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COMPLETE CONVERGENCE FOR ARRAYS OF ROWWISE ASYMPTOTICALLY NEGATIVELY ASSOCIATED RANDOM VARIABLES

Hyun-Chull Kim*

Abstract. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise asymptotically negatively associated random variables and $\{a_{ni}, i \geq 1, n \geq 1\}$ an array of constants. Some results concerning complete convergence of weighted sums $\sum_{i=1}^{n} a_{ni}X_{ni}$ are obtained. They generalize some previous known results for arrays of rowwise negatively associated random variables to the asymptotically negative association case.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and $\{X_n, n \geq 1\}$ be a sequence of random variables defined on this space.

A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated(NA) if for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$, and any coordinatewise nondecreasing functions f on $\mathbb{R}^A$ and g on $\mathbb{R}^B$

$$Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0,$$

whenever f and g are such that covariance exists.

An infinite family of random variables is negatively associated if every finite subfamily is negatively associated. The notion of negative association was first introduced by Block *et al.* (1982). Joag-Dev and Proschan (1983) showed that many well known multivariate distributions possess the negative association property. The negative association property has aroused wide interest because of numerous applications in reliability theory, percolation theory and multivariate statistical analysis. In the past

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decades, many researchers studied in the different aspects: the moment inequalities by Su *et al.* (1997), Shao (2000) and Huang and Xu (2002), the almost sure convergence by Matula (1992), the weak convergence by Su *et al.* (1997), the complete convergence by Liang and Su (1999), Shao (2000), Huang and Xu (2002) and Li and Zhang (2004).

A new kind of dependence structure called asymptotically negative association was proposed by Zhang ([14], [15]), which is a useful weakening the definition of negative association.

DEFINITION 1.1. [13] A sequence $\{X_n, n \geq 1\}$ of random variables is said to be asymptotically negatively associated(ANA) if

$$\rho^-(r) = sup\{\rho^-(S,T) : S,T \subset N,\ dist(S,T) \geq r\} \to 0\ as\ r \to \infty,$$

where

$$\rho^-(S,T) = 0 \vee \{\frac{Cov(f(X_i, i \in S), g(X_j, j \in T))}{(Var(f(X_i, i \in S)))^{1/2}(Var(g(X_j, j \in T)))^{1/2}};\ f\ on\ \mathbb{R}^S,\ g\ on\ \mathbb{R}^T\}$$

Here, f and g are coordinatewise increasing functions.

It is obvious that a sequence of asymptotically negatively associated random variables is negatively associated if and only if $\rho^-(1) = 0$. Compared to negative association, asymptotically negative association defines a strictly larger class of random variables (for detail examples, see Zhang [14]). Consequently, the study of the limit theorems for asymptotically negatively associated sequences is of much interest. For example, Zhang [15] proved the central limit theorem, Wang and Lu [11] obtained some inequalities of maximum of partial sums and weak convergence, Wang and Zhang [12] established the law of the iterated logarithm and Yuan and Wu [13] showed limiting behavior of the maximum sums.

The aim of this paper is to obtain some results concerning complete convergence of weighted sums $\sum_{i=1}^n a_{ni}X_{ni}$, where $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of constants and $\{X_{ni}, i \geq 1, n \geq 1\}$ is an array of rowwise asymptotically negatively associated random variables and to generalize some previous results for arrays of rowwise negatively associated random variables to the asymptotically negative association case.

2. Preliminaries

We start our study from introducing a few definitions and lemmas needed in the main results.

Definition 2.1. An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant D such that

$$P\{|X_{ni}| > x\} \leq DP\{|X| > x\}$$

for all $x \geq 0, i \geq 1$ and $n \geq 1$.

Definition 2.2. A real-valued function $h(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be slowly varying if

$$\lim_{x\to\infty} \frac{h(x\lambda)}{h(x)} = 1$$

for each $\lambda > 0$.

From the definition of asymptotically negative association, we have the following lemma.

Lemma 2.3 ([13]). *Nondecreasing or nonincreasing functions defined on disjoint subsets of an ANA sequence $\{X_n, n \geq 1\}$ with mixing coefficients $\rho^-(s)$ is also ANA with mixing coefficients not greater than $\rho^-(s)$.*

Wang and Lu([11]) proved the following Rosenthal type inequality for asymptotically negatively associated random variables (see also [13]):

Lemma 2.4. *For a positive integer $N \geq 1$, real numbers $p \geq 2$ and $0 \leq r < (1/(6p))^{p/2}$, if $\{X_i, i \geq 1\}$ is a sequence of asymptotically negatively associated random variables with $\rho^-(N) \leq r$, $EX_i = 0$ and $E|X_i|^p < \infty$ for every $i \geq 1$, then for all $n \geq 1$, there is a positive constant $D = D(p, N, r)$ such that*

$$E \max_{1\leq i\leq n} |\sum_{j=1}^{i} X_j|^p \leq D(\sum_{i=1}^{n} E|X_i|^p + (\sum_{i=1}^{n} EX_i^2)^{p/2}).$$

The following lemmas are well known results.

Lemma 2.5 ([1]). *If $h(x) > 0$ is a slowly varying function as $x \to \infty$, then*

(i) $\lim_{x\to\infty} \frac{h(x+u)}{h(x)} = 1$ *for each* $u > 0$;
(ii) $\lim_{x\to\infty} \sup_{2^k \leq x < 2^{k+1}} \frac{h(x)}{h(2^x)} = 1$;
(iii) $\lim_{x\to\infty} x^{\delta} h(x) = \infty,\ \lim_{x\to\infty} x^{-\delta} h(x) = 0$ *for each* $\delta > 0$;
(iv) $C2^{kr}h(\epsilon 2^k) \leq \sum_{j=1}^{k} 2^{jr} h(\epsilon 2^j) \leq C2^{kr}h(\epsilon 2^k)$ *for every* $r > 0$, $\epsilon > 0$ *and positive integer* k;

(v) $C2^{kr}h(\epsilon 2^k) \le \sum_{j=k}^{\infty} 2^{jr}h(\epsilon 2^j) \le C2^{kr}h(2^k)$ *for every* $r > 0$, $\epsilon > 0$ *and positive* k.

Lemma 2.6 ([11]). *Let* $\{X_n, n \ge 1\}$ *be a sequence of random variables which is stochastically dominated by a random variable* X. *Then for any* $p > 0$ *and* $x > 0$,

(i) $E|X_n|^p I[|X_n| < x] \le C\{|X|^p I[|X| < x] + x^p P(|X| \ge x)\}$,
(ii) $E|X_n|^p I[|X_n| \ge x] \le CE|X|^p I[|X| \ge x]$.

3. Main results

Theorem 3.1. *Let* $q \ge 2$ *and integer* $N \ge 1$. *Let* $\{X_{ni}, i \ge 1, n \ge 1\}$ *be an array of rowwise ANA random variables with mixing coefficients* $\rho^-(s)$ *such that* $\rho^-(N) < (\frac{1}{6q})^{q/2}$ *and* $\{a_{ni}, i \ge 1, n \ge 1\}$ *be an array of real numbers. Let* $\{c_n, n \ge 1\}$ *be a sequence of positive real numbers. If for some* $q \ge 2, 0 < t < 2$ *and any* $\epsilon > 0$ *the following conditions are fulfilled*

(a) $\sum_{i=1}^{n} P[|a_{ni}X_{ni}| \ge \epsilon n^{\frac{1}{t}}] = o(1)$,
(b) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{n} P[|a_{ni}X_{ni}| > \epsilon n^{\frac{1}{t}}] < \infty$,
(c) $\sum_{n=1}^{\infty} c_n n^{-\frac{q}{t}} \sum_{i=1}^{n} |a_{ni}|^q E|X_{ni}|^q I[|a_{ni}X_{ni}| < \epsilon n^{\frac{1}{t}}] < \infty$,
(d) $\sum_{n=1}^{\infty} c_n n^{-\frac{q}{t}} (\sum_{i=1}^{n} a_{ni}^2 EX_{ni}^2 I[|a_{ni}X_{ni}| < \epsilon n^{\frac{1}{t}}])^{q/2} < \infty$,

then for any $\epsilon > 0$

(3.1)
$$\sum_{n=1}^{\infty} c_n P[\max_{1\le k\le b_n} |\sum_{i=1}^{k}(a_{ni}X_{ni} - a_{ni}EX_{ni}I[|a_{ni}X_{ni}| < \epsilon n^{\frac{1}{t}}])| > \epsilon n^{\frac{1}{t}}] < \infty.$$

Proof. The proof is similar to that of Theorem 2.1 in [9] for NA random variables. For completeness we repeat it here. It is obvious that if the series $\sum_{n=1}^{\infty} c_n$ is convergent, then (3.1) holds. Therefore we will consider only the case that the series $\sum_{i=1}^{\infty} c_n$ is divergent.

Let

$$Y_{ni} = X_{ni}I[|a_{ni}X_{ni}| < \epsilon n^{\frac{1}{t}}] + \frac{\epsilon n^{\frac{1}{t}}}{a_{ni}} I[a_{ni}X_{ni} \ge \epsilon n^{\frac{1}{t}}] - \frac{\epsilon n^{\frac{1}{t}}}{a_{ni}} I[a_{ni}X_{ni} \le -\epsilon n^{\frac{1}{t}}],$$

$$Z_{ni} = Y_{ni} - EY_{ni},$$

and

$$T_{nk} = \sum_{i=1}^{k} Z_{ni}.$$

Then $\{T_{nk}, n \geq 1, k \geq 1\}$ is an array of rowwise ANA random variables by Lemma 2.3. By (a), for sufficient large n we have

$$\begin{aligned}P[\max_{1\leq k\leq n} | \sum_{i=1}^{k}(a_{ni}X_{ni} - a_{ni}EX_{ni}I[|a_{ni}X_{ni}| < \epsilon n^{\frac{1}{t}}])| > \epsilon n^{\frac{1}{t}}] \\ \leq C[\sum_{i=1}^{n} P[|a_{ni}X_{ni}| > \epsilon n^{\frac{1}{t}}] + \epsilon^{-q} n^{-\frac{q}{t}} E(\max_{1\leq i\leq n} |T_{ni}|)^q].\end{aligned}$$

Using the C_r inequality we can estimate

$$\begin{aligned}&E|Y_{ni} - EY_{ni}|^r \\ &\leq C(E|X_{ni}|^r I[|a_{ni}X_{ni}| < \epsilon n^{\frac{1}{t}}] + \frac{n^{\frac{r}{t}}}{|a_{ni}|^r} P(|a_{ni}X_{ni}| \geq \epsilon n^{\frac{1}{t}})).\end{aligned}$$

Thus by the above estimations and Lemma 2.4 we obtain

$$\begin{aligned}&P[\max_{1\leq k\leq n} | \sum_{i=1}^{k}(a_{ni}X_{ni} - a_{ni}EX_{ni}I[|a_{ni}X_{ni}| < \epsilon n^{\frac{1}{t}}])| > \epsilon n^{\frac{1}{t}}] \\ &\leq C\{\sum_{i=1}^{n} P[|a_{ni}X_{ni}| > \epsilon n^{\frac{1}{t}}] \\ &+ n^{-\frac{q}{t}} (\sum_{i=1}^{n} |a_{ni}|^q E|X_{ni}|^q I[|a_{ni}X_{ni}| < \epsilon n^{\frac{1}{t}}]) \\ &+ n^{-\frac{q}{t}} (\sum_{i=1}^{n} a_{ni}^2 E|X_{ni}|^2 I[|a_{ni}X_{ni}| < \epsilon n^{\frac{1}{t}}])^{q/2}\}.\end{aligned} \tag{3.2}$$

Therefore from (b), (c), (d) and (3.2) the result (3.1) follows. □

Corollary 3.2. *Let $q \geq 2$ and integer $N \geq 1$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise ANA random variables with mixing coefficients $\rho^-(s)$ such that*

$$\rho^-(N) < (\tfrac{1}{6q})^{q/2} \text{ and } EX_{ni} = 0, \text{ and } E|X_{ni}|^p < \infty$$

for $i \geq 1, n \geq 1$ and $1 < p \leq 2$. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers satisfying the condition

$$\sum_{i=1}^{n} |a_{ni}|^p E|X_{ni}|^p = O(n^{\delta}) \text{ as } n \to \infty, \tag{3.3}$$

for some $0 < \delta < \frac{2}{q}$ *and* $q > 2$.

Then, for any $\epsilon > 0$ *and* $\alpha p \geq 1$

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P[\max_{1 \leq i \leq n} |\sum_{j=1}^{i} a_{nj} X_{nj}| > \epsilon n^{\alpha}] < \infty. \tag{3.4}$$

Proof. Put $c_n = n^{\alpha p - 2}$, $t = \frac{1}{\alpha}$ in Theorem 3.1. Then by (3.3) we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n \sum_{i=1}^{n} P[|a_{ni} X_{ni}| > \epsilon n^{\frac{1}{t}}] \\
&\quad < \epsilon^{-p} \sum_{n=1}^{\infty} n^{\alpha p - 2} \sum_{i=1}^{n} \frac{|a_{ni}|^p E|X_{ni}|^p}{n^{\alpha p}} \\
&\quad \leq C \sum_{n=1}^{\infty} n^{-2+\delta} < \infty,
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n n^{-\frac{q}{t}} \sum_{i=1}^{n} |a_{ni}|^q E|X_{ni}|^q I[|a_{ni} X_{ni}| > \epsilon n^{\frac{1}{t}}] \\
&\qquad \leq \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^{n} |a_{ni}|^p E|X_{ni}|^p \\
&\qquad \leq C \sum_{n=1}^{\infty} n^{-2+\delta} < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n n^{-\frac{q}{t}} (\sum_{i=1}^{n} a_{ni}^2 E|X_{ni}|^2 I[|a_{ni} X_{ni}| < \epsilon n^{\frac{1}{t}}])^{q/2} \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - \frac{\alpha p q}{2}} (\sum_{i=1}^{n} |a_{ni}|^p E|X_{ni}|^p)^{q/2} \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - \frac{\alpha p q}{2} + \frac{\delta q}{2}} \leq C \sum_{n=1}^{\infty} n^{\alpha p (1 - \frac{q}{2}) - 1} < \infty
\end{aligned}$$

because $\frac{\delta q}{2} < 1$.

To complete the proof, it is enough to show that the following is holds;

$$\frac{1}{n^\alpha}\sum_{j=1}^{i} a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon n^\alpha] \to 0 \tag{3.5}$$

as $n \to \infty$ for $1 \le i \le n$. Clearly, we can obtain (3.5) by the assumption $EX_{ni} = 0$ for $i \ge 1, n \ge 1$ and (3.3). $\square$

REMARK 3.3. The proof of Corollary 3.2 is similar to that of Corollary 2.2 of [9].

COROLLARY 3.4. *Let $q \ge 2$ and integer $N \ge 1$. Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise mean zero ANA random variables with mixing coefficients $\rho^-(s)$ such that*

$$\rho^-(N) < (\tfrac{1}{6q})^{q/2} \text{ and } E|X_i|^p < \infty$$

for $i \ge 1, n \ge 1$ and $1 < p \le 2$. Assume that the random variables in each row are stochastically dominated by a random variable X such that $E|X|^p < \infty$ for $1 < p \le 2$ and that $\{a_{ni}, i \ge 1, n \ge 1\}$ is an array of real numbers satisfying

$$\sum_{i=1}^{n} |a_{ni}|^p = O(n^\delta)$$

as $n \to \infty$, for some $0 < \delta < \frac{2}{q}$ and $q > 2$. Then for any $\epsilon > 0$ and $\alpha p \ge 1$ (3.4) holds.

COROLLARY 3.5. *Let $q \ge 2$ and integer $N \ge 1$. Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise ANA random variables with mixing coefficients $\rho^-(s)$ such that $\rho^-(N) < (\frac{1}{6q})^{q/2}$ and $EX_{ni} = 0$ for all $i \ge 1$ and $n \ge 1$. Let the random variables in each row be stochastically dominated by a random variable X. Assume that $h(x)$ is a slowly varying function as $n \to \infty$ and $E|X|^{rt}h(|X|^t) < \infty$, for some $r > 1, 0 < t < 2$, then for any $\epsilon > 0$*

$$\sum_{n=1}^{\infty} n^{r-2}h(n)P[\max_{1\le i\le n}|\sum_{j=1}^{i} X_{nj}| > \epsilon n^{\frac{1}{t}}] < \infty.$$

Proof. In Theorem 3.1 we put $c_n = n^{r-2}h(n), a_{ni} = 1$ and for any $1 < r' < r$, choose q such that $q > \max\{2, \frac{2(r-1)}{r'-1}, \frac{2t(r-1)}{2-t}, rt\}$. For any

$0 < \epsilon \leq 1$ we have

$$\begin{aligned}\sum_{i=1}^{n} P[|a_{ni}X_{ni}| > \epsilon n^{\frac{1}{t}}] &\leq D\sum_{i=1}^{n} P[|X| > \epsilon n^{\frac{1}{t}}] \text{ by Definition 2.1} \\ &\leq Dn\frac{E|X|^{r' t}}{n^{r'}} \text{ by Markov inequality} \\ &\leq Dn\frac{E|X|^{rt}h(|X|^t)}{n^r} \to 0 \text{ as } n \to \infty\end{aligned} \tag{3.6}$$

since $E|X|^{r' t} \leq CE|X|^{rt}h(|X|^t) < \infty$ by Lemma 2.5.

By Lemma 2.5, we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P[|a_{ni}X_{ni} \geq \epsilon n^{\frac{1}{t}}] \\ &= \sum_{n=1}^{\infty} n^{r-2}h(n)\sum_{i=1}^{n} P[|X_{ni}| > \epsilon n^{\frac{1}{t}}] \\ &\leq C\sum_{n=1}^{\infty} n^{r-2}h(n)\sum_{i=1}^{n} P[|X| > \epsilon n^{\frac{1}{t}}] \text{ by Definition 2.1} \\ &= C\sum_{n=1}^{\infty} n^{r-1}h(n)P[|X| > \epsilon n^{\frac{1}{t}}] \\ &\leq C\sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} 2^{i(r-1)}h(2^i)P[|X| > \epsilon 2^{\frac{i}{t}}] \\ &\leq C\sum_{i=0}^{\infty} 2^{ir}h(2^i)P[|X| > \epsilon 2^{\frac{i}{t}}] \\ &= C\sum_{i=0}^{\infty} 2^{ir}h(2^i)\sum_{k=i}^{\infty} P[\epsilon^t 2^k \leq |X|^t < \epsilon^t 2^{k+1}] \\ &= C\sum_{k=0}^{\infty} P[\epsilon^t 2^k \leq |X|^t < \epsilon^t 2^{k+1}]\sum_{i=0}^{k} 2^{ir}h(2^i) \\ &\leq C\sum_{k=0}^{\infty} 2^{kr}h(2^k)P[\epsilon^t 2^k \leq |X|^t < \epsilon^t 2^{k+1}] \\ &\leq CE|X|^{rt}h(|X|^t) < \infty.\end{aligned} \tag{3.7}$$

It follows from Lemma 2.5, Lemma 2.6 (i) and (3.7) that

$$\begin{aligned} I &= \sum_{n=1}^{\infty} c_n n^{-\frac{q}{t}} \sum_{i=1}^{n} |a_{ni}|^q E|X_{ni}|^q I[|a_{ni}X_{ni}| < \epsilon n^{\frac{1}{t}}] \\ &= \sum_{n=1}^{\infty} n^{r-2-\frac{q}{t}} h(n) \sum_{i=1}^{n} E|X_{ni}|^q I[|X_{ni}| < \epsilon n^{\frac{1}{t}}] \\ &\leq C\sum_{n=1}^{\infty} n^{r-2-\frac{q}{t}} h(n) \sum_{i=1}^{n} E|X|^q I[|X| < \epsilon n^{\frac{1}{t}}] \end{aligned}$$

$$\begin{aligned} &+C\sum_{n=1}^{\infty} n^{r-2} h(n) \sum_{i=1}^{n} P(|X| \geq \epsilon n^{\frac{1}{t}}) \\ &= I_1 + I_2. \end{aligned}$$

It is obvious that $I_2 < \infty$ by (3.7). It remains to prove $I_1 < \infty$.

$$\begin{aligned} I_1 &= C\sum_{n=1}^{\infty} n^{r-1-\frac{q}{t}} h(n) E|X|^q I[|X| < \epsilon n^{\frac{1}{t}}] \\ &\leq C\sum_{i=0}^{\infty} 2^{i(r-\frac{q}{t})} h(2^i) \sum_{k=0}^{i} E|X|^q I[\epsilon^t 2^k \leq |X|^t < \epsilon^t 2^{k+1}] \\ &\leq C\sum_{i=0}^{\infty} E|X|^q I[\epsilon^t 2^k \leq |X|^t < \epsilon^t 2^{k+1}] \sum_{i=k}^{\infty} 2^{i(r-\frac{q}{t})} h(2^i) \\ &\leq C\sum_{k=0}^{\infty} 2^{k(r-\frac{q}{t})} h(2^k) E|X|^q I[\epsilon^t 2^k \leq |X|^t < \epsilon^t 2^{k+1}] \\ &\leq CE|X|^{rt} h(|X|^t) < \infty. \end{aligned}$$

Hence, $I_1 + I_2 < \infty$, that is,

$$I < \infty. \tag{3.8}$$

By C_r inequality and Lemma 2.6 (i) we obtain

$$
\begin{aligned}
II &= \sum_{n=1}^{\infty} c_n n^{-\frac{q}{t}} (\sum_{i=1}^{n} |a_{ni}|^2 E|X_{ni}|^2 I[|a_{ni}X_{ni} < \epsilon n^{\frac{1}{t}}])^{\frac{q}{2}} \\
&= \sum_{n=1}^{\infty} n^{r-2-\frac{q}{t}} h(n) (\sum_{i=1}^{n} E|X_{ni}|^2 I[|X_{ni}| < \epsilon n^{\frac{1}{t}}])^{\frac{q}{2}} \\
&\leq C \sum_{n=1}^{\infty} n^{r-2-\frac{q}{t}} h(n) (\sum_{i=1}^{n} E|X|^2 I[|X| < \epsilon n^{\frac{1}{t}}] \\
&\quad + n^{\frac{2}{t}} \sum_{i=1}^{n} P(|X| \geq \epsilon n^{\frac{1}{t}}))^{\frac{q}{2}} \\
&\leq C \sum_{n=1}^{\infty} n^{r-2-\frac{q}{t}+\frac{q}{2}} h(n) (E|X|^2 I[|X| < \epsilon n^{\frac{1}{t}}])^{\frac{q}{2}} \\
&\quad + C \sum_{n=1}^{\infty} n^{r-2} h(n) \sum_{i=1}^{n} P(|X| \geq \epsilon n^{\frac{1}{t}}) = I_3 + I_4.
\end{aligned}
$$

It is obvious $I_4 < \infty$ by (3.7). It remains to prove $I_3 < \infty$.

If $rt \leq 2$, from the fact that $(E|X|^2 I[|X| < \epsilon n^{\frac{1}{t}}])^{q/2} \leq C n^{\frac{q}{t} - \frac{r^{'} q}{2}} (E|X|^{r^{'} t})^{q/2}$ for any $1 < r^{'} < r$, and $r - 2 + \frac{q}{2} - \frac{r^{'} q}{2} < -1$, we obtain

$$
\begin{aligned}
I_3 &\leq C \sum_{n=1}^{\infty} n^{r-2+\frac{q}{2}-\frac{r^{'} q}{2}} h(n) (E|X|^{r^{'} t})^{q/2} \\
&\leq C \sum_{n=1}^{\infty} n^{r-2+\frac{q}{2}-\frac{r^{'} q}{2}} h(n) (E|X|^{rt} h(|X|^t))^{q/2} \qquad (3.9) \\
&\leq C \sum_{n=1}^{\infty} n^{r-2+\frac{q}{2}-\frac{r^{'} q}{2}} h(n) < \infty.
\end{aligned}
$$

If $rt > 2$, by the fact that

$$(E|X|^2 I[|X| < \epsilon n^{\frac{1}{t}}])^{q/2} \leq C(E|X|^{rt} h(|X|^t))^{q/2}$$

and $r - 2 - \frac{q}{t} + \frac{q}{2} < -1$, we obtain

$$I_3 \leq C \sum_{n=1}^{\infty} n^{r-2-\frac{q}{t}+\frac{q}{2}} h(n) < \infty. \qquad (3.10)$$

By (3.8), (3.9) and (3.10), we obtain

$$II < \infty. \tag{3.11}$$

Thus, conditions (a), (b), (c) and (d) in theorem 3.1 are fulfilled by (3.6), (3.7), (3.8) and (3.11), respectively. To complete the proof it remains show that

$$III = \frac{1}{n^{\frac{1}{t}}} \max_{1\le k\le n} \sum_{i=1}^{k} EX_{ni} I[|X_{ni}| < \epsilon n^{\frac{1}{t}}] \to 0, \ as\ n \to \infty. \tag{3.12}$$

In fact, if $rt < 1$, by Lemma 2.6 (i) we observe

$$\begin{aligned} III &\le \frac{1}{n^{1/t}} \sum_{i=1}^{n} E|X_{ni}|I[|X_{ni}| < \epsilon n^{1/t}]| \\ &\le C\frac{1}{n^{1/t}}[\sum_{i=1}^{n} E|X|I[|X| < \epsilon n^{1/t}] + \sum_{i=1}^{n} n^{1/t} P[|X| \ge \epsilon n^{1/t})] \\ &\le C[n^{1-r'} E|X^{r't}| + nP[|X| \ge \epsilon n^{1/t}]] \\ &\le Cn^{1-r'} E|X|^{rt} h(|X|^t) \to 0, \ as\ n \to \infty. \end{aligned}$$

If $rt > 1$, by the assumption $EX_{ni} = 0$ for $i \ge 1, n \ge 1$ and by Lemma 2.6 (ii) we have

$$\begin{aligned} III &= \frac{1}{n^{1/t}} \max_{1\le k\le n} |\sum_{i=1}^{k} EX_{ni} I[|X_{ni}| \ge \epsilon n^{1/t}]| \\ &\le \frac{1}{n^{1/t}} \sum_{i=1}^{n} E|X_{ni}|I[|X_{ni}| \ge \epsilon n^{1/t}] \\ &\le C\frac{n}{n^{1/t}} E|X|I[|X| \ge \epsilon n^{1/t}] \\ &\le C\frac{n}{n^{1/t}} n^{(1-r't)/t} E|X|^{r't} \\ &\le Cn^{1-r'} E|X|^{rt} h(|X|^t) \\ &\le Cn^{1-r'} \to 0, \ as\ n \to \infty. \end{aligned}$$

Hence, the proof is complete. □

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References

[1] Z. Bai and C. Su, *The complete convergence for partial sums of i.i.d. random variables*, Sci. Sinica Ser. A, **28** (1985), 1261-1277

[2] H. W. Block, T. H. Savits, and M. Shaked, *Some concepts of negative dependence*, Ann. Probab. **10** (1982), 765-772

[3] W. T. Huang and B. Xu, *Some maximal inequalities and complete convergence of negatively associated random sequences*, Statist. Probab. Lett. **57** (2002), 183-191

[4] K. Joag-Dev and F. Proschan, *Negative association of random variables with applications*, Ann. Statist. **11** (1983), 286-295

[5] A. Kuczmaszewska, *On complete convergence for arrays of rowwise negatively associated random variables*, Statist. Probab. Lett. **79** (2009), 116-124

[6] Y. X. Li and L. X. Zhang, *Complete moment convergence of moving average processes under dependence assumptions*, Statist. Probab. Lett. **70** (2004), 191-197

[7] H. Y. Liang and C. Su, *Complete convergence for weighted sums of NA sequences*, Statist. Probab. Lett. **45** (1999), 85-95

[8] P. Matula, *A note on the almost sure convergence of sums of negatively dependent random variables* Statist. Probab. Lett. **15** (1992), 209-213

[9] Q. M. Shao, *A comparison theorem on moment inequalities between negatively associated and independent random variables,* J. Theoret. Probab. **13** (2000), 343-356

[10] C. Su, L. C. Zhao, and Y. B. Wang, *Moment inequalities and weak convergence for negatively associated sequences*, Sci. China Ser. A, **40** (1997), 172-182

[11] J. F. Wang and F. B. Lu, *Inequalities of maximum partial sums and weak convergence for a class of weak dependent random variables*, Acta. Math. Sin. Engl. Ser. **23** (2006), 127-136

[12] J. F. Wang and L. X. Zhang, *A Berry-Esseen theorem and a law of the iterated logarithm for asymptotically negatively associated sequences*, Acta. Math. Sin. Engl. Ser. **22** (2007), 693-700

[13] D. M. Yuan and X. S. Wu, *Limiting behavior of the maximum of the partial sum for asymptotically negatively associated random variables under residual* Cesàro *alpha-integrability assumption*, J. Statistical Plan. Infer. **140** (2010), 2395-2402

[14] L. X. Zhang, *A functional central limit theorem for asymptotically negatively dependent random fields*, Acta. Math. Hung. **86** (2000), 237-259

[15] L. X. Zhang, *Central limit theorems for asymptotically negative dependent random field*, Acta. Math. Sin. Engl. Ser. **16** (2000), 691-710

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Department of Mathematics Education
Sehan University
Jeonnam 58447, Republic of Korea
E-mail: `kimhc@sehan.ac.kr`