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EXTINCTION AND POSITIVITY OF SOLUTIONS FOR A CLASS OF SEMILINEAR PARABOLIC EQUATIONS WITH GRADIENT SOURCE TERMS

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Abstract. In this paper, we investigated the extinction, positivity, and decay estimates of the solutions to the initial-boundary value problem of the semilinear parabolic equation with nonlinear gradient source and interior absorption terms by using the integral norm estimate method. We found that the decay estimates depend on the choices of initial data, coefficients and domain, and the first eigenvalue of the Laplacean operator with homogeneous Dirichlet boundary condition plays an important role in the proofs of main results.

1. Introduction

We consider the initial-boundary value problem of the semilinear parabolic equation with nonlinear gradient source and interior absorption terms

$$u_t = \Delta u + \lambda|\nabla u|^r - \beta u^q, \quad (x,t) \in \Omega \times (0,+\infty), \tag{1.1}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where λ, $\beta, q > 0$, $0 < r \leq 1$, $\Omega \subset R^N (N \geq 1)$ is a bounded domain with smooth boundary and the initial function satisfies that $0 \not\equiv u_0(x) \in C^\gamma(\overline{\Omega})$ $(0 < \gamma < 1)$ and $u_0(x) = 0$ on $\partial\Omega$. The notations $\|\cdot\|_s$ and $\|\cdot\|_{1,s}$ denote $L^s(\Omega)$- and $W^{1,s}(\Omega)$- norm, respectively, where $s \geq 1$, and $|\Omega|$ denotes the measure of Ω.

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Semilinear parabolic equations like (1.1) have been used as mathematical models in the study of the heat conduction, combustion, filtration phenomena, and diffusion theory. In diffusion theory, the nonlinear gradient term $\lambda|\nabla u|^r$ is physically called source term and $-\beta u^q$ represents absorption term or cooling source. The nonlinear gradient source and absorption terms cooperate and interact with each other during diffusion, see [3, 14, 15].

In the last decades, extinction and positivity of the solutions for semilinear parabolic equations without gradient terms have been extensively studied, see [7, 8, 9, 11] and the references therein. In the case of $\lambda = 0$, it is well known that the solutions of problem (1.1)-(1.3) with $0 < q < 1$ vanish in finite time. Evans and Knerr [8] investigated extinction behavior of solutions for the Cauchy problem of semilinear parabolic equation by constructing a suitable comparison function. Fukuda [9] considered the homogeneous Dirichlet boundary value problem of the semilinear parabolic equation

$$u_t = \Delta u + \lambda u - u^q, \quad (x,t) \in \Omega \times (0, +\infty), \tag{1.4}$$

with (1.2), (1.3), $\lambda \geq 0$ and $0 < q < 1$, and obtained some sufficient conditions for extinction of the solution by using the energy method.

Recently, many researchers have been devoted to the studies of blow-up and extinction of solutions for nonlinear parabolic equations with gradient terms. For example, Chipton and Weisster [7] firstly studied the initial-boundary value problem of the semilinear heat equation with gradient absorption term

$$u_t = \Delta u - |\nabla u|^r + |u|^{q-1}u, \quad (x,t) \in \Omega \times (0, +\infty), \tag{1.5}$$

where $r, q > 1$, and found the solution of (1.5) blows up in finite time under appropriate conditions on r, q, and u_0 by using the energy method. Hesaarki and Moameni [12] investigated the homogeneous Dirichlet boundary value problem of the semilinear heat equation with gradient term

$$u_t = \Delta u + b|\nabla u|^r + au^q, \quad (x,t) \in \Omega \times (0, +\infty), \tag{1.6}$$

where $r, q > 1$, and proved that the solution either exists globally or blows up in finite time under some assumptions on a, b, r, q, Ω, and the initial data. The blow-up phenomena, blow-up rates, and blow-up sets of the solutions to equation (1.1) with $r, q \in (1, +\infty)$ have been extensively studied, see [1, 2] and the references therein. The extinction phenomena of solutions for these kind of equations were also investigated. For instance, Benachour *et al.* [4] considered the semilinear heat equation with absorption term

$$u_t = \Delta u - \lambda |\nabla u|^r, \quad (x,t) \in \Omega \times (0,+\infty), \tag{1.7}$$

under (1.2) and (1.3), and proved that a sufficient condition for extinction to occur is $0 < r < 1$ by using the upper and lower solution method. For equation (1.1) with $\lambda = 1$, $0 < r < 2$, and $0 < q < 1$, some researches on extinction properties of solutions have been performed, but all the results are limited to local range and higher dimensional space, while precise decay estimate has not been given (cf. [16]). As far as we know, there are fewer papers on the extinction phenomena for nonlinear parabolic equations with gradient source term.

The existence and regularity of the solution for problem (1.1)-(1.3) have been studied by Ladyzenska *et al,* see [13]. Thus, in this paper, our work is to establish sufficient conditions on extinction and positivity of the solution for problem (1.1)-(1.3) in the whole dimensional space. The main tool is the L^p-integral norm estimate method, which can be applied in many research fields, especially for those situations in which traditional methods based on the comparison principles have failed. We found that the exponential decay estimates depend on the choices of initial data, coefficients and domain, and the first eigenvalue λ_1 of $-\Delta$ with homogeneous Dirichlet boundary condition plays an important role in the proofs of Theorems 1.1 and 1.2. More precisely, we give the following results:

THEOREM 1.1. *Assume that $0 < q < 1$ and $r = 1$. Then the solution of problem (1.1)-(1.3) vanishes in finite time for any initial data u_0 provided that λ is sufficiently small, and we have*

$$\begin{aligned} &\|u(\cdot,t)\|_2 \\ &\quad \leq \left[\left(\|u_0\|_2^{2-k_1} + \frac{C(N,q)^{-k_1}\beta}{C(\eta)C_1} \right) e^{(k_1-2)C_1 t} - \frac{C(N,q)^{-k_1}\beta}{C(\eta)C_1} \right]^{\frac{1}{2-k_1}}, \\ &t \in [0,T_1); \quad \|u(\cdot,t)\|_2 \equiv 0, \quad t \in [T_1,+\infty), \end{aligned}$$

where $k_1 = \frac{2N(1-q)+4(q+1)}{N(1-q)+4}$, and C_1 and T_1 are given by (3.5) and (3.6), respectively.

THEOREM 1.2. *Assume that $0 < q < 1$ and $\frac{N(1-q)+4q}{N(1-q)+2(q+1)} < r < 1$. Then the solution of problem (1.1)-(1.3) vanishes in finite time provided that either $\|u_0\|_2$, λ, or $|\Omega|$ is sufficiently small, or β is sufficiently large, and we have*

$$\|u(\cdot,t)\|_2 \leq \|u_0\| e^{-\alpha t}, \quad t \in [0,T_2),$$

$$\|u(\cdot,t)\|_2 \le \left[\left(\|u(\cdot,T_2)\|_2^{2-k_1} + \frac{C_3}{C_2}\right) e^{(k_1-2)C_2(t-T_2)} - \frac{C_3}{C_2}\right]^{\frac{1}{2-k_1}},$$
$$t \in [T_2, T_3); \quad \|u(\cdot,t)\|_2 \equiv 0, \quad t \in [T_3, +\infty),$$

where k_1 is the same as the above, and C_2, C_3, and T_3 are given by (4.3)-(4.5), respectively.

THEOREM 1.3. *Assume that $0 < q < 1$ and $\frac{2q}{q+1} < r \le \frac{N(1-q)+4q}{N(1-q)+2(q+1)}$. Then the solution of problem (1.1)-(1.3) vanishes in finite time for any initial data u_0 provided that either λ and $|\Omega|$ are sufficiently small or β is sufficiently large, and we have*

$$\|u(\cdot,t)\|_{s+1} \le \left[\|u_0\|_{s+1}^{\frac{2(1-r)}{2-r}} - \frac{2(1-r)}{2-r}t\right]^{\frac{2-r}{2(1-r)}}, \quad t \in [0, T_4),$$
$$\|u(\cdot,t)\|_{s+1} \equiv 0, \quad t \in [T_4, +\infty),$$

where T_4 and s are given by (5.8) and (5.11), respectively.

REMARK 1.4. Theorems 1.1-1.3 all require that either $|\Omega|$, λ, or $\|u_0\|_2$ should be sufficiently small, or β should be sufficiently large, and we will give more concrete conditions which they satisfy in the later proofs.

REMARK 1.5. In fact, with a slight change of the proofs of Theorems 1.1-1.3, one can see that the behavior of the solutions of problem (1.1)-(1.3) will also change if signs of the coefficients of the nonlinear gradient source and absorption terms are changed. For instance, if $\lambda < 0$ and $\beta > 0$, the solution of problem (1.1)-(1.3) vanishes in finite time for any initial data, and if $\lambda < 0$ and $\beta < 0$, the solution of problem (1.1)-(1.3) vanishes in finite time provided that $\|u_0\|_2$ is sufficiently small or β is sufficiently large. When $\lambda > 0$ and $\beta < 0$, the solution of problem (1.1)-(1.3) blows up in infinite time for any initial data provided that β is sufficiently small.

THEOREM 1.6. *Assume that $q \ge 1$. Then the solution of problem (1.1)-(1.3) is positive and does not vanish in finite time for any $r > 0$ and nonnegative initial data $u_0(x) \in W_0^{1,2} \cap L^{q+1}(\Omega)$, and we have the inequality*

$$\|u(\cdot,t)\|_2 \ge Ce^{-\rho^{1-q}t}, \quad t \in (T, +\infty),$$

where C, ρ, and T are positive constants which are independent of $u(x,t)$.

REMARK 1.7. According to Theorems 1.1-1.3 and 1.6, we observe that $q = 1$ is the critical exponent for extinction of the solution to problem (1.1)-(1.3), when $\frac{2q}{q+1} < r \le 1$.

The outline of this paper is as follows: In Section 2, we give some preliminary lemmas and provide proofs for Theorems 1.1-1.3 and 1.6 in Sections 3-6, respectively.

2. Preliminary results

Before proving our main results, we give some preliminary lemmas and the Gagliardo-Nirenberg inequality, which are very important in the following proofs. Since the lemmas can be similarly showed as the proofs given in [5, 6, 13], we will not give the proofs.

LEMMA 2.1. *Let $y(t)$ be a nonnegative absolutely continuous function on $[0,+\infty)$ satisfying*

$$\frac{dy}{dt} + \alpha y^k \leq 0, \quad t \geq 0; \quad y(0) \geq 0,$$

where $\alpha > 0$ is a constant and $k \in (0,1)$. Then we have the decay estimate

$$y(t) \leq [y^{1-k}(0) - \alpha(1-k)t]^{\frac{1}{1-k}}, \quad t \in [0, T_*),$$

$$y(t) \equiv 0, \quad t \in [T_*, +\infty),$$

where $T_* = \frac{y^{1-k}(0)}{\alpha(1-k)}$.

LEMMA 2.2. *Let $y(t)$ be a nonnegative absolutely continuous function on $[0,+\infty)$ satisfying*

$$\frac{dy}{dt} + \alpha y^k + \beta y \leq 0, \quad t \geq T_0; \quad y(T_0) \geq 0,$$

where $\alpha, \beta > 0$ are constants and $k \in (0,1)$. Then we have the decay estimate

$$y(t) \leq \left[\left(y^{1-k}(T_0) + \frac{\alpha}{\beta}\right) e^{(k-1)\beta(t-T_0)} - \frac{\alpha}{\beta}\right]^{\frac{1}{1-k}}, \quad t \in [T_0, T_*),$$

$$y(t) \equiv 0, \quad t \in [T_*, +\infty),$$

where $T_* = \frac{1}{(1-k)\beta} \ln(1 + \frac{\beta}{\alpha} y^{1-k}(T_0)) + T_0$.

LEMMA 2.3. *Suppose that $0 < k < m \leq 1$ and $y(t)$ is a nonnegative solution of the differential inequality*

$$\frac{dy}{dt} + \alpha y^k + \beta y \leq \gamma y^m, \quad t \geq 0; \quad y(0) = y_0 > 0,$$

where $\alpha, \beta > 0$ and γ is a positive constant such that $\gamma < \alpha y_0^{k-m}$. Then there exists $\eta > \beta$ such that

$$0 \le y(t) \le y_0 e^{-\eta t}, \quad t \ge 0.$$

LEMMA 2.4. *(The Gagliardo-Nirenberg inequality) Suppose that $u \in W_0^{k,m}(\Omega)$, $1 \le m \le +\infty$, $0 \le j < k$, and $1 \ge \frac{1}{r} \ge \frac{1}{m} - \frac{k}{N}$. Then we have the inequality*

$$\|D^j u\|_q \le C\|D^k u\|_m^{\theta}\|u\|_r^{1-\theta},$$

where C is a constant depending on N, m, r, j, k, q, and $\frac{1}{q} = \frac{j}{N} + \theta(\frac{1}{m} - \frac{k}{N}) + \frac{1-\theta}{r}$. While if $m < \frac{N}{k-j}$, then $q \in [\frac{Nr}{N+rj}, \frac{Nm}{N-(k-j)m}]$, and if $m \ge \frac{N}{k-j}$, then $q \in [\frac{Nr}{N+rj}, +\infty]$.

Finally, we need the following comparison principle, which can be found in [2, 10, 13]:

LEMMA 2.5. *Suppose that $v(x,t)$ and $u(x,t)$ are sub and super solutions of problem (1.1)-(1.3), respectively. Then $v(x,t) \le u(x,t)$ a.e. in $\Omega \times (0,T)$, where $0 < T < \infty$.*

3. Proof of Theorem 1.1

Proof. Multiplying both sides of (1.1) by u and integrating the result over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_2^2 + \beta\|u\|_{q+1}^{q+1} = \lambda \int_\Omega u|\nabla u| dx. \tag{3.1}$$

By Young's inequality, we have the inequality

$$\int_\Omega u|\nabla u| dx \le \varepsilon\|\nabla u\|_2^2 + C(\varepsilon)\|u\|_2^2. \tag{3.2}$$

From Lemma 2.4, we get the inequality

$$\|u\|_2 \le C(N,q)\|u\|_{q+1}^{1-\theta_1}\|\nabla u\|_2^{\theta_1},$$

where $\theta_1 = (\frac{1}{q+1} - \frac{1}{2})(\frac{1}{N} - \frac{1}{2} + \frac{1}{q+1})^{-1} = \frac{N(1-q)}{N(1-q)+2(q+1)}$.

Since $0 < q < 1$, one can easily see that $0 < \theta_1 < 1$. By Young's inequality again, we obtain the inequalities

$$\begin{aligned}\|u\|_2^{k_1} &\le C(N,q)^{k_1}\|u\|_{q+1}^{(1-\theta_1)k_1}\|\nabla u\|_2^{\theta_1 k_1} \\ &\le C(N,q)^{k_1}\left(\eta\|\nabla u\|_2^2 + C(\eta)\|u\|_{q+1}^{\frac{2k_1(1-\theta_1)}{2-k_1\theta_1}}\right),\end{aligned}$$

where $\eta > 0$ and $k_1 > 1$ will be determined later. Here, we choose $k_1 = \frac{2N(1-q)+4(q+1)}{N(1-q)+4}$ and then $1 < k_1 < 2$ and $\frac{2k_1(1-\theta_1)}{2-k_1\theta_1} = q+1$. Thus, one can obtain the inequality

$$\frac{C(N,q)^{-k_1}\beta}{C(\eta)}\|u\|_2^{k_1} \leq \frac{\eta\beta}{C(\eta)}\|\nabla u\|_2^2 + \beta\|u\|_{q+1}^{q+1}. \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we get the inequality

(3.4)

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \left[1 - \lambda\varepsilon - \frac{\eta\beta}{C(\eta)}\right]\|\nabla u\|_2^2 + \frac{C(N,q)^{-k_1}\beta}{C(\eta)}\|u\|_2^{k_1} \leq \lambda C(\varepsilon)\|u\|_2^2.$$

Here, we can choose ε and η small enough for which $1 - \lambda\varepsilon - \frac{\eta\beta}{C(\eta)} > 0$. By Poincare's inequality, we have the inequality

$$\lambda_1\|u\|_2^2 \leq \|\nabla u\|_2^2.$$

Substituting the above inequality into (3.4), we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + C_1\|u\|_2^2 + \frac{C(N,q)^{-k_1}\beta}{C(\eta)}\|u\|_2^{k_1} \leq 0,$$

where

$$C_1 = \lambda_1\left[1 - \lambda\varepsilon - \frac{\eta\beta}{C(\eta)}\right] - \lambda C(\varepsilon). \tag{3.5}$$

Once ε and η are fixed, we can choose λ small enough so that $C_1 > 0$. Then

$$\frac{d}{dt}\|u\|_2 + \frac{C(N,q)^{-k_1}\beta}{C(\eta)}\|u\|_2^{k_1-1} + C_1\|u\|_2 \leq 0.$$

By Lemma 2.2, we can obtain the desired decay estimate for

$$T_1 = \frac{1}{(2-k_1)C_1}\ln\left(1 + \frac{C(\eta)C_1}{C(N,q)^{-k_1}\beta}\|u_0\|_2^{2-k_1}\right). \tag{3.6}$$

This completes the proof. $\square$

4. Proof of Theorem 1.2

Proof. Multiplying both sides of (1.1) by u and integrating the result over Ω, we have the equation

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_2^2 + \beta\|u\|_{q+1}^{q+1} = \lambda\int_\Omega u|\nabla u|^r dx. \tag{4.1}$$

By Young's inequality, we have

$$\int_\Omega u|\nabla u|^r dx \le \varepsilon\|\nabla u\|_2^2 + C(\varepsilon)|\Omega|^{\frac{1-r}{2-r}}\|u\|_2^{1+\frac{r}{2-r}}. \tag{4.2}$$

Substituting (3.3) and (4.2) into (4.1), we get the inequality

$$\frac{d}{dt}\|u\|_2 + \frac{C(N,q)^{-k_1}\beta}{C(\eta)}\|u\|_2^{k_1-1} + C_2\|u\|_2 \le \lambda C(\varepsilon)|\Omega|^{\frac{1-r}{2-r}}\|u\|_2^{\frac{r}{2-r}},$$

where

$$C_2 = \lambda_1\left[1 - \lambda\varepsilon - \frac{\eta\beta}{C(\eta)}\right]. \tag{4.3}$$

Here, one can choose ε and η small enough for which $C_2 > 0$. By Lemma 2.3, there exists $\alpha > C_2$ such that

$$0 \le \|u\|_2 \le \|u_0\|e^{-\alpha t}, \quad t \ge 0,$$

provided that

$$\|u_0\|_2 < \left[\frac{C(N,q)^{-k_1}\beta}{C(\eta)\lambda C(\varepsilon)|\Omega|^{\frac{1-r}{2-r}}}\right]^{\frac{1}{\frac{r}{2-r}-k_1+1}}.$$

Furthermore, there exists $T_2 > 0$ such that

$$\begin{aligned}
&\frac{C(N,q)^{-k_1}\beta}{C(\eta)} - \lambda C(\varepsilon)|\Omega|^{\frac{1-r}{2-r}}\|u\|_2^{\frac{r}{2-r}-k_1+1} \\
&\ge \frac{C(N,q)^{-k_1}\beta}{C(\eta)} - \lambda C(\varepsilon)|\Omega|^{\frac{1-r}{2-r}}(\|u_0\|_2 e^{-\alpha T_2})^{\frac{r}{2-r}-k_1+1} =: C_3 > 0,
\end{aligned} \tag{4.4}$$

for all $t \in [T_2, +\infty)$. Therefore, when $t \in [T_2, +\infty)$, we have

$$\frac{d}{dt}\|u\|_2 + C_3\|u\|_2^{k_1-1} + C_2\|u\|_2 \le 0.$$

By Lemma 2.2, we can obtain the desired decay estimate for

$$T_3 = \frac{1}{(2-k_1)C_2}\ln\left(1 + \frac{C_2}{C_3}\|u(\cdot,T_2)\|_2^{2-k_1}\right) + T_2. \tag{4.5}$$

This completes the proof. □

5. Proof of Theorem 1.3

Proof. Multiplying both sides of (1.1) by u^s, where $s \geq 1$ will be determined, and integrating the result over Ω, we have the equation

(5.1)
$$\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1} + \frac{4s}{(s+1)^2}\|\nabla u^{\frac{s+1}{2}}\|_2^2 + \beta\|u\|_{q+s}^{q+s} = \lambda\int_\Omega u^s|\nabla u|^r dx.$$

Since $0 < r < 1$, one can see that $0 < \frac{r}{2-r} < 1$. By Young's inequality, we have

$$\begin{aligned}
\int_\Omega u^s|\nabla u|^r dx &= \left(\frac{2}{s+1}\right)^r \int_\Omega u^{s-\frac{(s-1)r}{2}}|\nabla u^{\frac{s+1}{2}}|^r dx \\
&\leq \varepsilon\|\nabla u^{\frac{s+1}{2}}\|_2^2 + C(\varepsilon)\int_\Omega u^{s+\frac{r}{2-r}}dx \qquad (5.2)\\
&\leq \varepsilon\|\nabla u^{\frac{s+1}{2}}\|_2^2 + C(\varepsilon)|\Omega|^{\frac{2(1-r)}{(s+1)(2-r)}}\|u\|_{s+1}^{s+\frac{r}{2-r}}.
\end{aligned}$$

Substituting (5.2) into (5.1), we get the inequality

$$\begin{aligned}
\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1} &+ \left[\frac{4s}{(s+1)^2} - \lambda\varepsilon\right]\|\nabla u^{\frac{s+1}{2}}\|_2^2 + \beta\|u\|_{q+s}^{q+s} \qquad (5.3)\\
&\leq \lambda C(\varepsilon)|\Omega|^{\frac{2(1-r)}{(s+1)(2-r)}}\|u\|_{s+1}^{s+\frac{r}{2-r}}.
\end{aligned}$$

By Lemma 2.4, we have the inequality

$$\|u\|_{s+1}^{\frac{s+1}{2}} = \|u^{\frac{s+1}{2}}\|_2 \leq C(N,s,q)\|\nabla u^{\frac{s+1}{2}}\|_2^{\theta_2}\|u^{\frac{s+1}{2}}\|_{\frac{2(q+s)}{s+1}}^{1-\theta_2},$$

where $\theta_2 = \frac{N(1-q)}{N(1-q)+2(q+s)}$. If we choose $\sigma > 0$ such that

$$0 < \sigma\theta_2 < 2, \tag{5.4}$$

$$\frac{\sigma(1-\theta_2)(s+1)}{2-\sigma\theta_2} = q+s, \tag{5.5}$$

then, by Young's inequality, we obtain

$$\begin{aligned}
\|u\|_{s+1}^{\frac{\sigma(s+1)}{2}} &\leq C(N,s,q)^\sigma\|\nabla u^{\frac{s+1}{2}}\|_2^{\sigma\theta_2}\|u^{\frac{s+1}{2}}\|_{\frac{2(q+s)}{s+1}}^{\sigma(1-\theta_2)} \\
&= C(N,s,q)^\sigma\|\nabla u^{\frac{s+1}{2}}\|_2^{\sigma\theta_2}\|u\|_{q+s}^{\frac{\sigma(1-\theta_2)(s+1)}{2}} \\
&\leq C(N,s,q)^\sigma\left(\delta\|\nabla u^{\frac{s+1}{2}}\|_2^2 + C(\delta)\|u\|_{q+s}^{q+s}\right).
\end{aligned}$$

Therefore, we have the inequality

$$\frac{\beta C(N,s,q)^{-\sigma}}{C(\delta)}\|u\|_{s+1}^{\frac{\sigma(s+1)}{2}} - \frac{\beta\delta}{C(\delta)}\|\nabla u^{\frac{s+1}{2}}\|_2^2 \le \beta\|u\|_{q+s}^{q+s}.$$

Substituting the above inequality into (5.3), we get the inequality

$$\begin{aligned}\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1} &+ \left[\frac{4s}{(s+1)^2} - \lambda\varepsilon - \frac{\beta\delta}{C(\delta)}\right]\|\nabla u^{\frac{s+1}{2}}\|_2^2 \\ &+ \frac{\beta C(N,s,q)^{-\sigma}}{C(\delta)}\|u\|_{s+1}^{\frac{\sigma(s+1)}{2}} \le \lambda C(\varepsilon)|\Omega|^{\frac{2(1-r)}{(s+1)(2-r)}}\|u\|_{s+1}^{s+\frac{r}{2-r}}.\end{aligned}$$

We can choose ε and δ small enough for which $\frac{4s}{(s+1)^2} - \lambda\varepsilon - \frac{\beta\delta}{C(\delta)} \ge 0$, and hence, we obtain the inequality

$$\begin{aligned}\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1} &+ \frac{\beta C(N,s,q)^{-\sigma}}{C(\delta)}\|u\|_{s+1}^{\frac{\sigma(s+1)}{2}} \\ &\le \lambda C(\varepsilon)|\Omega|^{\frac{2(1-r)}{(s+1)(2-r)}}\|u\|_{s+1}^{s+\frac{r}{2-r}}.\end{aligned}$$

Once ε and δ are fixed, we can choose λ and $|\Omega|$ small enough or β large enough so that

$$\frac{\beta C(N,s,q)^{-\sigma}}{C(\delta)} \ge \lambda C(\varepsilon)|\Omega|^{\frac{2(1-r)}{(s+1)(2-r)}} + 1. \tag{5.6}$$

If we can choose s such that

$$\frac{\sigma(s+1)}{2} = s + \frac{r}{2-r}, \quad s \ge 1, \tag{5.7}$$

we have the inequality

$$\frac{d}{dt}\|u\|_{s+1} + \|u\|_{s+1}^{\frac{r}{2-r}} \le 0.$$

From $0 < r < 1$ and (5.7), one can easily see that $0 < \frac{r}{2-r} < 1$. By Lemma 1, we can obtain the desired decay estimate for

$$T_4 = \frac{(2-r)\|u_0\|_{s+1}^{\frac{2(1-r)}{2-r}}}{2(1-r)}. \tag{5.8}$$

Therefore, if (5.4), (5.5), and (5.7) are all established, the solution of problem (1.1)-(1.3) will vanish in finite time under the condition that β, λ, and $|\Omega|$ satisfy (5.6).

What conditions for r and q should be satisfied in order to get $s \ge 1$ for which (5.4), (5.5), and (5.7) are all satisfied? It follows from (5.7) that

$$s = \frac{\sigma(2-r) - 2r}{(2-\sigma)(2-r)}. \tag{5.9}$$

From (5.5), we get $\sigma[1-(1-q)\theta_2] = (2-\sigma)s + 2q$.

Substituting (5.9) into the equation above, we get

$$\sigma = \frac{2[r-q(2-r)]}{\theta_2(1-q)(2-r)} = \frac{2[r-q(2-r)][N(1-q)+2(s+q)]}{N(2-r)(1-q)^2}, \tag{5.10}$$

and hence, $\sigma\theta_2 = \frac{2[r-q(2-r)]}{(1-q)(2-r)} < 2$. Substituting (5.10) into (5.9), we obtain

$$s = \frac{[r-q(2-r)][N(1-q)+2(s+q)] - Nr(1-q)^2}{N(2-r)(1-q)^2 - [r-q(2-r)][N(1-q)+2(s+q)]}.$$

Thus, we have the equation

$$\begin{aligned} &2[q(2-r)-r]s^2 + \{N(2-r)(1-q)^2 \\ &\quad - [r-q(2-r)][N(1-q)+2q+2]\}s \\ &\quad - \{[r-q(2-r)][N(1-q)+2q] - Nr(1-q)^2\} = 0. \end{aligned} \tag{5.11}$$

We write equation (5.11) as $as^2+bs+c=0$ and set $\Delta = b^2-4ac$. Since $0<r<1$, one can easily see that $b>0$.

1) If $0<r<\frac{2q}{q+1}$, then $a,b,c>0$. Equation (5.11) will have two negative real roots if it has real roots. It does not satisfy the requirement that $s\geq 1$.

2) If $\frac{2q}{q+1}<r<1$, then $a<0$ and $c=-2qr[N(1-q)+(q+1)]+2q[N(1-q)+2q]$. If $r\leq \frac{N(1-q)+2q}{N(1-q)+q+1} = r_1 < 1$, then $c\geq 0$. Since $b>0$, we have $\Delta>0$ and equation (5.11) has two real roots which have opposite signs by Vieta's theorem. Since $a<0$, $s=\frac{-b-\sqrt{\Delta}}{2a}$ is the positive real root, and $s\geq 1$ is equivalent to $\sqrt{\Delta}\geq -b-2a$. It can be easily seen that $-b-2a\geq 0$ when $r\geq \frac{N(1-q)+2q(q+3)}{N(1-q)+(q+1)(q+3)} = r_2$ and $\frac{2q}{q+1}<r_2<r_1$. If $\frac{2q}{q+1}<r\leq r_2$, we have $\sqrt{\Delta}\geq 0\geq -b-2a$ and $s\geq 1$. If $r_2<r\leq r_1$, then $s\geq 1$ is equivalent to $\Delta\geq (2a+b)^2$ or $a+b+c\geq 0$. We then easily obtain $r\leq \frac{N(1-q)+4q}{N(1-q)+2(q+1)} = r_0$ and $r_2<r_0<r_1$. Therefore, we always have $s\geq 1$ when $r_2<r\leq r_0$. In summary, if $\frac{2q}{q+1}<r<r_0$, we have $s\geq 1$, which satisfies the requirement. If $r_1<r<1$, then $c<0$ and $b>0$. Equation (5.11) does not have real root when $\Delta<0$, while it may have two positive roots when $\Delta\geq 0$, and the bigger root is $s=\frac{-b-\sqrt{\Delta}}{2a}$. Since $r>r_1>r_2$, we get that $s\geq 1$ is equivalent to

$r \le r_0 < r_1$, which contradict to the fact that $r > r_1$, and hence, if $r_1 < r < 1$, we cannot get s which satisfies the requirement.

3) If $r = \frac{2q}{q+1}$, we get $a = 0$ and $s = -\frac{Nr}{N(2-r)} < 0$, which does not satisfy the requirement. □

REMARK 5.1. When $0 < q < 1$ and $0 < r \le \frac{2q}{q+1}$, it can be shown that one cannot determine if the solution of problem (1.1)-(1.3) vanishes or not by using the L^p-integral norm estimate method. We conjecture the solution of problem (1.1)-(1.3) vanishes in finite time.

6. Proof of Theorem 1.6

Proof. Suppose that $v(x,t)$ is a solution of the following initial-bonundary value problem:

$$v_t = \Delta v - \beta v^q, \quad (x,t) \in \Omega \times (0,+\infty),$$
$$v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty),$$
$$v(x,0) = u_0(x), \quad x \in \Omega.$$

By the maximum principle, there exists $T > 0$ such that $v(x,t) \ge 0$, $(x,t) \in \overline{\Omega} \times (0,T]$. By Theorem 3.1 in [11], $v(x,t)$ cannot vanish in finite time and

$$\|v\|_2 \ge Ce^{-\rho^{1-q}t}, \quad (x,t) \in \Omega \times (T,+\infty),$$

where $C, \rho, T > 0$ are constants which are independent of $u(x,t)$. By Lemma 2.5, we have $u(x,t) \ge v(x,t)$, $(x,t) \in \overline{\Omega} \times (0,T]$. □

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