# THE MINIMAL GRADED FREE RESOLUTION OF THE UNION OF TWO STAR CONFIGURATIONS IN $\mathbb{P}^{n}$ AND THE WEAK LEFSCHETZ PROPERTY 

Yong-Su Shin*


#### Abstract

We find a graded minimal free resolution of the union of two star configurations $\mathbb{X}$ and $\mathbb{Y}$ (not necessarily linear star configurations) in $\mathbb{P}^{n}$ of type $s$ and $t$ for $s, t \geq 2$, and $n \geq 3$. As an application, we prove that an Artinian ring $R /\left(I_{\mathrm{X}}+I_{\mathbb{Y}}\right)$ of two linear star configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^{3}$ of type $s$ and $t$ has the weak Lefschetz property for $s \geq\left\lfloor\frac{1}{2}\binom{t}{2}\right\rfloor$ and $t \geq 2$.


## 1. Introduction

A star configuration set of points in $\mathbb{P}^{2}($ see $[5])$, which was introduced by Geramita, Migliore, and Sabourin in 2006, will be called a linear star configuration in this paper. Configurations of this type and their natural generalizations to $\mathbb{P}^{n}$ have been proved to be a very interesting family of points, hypersurfaces, and so on. For example, it is easy to describe their defining ideals algebraically (see $[6,8]$ ). Moreover, the graded Betti numbers and shifts of a graded minimal free resolutions of star configurations in $\mathbb{P}^{n}$ can be described in terms of the number and the degrees of defining forms in $R=\mathbb{k}\left[x_{0}, x_{1}, \ldots, x_{n}\right]$ (see [8]). In addition, Catalisano, Geramita, Gimigliano, Migliore, Nagel, and Shin [3] have studied star configurations in $\mathbb{P}^{n}$ to calculate the dimensions of the secant varieties to the varieties of reducible curves (see also [2]). There have been continual efforts, which have further developed the properties of star configurations in $\mathbb{P}^{n}$ (see $[1,2,6,7,9]$ ).

We briefly recall generic Hilbert function and the weak Lefschetz property. Let $\mathbb{k}$ be an infinite field of characteristic free and $R=$

[^0]$\mathbb{k}\left[x_{0}, x_{1}, \ldots, x_{n}\right]$ be an $(n+1)$-variable polynomial ring over a field $\mathbb{k}$. If $I$ is a homogeneous ideal in $R$, the numerical function $\mathbf{H}_{R / I}(t):=$ $\operatorname{dim}_{\mathbb{k}} R_{t}-\operatorname{dim}_{\mathbb{k}} I_{t}$ is called the Hilbert function of the ring $R / I$. If $I:=I_{\mathbb{X}}$ is the ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^{n}$, then we denote the Hilbert function of $\mathbb{X}$ by $\mathbf{H}_{\mathbb{X}}(t):=\mathbf{H}_{R / I_{\mathbb{X}}}(t)$. In particular, if $\mathbb{X}$ is a finite set of points in $\mathbb{P}^{n}$, then we say that $\mathbb{X}$ has generic Hilbert function if $\mathbf{H}_{\mathbb{X}}(t)=\min \left\{|\mathbb{X}|,\binom{t+n}{n}\right\}$ for every $t \geq 0$. In addition, for a finite set $\mathbb{X}$ of points in $\mathbb{P}^{n}$, we define $\sigma(\mathbb{X})=\min \left\{i \mid \mathbf{H}_{\mathbb{X}}(i-1)=\mathbf{H}_{\mathbb{X}}(i)\right\}$.

Let $R / I$ be a standard graded Artinian algebra. We say that $R / I$ has the weak Lefschetz property if, for a general linear form $L \in R$ and for every $d \geq 0$, the multiplication map by $L,[R / I]_{d} \xrightarrow{\times L}[R / I]_{d+1}$, has maximal rank. In this case, $L$ is said to be a Lefschetz element. Over the years, there have been several papers which have devoted to a classification of possible Artinian quotients having the weak Lefschetz property (see [4, 9]). Note that since $R / I$ is Artinian, $A_{d}=0$ for $d \gg 0$, and so only a finite number of maps have to be considered. The strong Lefschetz property says that for every $i \geq 1$ the multiplication map by $L^{i},[R / I]_{d} \xrightarrow{\times L^{i}}[R / I]_{d+i}$, has maximal rank for every $d \geq 0$.

The Lefschetz properties for a standard graded Artinian $\mathbb{k}$-algebra are algebraic abstractions introduced by Stanley [12]. The weak Lefschetz property has recently received more attendtion, and is a very fundamental and natural property of Artinian algebras (see [4, 7, 12]).

The goal of this paper is to find a graded minimal free resolution of the union $\mathbb{X} \cup \mathbb{Y}$ of two star configurations $\mathbb{X}$ and $\mathbb{Y}$ (not necessarily linear star configurations) in $\mathbb{P}^{n}$ of type $s, t$ with $s, t \geq 2$ and $n \geq$ 3 (see Theorem 3.1) using the Künneth formula and a mapping cone construction. Furthermore, we show that an Artinian ring $R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right)$ of two linear star configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^{3}$ of type $s, t$ with $s \geq\left\lfloor\frac{1}{2}\binom{t}{2}\right\rfloor$ and $t \geq 2$ has the weak Lefschetz property (see Theorem 4.4).

## 2. A Graded Minimal Free Resolution of A Star configuration in $\mathbb{P}^{n}$

We first introduce notions of a star configuration in $\mathbb{P}^{n}$.
Definition 2.1. Let $R=\mathbb{k}\left[x_{0}, x_{1}, \ldots, x_{n}\right]$ be a polynomial ring over a field $\mathbb{k}$. For positive integers $r$ and $s$ with $1 \leq r \leq \min \{n, s\}$, suppose $F_{1}, \ldots, F_{s}$ are general forms in $R$ of degrees $d_{1}, \ldots, d_{s}$, respectively. We
call the variety $\mathbb{X}$ defined by the ideal

$$
\bigcap_{1 \leq i_{1}<\cdots<i_{r} \leq s}\left(F_{i_{1}}, \ldots, F_{i_{r}}\right)
$$

a star configuration in $\mathbb{P}^{n}$ of type $(r, s)$. In particular, if $\mathbb{X}$ is a star configuration in $\mathbb{P}^{n}$ of type $(2, s)$, then we simply call a star configuration in $\mathbb{P}^{n}$ of type $s$ for short.

Notice that, for $s \geq n$, each $n$-forms $F_{i_{1}}, \ldots, F_{i_{n}}$ of $s$-general forms $F_{1}, \ldots, F_{s}$ in $R$ define $d_{i_{1}} \cdots d_{i_{n}}$ points in $\mathbb{P}^{n}$ for each $1 \leq i_{1}<\cdots<$ $i_{n} \leq s$. Thus the ideal $\bigcap_{1 \leq i_{1}<\cdots<i_{n} \leq s}\left(F_{i_{1}}, \ldots, F_{i_{n}}\right)$ defines a finite set $\mathbb{X}$ of points in $\mathbb{P}^{n}$ with

$$
\operatorname{deg}(\mathbb{X})=\sum_{1 \leq i_{1}<i_{2}<\cdots<i_{n} \leq s} d_{i_{1}} d_{i_{2}} \cdots d_{i_{n}}
$$

Furthermore, if $F_{1}, \ldots, F_{s}$ are general linear forms in $R$, then we call $\mathbb{X}$ a linear star configuration in $\mathbb{P}^{n}$ of type $s$, respectively.

Theorem 2.2 ([8, Theorem 3.4]). Let $\mathbb{X}$ be a star configuration in $\mathbb{P}^{n}$ of type $(r, s)$ defined by general forms $F_{1}, \ldots, F_{s}$ in $R=\mathbb{k}\left[x_{0}, x_{1}, \ldots, x_{n}\right]$ of degrees $d_{1}, d_{2}, \ldots, d_{s}$, where $2 \leq r \leq \min \{s, n\}$, and let $d=d_{1}+\cdots+$ $d_{s}$. Then a graded minimal free resolution of $I_{\mathbb{X}}$ is

$$
\begin{equation*}
0 \rightarrow \mathbb{F}_{r}^{(r, s)} \rightarrow \mathbb{F}_{r-1}^{(r, s)} \rightarrow \cdots \rightarrow \mathbb{F}_{1}^{(r, s)} \rightarrow I_{\mathbb{X}} \rightarrow 0 \tag{2.1}
\end{equation*}
$$

where

$$
\begin{aligned}
\mathbb{F}_{r}^{(r, s)} & =R^{\alpha_{r}^{(r, s)}}(-d), \\
\mathbb{F}_{r-1}^{(r, s)} & =\bigoplus_{1 \leq i_{1} \leq s} R^{\alpha_{r-1}^{(r, s)}}\left(-\left(d-d_{i_{1}}\right)\right), \\
& \vdots \\
\mathbb{F}_{\ell}^{(r, s)} & =\bigoplus_{1 \leq i_{1}<\cdots<i_{r-\ell} \leq s} R^{\alpha_{\ell}^{(r, s)}}\left(-\left(d-\left(d_{i_{1}}+\cdots+d_{i_{r-\ell}}\right)\right)\right), \\
& \vdots \\
\mathbb{F}_{2}^{(r, s)} & =\bigoplus_{1 \leq i_{1}<\cdots<i_{r-2} \leq s} R^{\alpha_{2}^{(r, s)}}\left(-\left(d-\left(d_{i_{1}}+\cdots+d_{i_{r-2}}\right)\right)\right), \\
\mathbb{F}_{1}^{(r, s)} & =\bigoplus_{1 \leq i_{1}<\cdots<i_{r-1} \leq s} R^{\alpha_{1}^{(r, s)}}\left(-\left(d-\left(d_{i_{1}}+\cdots+d_{i_{r-1}}\right)\right)\right),
\end{aligned}
$$

with

$$
\begin{gathered}
\alpha_{\ell}^{(r, s)}=\binom{s-r+\ell-1}{\ell-1} \text { and } \\
\operatorname{rank} \mathbb{F}_{\ell}^{(r, s)}=\binom{s-r+\ell-1}{\ell-1} \cdot\binom{s}{r-\ell}
\end{gathered}
$$

for $1 \leq \ell \leq r$. In particular, the last free module $\mathbb{F}_{r}^{(r, s)}$ has only one shift d, i.e., a star configuration $\mathbb{X}$ in $\mathbb{P}^{n}$ is level. Furthermore, any star configuration $\mathbb{X}$ in $\mathbb{P}^{n}$ is arithmetically Cohen-Macaulay.

Theorem 2.3 ([10, Proposition 2.5]). Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star configurations in $\mathbb{P}^{2}$ of type $s$ and $t$ defined by general linear forms $L_{1}, \ldots, L_{s}$ and $M_{1}, \ldots, M_{t}$ in $R=\mathbb{k}\left[x_{0}, x_{1}, x_{2}\right]$ with $s \geq\left\lfloor\frac{1}{2}\binom{t}{2}\right\rfloor$ and $t \geq 2$. Then the union $\mathbb{X} \cup \mathbb{Y}$ of two linear star configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^{2}$ has generic Hilbert function.

## 3. A Graded Minimal Free Resolution of The Union of Two Star Configurations in $\mathbb{P}^{n}$

In this section, we shall fins a graded minimal free resolution of $R /\left(I_{\mathbb{X}} \cap I_{\mathbb{Y}}\right)$ where $\mathbb{X}$ and $\mathbb{Y}$ are star configurations in $\mathbb{P}^{n}$ of type $s$ and $t$ with $s, t \geq 2$.

Theorem 3.1. Let $\mathbb{X}$ and $\mathbb{Y}$ be star configurations in $\mathbb{P}^{n}$ of type $s$ and $t$ defined by general forms of degrees $d_{1}, \ldots, d_{s}$ and $e_{1}, \ldots, e_{t}$ with $s, t \geq 2$. Let $d=d_{1}+\cdots+d_{s}$ and $e=e_{1}+\cdots+e_{t}$. Then, for $n \geq 3$, a graded minimal free resolution of $R /\left(I_{\mathbb{X}} \cap I_{\mathbb{Y}}\right)$ is

$$
\begin{aligned}
0 & \rightarrow R^{(s-1)(t-1)}(-(d+e)) \rightarrow\left[\begin{array}{c}
\bigoplus_{1 \leq i \leq s} R^{t-1}\left(-\left(d+e-d_{i}\right)\right) \\
\oplus \\
\bigoplus_{1 \leq i \leq t} R^{s-1}\left(-\left(d+e-e_{i}\right)\right)
\end{array}\right] \\
& \rightarrow \bigoplus_{\substack{1 \leq i \leq s \\
1 \leq j \leq t}} R\left(-\left(d+e-d_{i}-e_{j}\right)\right) \xrightarrow{\rightarrow} R \rightarrow R /\left(I_{\mathbb{X}} \cap I_{\mathbb{Y}}\right) \rightarrow 0
\end{aligned}
$$

In particular, if $\mathbb{X}$ and $\mathbb{Y}$ are linear star configuration in $\mathbb{P}^{n}$ of type $s$ and $t$ with $s, t \geq 3$ and $n \geq 3$, then the initial degree of $I_{\mathbb{X}} \cap I_{\mathbb{Y}}$ is $s+t-2$.

Proof. We first recall that

$$
\begin{align*}
& 0 \rightarrow R^{s-1}(-d) \rightarrow \bigoplus_{1 \leq i \leq s} R\left(-\left(d-d_{i}\right)\right) \rightarrow R \rightarrow R / I_{\mathbb{X}} \rightarrow 0  \tag{3.1}\\
& 0 \rightarrow R^{t-1}(-e) \rightarrow \bigoplus_{1 \leq i \leq t} R\left(-\left(e-e_{i}\right)\right) \rightarrow R \rightarrow R / I_{\mathbb{Y}} \rightarrow 0
\end{align*}
$$

are a graded minimal free resolutions of the Cohen-Macaulay rings $R / I_{\mathbb{X}}$ and $R / I_{\mathbb{Y}}$ of codimension 2 , respectively (see Theorem 2.2 ).

Notice that $R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right)$ is a Cohen-Macaulay ring of codimension 4 (see [3, Proposition 3.1]), which implies a projective dimension 4. Hence, by a mapping cone construction, the projective dimension of $R /\left(I_{\mathbb{X}} \cap I_{\mathbb{Y}}\right)$ is 3 . So we obtain the following diagram.

where

$$
0 \rightarrow \mathbb{F}_{3} \rightarrow \mathbb{F}_{2} \rightarrow \mathbb{F}_{1} \rightarrow R \rightarrow R /\left(I_{\mathbb{X}} \cap I_{\mathbb{Y}}\right) \rightarrow 0
$$

is a graded minimal free resolution of $R /\left(I_{\mathbb{X}} \cap I_{\mathbb{Y}}\right)$. Since $n \geq 3$, by Künneth formula (see [3, Theorem 2.14]), a graded minimal free resolution of $R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right)$ is

$$
\begin{align*}
& 0 \rightarrow \quad R^{(s-1)(t-1)}(-(d+e)) \quad \rightarrow\left[\begin{array}{l}
\bigoplus_{1 \leq i \leq s} R^{t-1}\left(-\left(d+e-d_{i}\right)\right) \\
\oplus \\
\bigoplus_{1 \leq i \leq t} R^{s-1}\left(-\left(d+e-e_{i}\right)\right)
\end{array}\right] \tag{3.3}
\end{align*}
$$

$$
\begin{aligned}
& \rightarrow \quad R \rightarrow R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right) \rightarrow 0 .
\end{aligned}
$$

By equations 3.2 and 3.3, we have

$$
\begin{aligned}
& \mathbb{F}_{3}=R^{(s-1)(t-1)}(-(d+e)), \\
& \mathbb{F}_{2}=\bigoplus_{1 \leq i \leq s} R^{t-1}\left(-\left(d+e-d_{i}\right)\right) \oplus \bigoplus_{1 \leq i \leq t} R^{s-1}\left(-\left(d+e-e_{i}\right)\right), \\
& \mathbb{F}_{1}=\bigoplus_{1 \leq i \leq s} R\left(-\left(d+e-d_{i}-e_{j}\right)\right),
\end{aligned}
$$

as we wished. In particular, if $\mathbb{X}$ and $\mathbb{Y}$ are linear star configurations in $\mathbb{P}^{n}$ with $n \geq 3$, then it is immediate from a graded minimal free
resolution of $R /\left(I_{\mathbb{X}} \cap I_{\mathbb{Y}}\right)$ that the initial degree of $I_{\mathbb{X}} \cap I_{\mathbb{Y}}$ is $(s+t-2)$. This completes the proof of this theorem.

## 4. An Artinian Ring of Codimension 4 and the Weak Lefschetz Property

We shall prove that a graded Artinian ring $R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right)$ of two linear star configurations in $\mathbb{P}^{3}$ of type $s$ and $t$ with $s \geq\left\lfloor\frac{1}{2}\binom{t}{2}\right\rfloor$ and $t \geq 2$ has the weak Lefschetz property. There is a useful numerical characterization of Lefschetz elements, which need some notations.

Definition 4.1. Let $\sum_{i \geq 0} a_{i} t^{i}$ be a formal power series, where $a_{i} \in$ $\mathbb{Z}$. Then we define an associated power series with non-negative coefficients by

$$
\left|\sum_{i \geq 0} a_{i} t^{i}\right|^{+}=\sum_{i \geq 0} b_{i} t^{i}
$$

where

$$
b_{i}= \begin{cases}a_{i}, & \text { if } a_{j}>0 \text { for all } j \leq i \\ 0, & \text { otherwise }\end{cases}
$$

The following lemma is immediate from the definition of the weak Lefschetz property, so we omit the proof here.

Lemma 4.2. Let $A$ be a standard artinian graded algebra, and let $L \in A$ be a linear form. Then the following conditions are equivalent:
(a) $L$ is a Lefschetz element of $A$.
(b) The Hilbert function of $A / L A$ is given by
$\operatorname{dim}_{\mathbb{k}}[A / L A]_{i}=\max \left\{0, \operatorname{dim}_{\mathbb{k}}[A]_{i}-\operatorname{dim}_{\mathbb{k}}[A]_{i-1}\right\}$ for all integers $i$.
(c) The Hilbert series $\mathbf{H S}(A / L A)$ of $A / L A$ is

$$
\mathbf{H S}(A / L A)=|(1-t) \cdot \mathbf{H S}(A)|^{+}
$$

Lemma 4.3. Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^{2}$ of type $s$ and $t$ with $s \geq\left\lfloor\frac{1}{2}\binom{t}{2}\right\rfloor$ and $t \geq 2$. Then

$$
\begin{equation*}
\sigma(\mathbb{X} \cup \mathbb{Y})<(s+t)-1 \tag{4.1}
\end{equation*}
$$

Proof. Recall that the union $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function for $s \geq\left\lfloor\frac{1}{2}\binom{t}{2}\right\rfloor$ and $t \geq 2$ (see Theorem 2.3). It is enough to show that

$$
\operatorname{deg}(\mathbb{X} \cup \mathbb{Y})=\binom{s}{2}+\binom{t}{2} \leq\binom{(s+t-3)+2}{2}
$$

This holds by a simple calculation, as we wished.

Theorem 4.4. Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star configurations in $\mathbb{P}^{3}$ of type $s$ and $t$ with $s \geq\left\lfloor\frac{1}{2}\binom{t}{2}\right\rfloor$ and $t \geq 2$, and let $R=\mathbb{k}\left[x_{0}, x_{1}, x_{2}, x_{3}\right]$. Then an Artinian ring $R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right)$ has the weak Lefschetz property.

For convenience, we shall use the following notations in the proof of Theorem 4.4.

1. $R=\mathbb{k}\left[x_{0}, x_{1}, x_{2}, x_{3}\right]$.
2. $S=\mathbb{k}\left[x_{0}, x_{1}, x_{2}\right] \simeq R /(L)$ where $L$ is a general linear form in $R$.
$3 . \mathbb{X}$ and $\mathbb{Y}$ are linear star configurations in $\mathbb{P}^{3}$ defined by general linear forms in $R=\mathbb{k}\left[x_{0}, x_{1}, x_{2}, x_{3}\right]$.
3. $\overline{\mathbb{X}}$ and $\overline{\mathbb{Y}}$ are linear star configurations in $\mathbb{P}^{2}$, where $\overline{\mathbb{X}}$ and $\overline{\mathbb{Y}}$ are obtained from the restriction of $\mathbb{X}$ and $\mathbb{Y}$ by a general hyperplane $\mathbb{H}$, respectively. So we can think of $\overline{\mathbb{X}}$ and $\overline{\mathbb{Y}}$ as two linear star configurations in $\mathbb{H} \cong \mathbb{P}^{2}$ (see Theorem 2.2).

Proof of Theorem 4.4. With notations as above, we define $\sigma:=\sigma(\overline{\mathbb{X}} \cup$ $\overline{\mathbb{Y}})<s+t-1$ (see Lemma 4.3). For every $d \geq \sigma-1$

$$
\begin{equation*}
\mathbf{H}\left(S /\left(I_{\overline{\mathbb{X}}} \cap I_{\overline{\mathbb{Y}}}\right), d\right)=\mathbf{H}_{\overline{\mathbb{X}}}(d)+\mathbf{H}_{\overline{\mathbb{Y}}}(d)=\operatorname{deg}(\overline{\mathbb{X}})+\operatorname{deg}(\overline{\mathbb{Y}})=\binom{s}{2}+\binom{t}{2} \tag{4.2}
\end{equation*}
$$

Recall that $L$ is a general linear form in $R$, and that, by Theorem $2.2, \mathbb{X}$ and $\mathbb{Y}$ are arithmetically Cohen-Macaulay schemes in $\mathbb{P}^{3}$ of codimension 2. So for every $d \geq 0$

$$
\Delta \mathbf{H}\left(R / I_{\mathbb{X}}, d\right)=\mathbf{H}\left(S / I_{\mathbb{X}}, d\right) \quad \text { and } \quad \Delta \mathbf{H}\left(R / I_{\mathbb{Y}}, d\right)=\mathbf{H}\left(S / I_{\mathbb{Y}}, d\right)
$$

Hence we obtain the exact sequence

$$
\begin{array}{ccccc}
R /\left[\left(I_{\mathbb{X}}, L\right) \cap\left(I_{\mathbb{Y}}, L\right)\right] & \hookrightarrow & R /\left(I_{\mathbb{X}}, L\right) \bigoplus R /\left(I_{\mathbb{Y}}, L\right) & \rightarrow & R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}, L\right)  \tag{4.3}\\
\| & & \| & \| \\
S /\left(I_{\overline{\mathbb{X}}} \cap I_{\overline{\mathbb{Y}}}\right) & \hookrightarrow & S / I_{\overline{\mathbb{X}}} \bigoplus S / I_{\overline{\mathbb{Y}}} & \rightarrow & S /\left(I_{\overline{\mathbb{X}}}+I_{\overline{\mathbb{Y}}}\right) .
\end{array}
$$

It is from equations (4.2) and (4.3) that $\left[R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}, L\right)\right]_{d}=0$ for every $d \geq \sigma-1$, so the multiplication map by $L$

$$
\left[R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right)\right]_{d-1} \quad \xrightarrow{\times L} \quad\left[R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right)\right]_{d} \quad \rightarrow \quad\left[R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}, L\right)\right]_{d}=0
$$

is surjective for such $d$. Hence, by Lemma 4.2, it suffices to show that the multiplication map by $L$

$$
\left[R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right)\right]_{d-1} \quad \xrightarrow{\times L} \quad\left[R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right)\right]_{d}
$$

is injective for $d<\sigma-1$. In other words, $R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right)$ has the weak Lefschetz property if and only if
$\Delta\left(\mathbf{H}\left(R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right), d\right)=\mathbf{H}\left(R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}, L\right), d\right) \quad\right.$ for every $\quad d<\sigma-1$.
Recall that from equation (4.3)

$$
\mathbf{H}\left(R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}, L\right), d\right)=\mathbf{H}\left(S / I_{\overline{\mathbb{X}}}, d\right)+\mathbf{H}\left(S / I_{\overline{\mathbb{V}}}, d\right)-\mathbf{H}\left(S /\left(I_{\overline{\mathbb{X}}} \cap I_{\overline{\mathbb{Y}}}\right), d\right)
$$

for every $d<\sigma-1$. Furthermore, notice that

- the initial degree of $I_{\mathbb{X}} \cap I_{\mathbb{Y}}$ is $(s+t-2)$ (see Theorem 3.1), and
- $\sigma<s+t-1$ (see Lemma 4.3).

Hence for every $d<\sigma-1<s+t-2$

$$
\begin{equation*}
\mathbf{H}\left(R /\left(I_{\mathbb{X}} \cap I_{\mathbb{Y}}\right), d\right)=\binom{d+3}{3} . \tag{4.5}
\end{equation*}
$$

Using the exact sequence

$$
\begin{equation*}
0 \rightarrow R /\left(I_{\mathbb{X}} \cap I_{\mathbb{Y}}\right) \rightarrow R / I_{\mathbb{X}} \oplus R / I_{\mathbb{Y}} \rightarrow R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right) \rightarrow 0, \tag{4.6}
\end{equation*}
$$

we obtain that for every $d<\sigma-1$

$$
\begin{aligned}
\Delta & \mathbf{H}\left(R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}\right), d\right) \\
& =\Delta \mathbf{H}\left(R / I_{\mathbb{X}}, d\right)+\Delta \mathbf{H}\left(R / I_{\mathbb{Y}}, d\right)-\Delta \mathbf{H}\left(R /\left(I_{\mathbb{X}} \cap I_{\mathbb{Y}}\right), d\right) \\
& \left.=\Delta \mathbf{H}\left(R / I_{\mathbb{X}}, d\right)+\Delta \mathbf{H}\left(R / I_{\mathbb{Y}}, d\right)-\binom{d+2}{2} \quad \text { (by equation }(4.5)\right) \\
& =\mathbf{H}\left(S / I_{\mathbb{\mathbb { X }}}, d\right)+\mathbf{H}\left(S / I_{\mathbb{\mathbb { Y }}}, d\right)-\binom{d+2}{2} \\
& =\mathbf{H}\left(S / I_{\mathbb{\mathbb { X }}}, d\right)+\mathbf{H}\left(S / I_{\mathbb{\mathbb { Y }}}, d\right)-\mathbf{H}\left(S / I_{\mathbb{X}} \cap I_{\mathbb{\mathbb { Y }}}, d\right) \\
& =\mathbf{H}\left(S /\left(I_{\mathbb{X}}+I_{\widetilde{\mathbb{Y}}}\right), d\right) \\
& =\mathbf{H}\left(R /\left(I_{\mathbb{X}}+I_{\mathbb{Y}}, L\right), d\right),
\end{aligned}
$$

as we wished. This completes the proof of this theorem.

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*
Department of Mathematics
Sungshin Women's University
Seoul 02844, Republic of Korea
E-mail: ysshin@sungshin.ac.kr


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