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ARITHMETIC OF MODULAR FORMS

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Abstract. We investigate congruence properties of Fourier coefficients of modular forms for $\Gamma_0^+(2)$.

1. Introduction and statement of results

The congruence properties of modular forms were investigated by many mathematicians. In particular, Choie, Kohnen and Ono [4]found congruence properties of the coefficients of modular forms for $SL_2(\mathbb{Z})$. Choi [1] generalized their result to modular forms for congruence subgroups. Let $\Gamma_0^+(p)$ be the group generated by the Hecke group $\Gamma_0(p)$ and the Fricke involution $W_p = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$. Choi [2] found some congruence properties of coefficients of modular forms for $\Gamma_0^+(5)$. In this paper, following the argument in [2, 4], we obtain congruence properties of coefficients of modular forms for $\Gamma_0^+(2)$. The main result of this paper is the following theorem.

Theorem 1.1. *Let k be a positive integer with $k \equiv 0 \pmod 8$. Let f be a modular form of weight k for $\Gamma_0^+(2)$ having a Fourier expansion of the form*

$$f(z) = \sum_{n=0}^{\infty} a_f(n)q^n, \qquad (q = e^{2\pi iz}).$$

Suppose that $f(z)$ has integral Fourier coefficients. Then for any integer b with $2^b \geq k/8 + 1$ we have

$$a_f(2^b) \equiv 0 \pmod 2.$$

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2. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. Let $k > 2$ be an even integer. Let E_k be the normalized Eisenstein series of weight k for $SL_2(\mathbb{Z})$ defined by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \qquad (q = e^{2\pi i z}),$$

where $\sigma_{k-1}(n)$ is the usual divisor sum of n and B_k is the k-th Bernoulli number. Let

$$E_k^+(z) := \frac{E_k(z) + 2^{k/2} E_k(2z)}{1 + 2^{k/2}}.$$

Then E_k^+ is a modular form of weight k for $\Gamma_0^+(2)$ (see [3]). For Dedekind eta function $\eta(z) = q^{1/(24)} \prod_{n=1}^{\infty}(1 - q^n)$, let

$$\Delta_2^+(z) := \eta(z)^8 \eta(2z)^8 = q - 8q^2 + O(q^3).$$

Then Δ_2^+ which is a cusp form of weight 8 for $\Gamma_0^+(2)$ has no zeros on the complex upper half plane and has integral coefficients (see [3]). We now consider a hauptmodul for $\Gamma_0^+(2)$ defined by

$$j_2^+(z) = \frac{E_4^+(z)^2}{\Delta_2^+(z)} = \frac{1}{q} + O(1).$$

Then j_2^+ has integral Fourier coefficients. Let $r := k/8 + 1$. We notice

$$\frac{-1}{2\pi i} \frac{dj_2^+(z)}{dz} = \frac{E_{10}^+(z)}{\Delta_2^+(z)}$$

and the functions

$$j^m \frac{dj_2^+(z)}{dz} = \frac{1}{m+1} \frac{dj_2^+(z)^{m+1}}{dz}$$

has zero constant term for each nonnegative integer m. Let b an integer with $2^b \geq r$. Since

$$(j_2^+)^{2^b - r} \frac{f}{(\Delta_2^+)^{r-1}}$$

is a polynomial in j_2^+, by linearity we obtain that the constant term of

$$(j_2^+)^{2^b - r} \frac{f}{(\Delta_2^+)^{r-1}} \frac{1}{2\pi i} \frac{dj_2^+(z)}{dz}$$

is zero. Thus we have that the constant term of

$$(j_2^+)^{2^b - r} \frac{-33f}{(\Delta_2^+)^{r-1}} \frac{1}{2\pi i} \frac{dj_2^+(z)}{dz} = (j_2^+)^{2^b - r} \frac{f}{(\Delta_2^+)^{r-1}} \frac{33E_{10}^+(z)}{\Delta_2^+(z)}$$

is zero $\pmod 2$. Since

$$\begin{aligned}(j_2^+)^{2^b-r}\frac{f}{(\Delta_2^+)^{r-1}}\frac{33E_{10}^+(z)}{\Delta_2^+(z)} &\equiv (j_2^+)^{2^b-r}\frac{33E_{10}^+(z)}{\Delta_2^+(z)^r}f \\ &\equiv \frac{(E_4^+(z))^{2^{b+1}-2r}33E_{10}^+(z)}{\Delta_2^+(2^bz)}f \pmod 2\end{aligned}$$

and $33E_{10}^+(z) \equiv 1 \equiv E_4^+(z) \pmod 2$, we have that the constant term of

$$\frac{(E_4^+(z))^{2^{b+1}-2r}33E_{10}^+(z)}{\Delta_2^+(2^bz)}f$$

is congruent to the constant term of

$$\frac{f}{\Delta_2^+(2^bz)} \pmod 2.$$

From the fact that $1/\Delta_2^+(2^bz) = q^{-2^b} + 8 + O(q)$, we have that the constant term of

$$\frac{f}{\Delta_2^+(2^bz)}$$

is $a_f(2^b) + 8a_f(0)$ which gives

$$a_f(2^b) \equiv 0 \pmod 2.$$

This completes the proof of Theorem 1.1.

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