (N m^2)	4.119E+2	5.706E+2	3.965E+2 (–30.5)

1–extension; 2,3–shear; 4–torsion; 5,6–bending

[a] Percentage difference with respect to the values computed without warping

composite box section is illustrated in Table 2. The stiffness values computed from displacement-based analysis [9] are also included for reference purpose. The maximum difference of 64.6% is noted for shear stiffness K_{33} compared to that without warping. The extension-shear coupling stiffness K_{12} and bending-torsion coupling stiffness K_{45} report the differences of 24.8% and 14.1%, respectively, with reference to those computed with warping. Overall, difference of larger than 10% is found for all stiffness values computed without warping. Such large deviations in stiffness constants essentially lead to significant underestimations of global beam response.

4. Concluding Remarks

In the present study, the influence of three-dimensional warping is investigated for composite beams with closed sections. A finite element based multifield variational cross-sectional analysis is developed to compute the generalized Timoshenko level stiffness constants. The 3D warping deformation modes are illustrated for beams with different elastic couplings. The effect of 3D warping on stiffness predictions is demonstrated to be significant with differences ranging from 12.3% to 93.3% in reference to that without warping. The elastic coupling stiffness constants indicate large errors for the cases when warping is neglected. The modeling of 3D warping is therefore essential for correct estimation of stiffness constants and accurate prediction for global structural behavior of elastically coupled composite beams.

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