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COVARIANT DERIVATIVE OF CERTAIN STRUCTURES IN TANGENT BUNDLE

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Abstract. Our aim is to study covariant derivative with respect to complete and vertical lifts of generalized almost r-contact structure in tangent bundle.

1. Introduction

The differential geometry of tangent bundles has valuable vicinity in the differential geometry because it provides numerous innovative problems in the study of modern differential geometry. Numerous investigators made valuable contributions on differential geometry of tangent bundles including Davies [4], Yano and Ishihara [13], Yano and Davies [14], Innus and Udriste [5]. The complete, vertical and horizontal lifts of tensor fields and connections on any manifold M to tangent manifold TM has been studied by Yano and Ishihara [15]. Furthermore, Das and the author [1] have obtained almost product structure by means of the complete, vertical and horizontal lifts of almost r-contact structures on tangent bundles. The author [6, 7] has studied lifts of hypersurface with connections to tangent bundles and Kaehler manifold. Tekkoyun [11] produced almost para-complex structures on tangent bundle by using lifting theory.

The rest of the paper is organized as follows: In Section 2, we define tangent bundle, complete, vertical lifts, Hsu-structure and generalized almost r-contact structure. We have considered the generalized almost r-contact structure in manifold and then defined Hsu-structure in tangent bundle in Section 3. In Section 4, we prove theorems on covariant derivative of generalized almost r-contact structure in tangent bundle.

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2. Preliminaries

Let M be an n-dimensional differentiable manifold and let $T(M) = \bigcup_{p\in M} T_p(M)$ be its tangent bundle. Then $T(M)$ is also a differentiable manifold. Let $X = \sum_{i=1}^{n}(\partial/\partial x^i)$ and $\eta = \sum_{i=1}^{n}\eta^i dx^i$ be the expressions in local coordinates for the vector field X and the 1-form η in M. Let (x^i, y^i) be local coordinates of point in $T(M)$ induced naturally from the coordinate chart (U, x^i) in M [15].

2.1. Vertical lifts

If f is a function in M, we write f^V for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \to M$ and $f : M \to R$, so that

$$f^V = f \circ \pi. \tag{2.1}$$

Thus, if a point $\tilde{p} \in \pi^{-1}(U)$ has induced coordinates (x^h, y^h), then

$$f^V(\tilde{p}) = f^V(x, y) = f \circ \pi(\tilde{p}) = f(p) = f(x). \tag{2.2}$$

Thus the value of $f^V(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^V the vertical lift of the function f.

Furthermore, the vertical lifts of tensor fields obey the general properties [5, 10, 15]:

(a) $(f \cdot g)^V = f^V g^V, (f + g)^V = f^V + g^V$,
(b) $(X + Y)^V = X^V + Y^V, (f \cdot X)^V = f^V X^V, X^V f^V = 0$,
$[X^V, Y^V] = 0$,
(c) $(f \cdot \eta)^V = f^V \eta^V, \eta^V(X^V) = 0, X^V(Y^V) = 0$,
for arbitrary $f, g \in \Im_0^0(M)$, $X, Y \in \Im_0^1(M), \eta \in \Im_1^0(M)$,
$F \in \Im_1^1(M)$.

2.2. Complete lifts

If f is a function in M, we write f^C for the function in $T(M)$ defined by [15]

$$f^C = i(df) \tag{2.3}$$

and call f^C the *complete lift* of the function f. The complete lift of a function f has the local expression

$$f^C = y^i \partial_i f = \partial f \tag{2.4}$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X \in \Im_0^1(M)$, that is X is a vector field in M. We define a vector field X^C in $T(M)$ by

$$X^C f^C = (Xf)^C, \tag{2.5}$$

where f is an arbitrary function in M and call X^C the complete lift of X in $T(M)$. The complete lift X^C of X with components x^h in M has components

$$X^C : \begin{bmatrix} x^h \\ \partial x^h \end{bmatrix} \tag{2.6}$$

with respect to the induced coordinates in $T(M)$.

Suppose that $\eta \in \Im_1^0(M)$, that is η is a 1-form in M. Then a 1-form η^C in $T(M)$ defined by

$$\eta^C(X^C) = (\eta(X))^C, \tag{2.7}$$

where X is an arbitrary vector field in M. We call η^C the complete lift of η.

Moreover, the complete lifts of tensor fields obey the general properties [9, 15]:

(a) $(fX)^C = f^C X^V + f^V X^C = (Xf)^C, X^C f^V = (Xf)^V$,
$V^V f^C = (Xf)^V$,
(b) $\phi^V X^C = (\phi X)^V, \phi^C X^V = (\phi X)^V, (\phi X)^C = \phi^C X^C$;
$\eta^V(X^C) = (\eta(X))^C, \eta^C(X^V) = (\eta(X))^V$,
(c) $[X^V, Y^C] = [X,Y]^C, [X^C, Y^C] = [X,Y]^C$;
$I^C = 1, I^V X^C = X^V$,
for arbitrary $f, g \in \Im_0^0(M)$, $X, Y \in \Im_0^1(M), \eta \in \Im_1^0(M)$,
$F \in \Im_1^1(M)$.

2.3. Hsu-structure

Let M be an n-dimensional differentiable manifold of C^∞ class. If there exists a tensor field F of type (1, 1) and of C^∞ class on M such that

$$F^2 = a^r I, \tag{2.8}$$

where I denotes the unit tensor field, r is an integer and a is a nonzero complex number. We say that the manifold M endowed with Hsu-structure [3].

2.4. Generalized almost r-contact structure

Let M be an n-dimensional differentiable manifold of C^∞ class. Suppose that there are given a tensor field F of type (1, 1), a vector field ξ_p and a 1-form $\eta_p, p = 1, 2, \cdots, r$ satisfying

$$F^2 = a^r I + \varepsilon \sum_{p=1}^{r} \xi_p \otimes \eta_p, \tag{2.9}$$

$$F\xi_p = 0, \tag{2.10}$$

$$\eta_p \circ F = 0, \tag{2.11}$$

$$\eta_p(\xi_q) = -\frac{a^r}{\varepsilon}\delta_{pq}, \tag{2.12}$$

where $p, q = 1, 2, \cdots, r$ and δ_{pq} denote the Kronecker delta while a and ε are nonzero complex numbers. The manifold M is called a *generalized almost r-contact manifold* and manifold with a generalized almost r-contact structure or in short with $(F, \eta_p, \xi_q, a, \varepsilon)$-structure [1, 2, 8, 12].

3. Induced structure on the tangent bundle

Let us suppose that the base space M admits the generalized almost r-contact structure. Then there exists a tensor field F of type (1, 1), $r(C^\infty)$ vector fields $\xi_1, \xi_2, \cdots, \xi_p$ and $r(C^\infty)$ 1-forms $\eta_1, \eta_2, \cdots, \eta_p$ such that equations from (2.9) to (2.11) are satisfied. Taking complete lifts of equations from (2.9) to (2.11), we obtain the following:

$$(F^C)^2 = a^r I + \varepsilon \sum_{p=1}^{r} \{\xi_p^V \otimes \eta_p^C + \xi_p^C \otimes \eta_p^V\}, \tag{3.1}$$

$$F^C \xi_p^V = 0, \ F^C \xi_p^C = 0, \tag{3.2}$$

$$\eta_p^V \circ F^C = 0, \eta_p^C \circ F^V = 0, \ \eta_p^C \circ F^C = 0, \eta_p^V \circ F^V = 0, \tag{3.3}$$

$$\eta_p^C(\xi_q^C) = \eta_p^V(\xi_q^V) = 0, \ \eta_p^C(\xi_q^V) = \eta_p^V(\xi_q^C) = -\frac{a^r}{\varepsilon}\delta_{pq}. \tag{3.4}$$

Let us define an element J of $J_1^1 T(M)$ by

$$J = F^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \{\xi_p^V \otimes \eta_p^V + \xi_p^C \otimes \eta_p^C\}. \tag{3.5}$$

Then in the view of equations from (3.1) to (3.4), it is easily shown that $J^2 X^V = a^r X^V$ and $J^2 X^C = a^r X^C$, which give that J is Hsu-structure on $T(M)$. Hence we have the following theorem.

Theorem 3.1. *Let M be a differentiable manifold endowed with the generalized almost r-contact structure $(F, \eta_p, \xi_q, a, \varepsilon)$. Then tensor field J defined by (3.5) gives a Hsu-structure on $T(M)$.*

Now in view of the equation (3.5), we have

(a) $JX^V = (FX)^V + \dfrac{\varepsilon}{a^{r/2}} \displaystyle\sum_{p=1}^{r} \{(\eta_p(X))^V \xi_p^C\},$

(b) $JX^C = (FX)^C + \dfrac{\varepsilon}{a^{r/2}} \displaystyle\sum_{p=1}^{r} \{(\eta_p(X))^V \xi_p^V + (\eta_p(X))^C \xi_p^C\}$

for arbitrary $X, Y \in \Im_0^1(M)$.

In particular, we have

(a) $JX^V = (FX)^V, JX^C = (FX)^C,$

(b) $J\xi_p^V = -\dfrac{a^{r/2}}{\varepsilon}\xi_p^C, J\xi_p^C = -\dfrac{a^{r/2}}{\varepsilon}\xi_p^V,$

where X is an arbitrary vector field in M such that $\eta_p(X) = 0$.

4. Covariant derivatives with respect to complete and vertical lifts

Let M be an n-dimensional differentiable manifold. Differential transformation of algebra $T(M)$, defined by

$$D = \nabla_X : T(M) \to T(M), X \in \Im_0^1(M) \tag{4.1}$$

is called as covariant derivation with respect to vector field X if

(a) $\nabla_{fX+gY} t = f\nabla_X t + g\nabla_Y t,$

(b) $\nabla_X f = Xf,$

where arbitrary $f, g \in \Im_0^0(M)$, $X, Y \in \Im_0^1(M), t \in \Im(M)$. On the other hand, a transformation defined by

$$\nabla : \Im_0^1(M) \times \Im_0^1(M) \longrightarrow \Im_0^1(M) \tag{4.2}$$

is called affine connection [6, 11].

Now we assume that M is a manifold with affine connection ∇. Then there exist a unique affine connection ∇^C in $\Im(M)$ which satisfies

$$\nabla^C_{X^C} Y^C = (\nabla_X Y)^C \tag{4.3}$$

for any $X, Y \in \Im_0^1(M)$. This affine connection is called the complete lift of the affine connection ∇ to $T(M)$ and denoted by ∇^C [15].

Proposition 4.1. *For any $X \in \Im_0^1(M)$, $f \in \Im_0^0(M)$ and ∇^C is the complete lift of the affine connection ∇ to $T(M)$* [15].

(a) $\nabla^C_{X^V} f^V = 0$,
(b) $\nabla^C_{X^V} f^C = (\nabla_X f)^V$,
(c) $\nabla^C_{X^C} f^V = (\nabla_X f)^V$,
(d) $\nabla^C_{X^C} f^C = (\nabla_X f)^C$.

Proposition 4.2. *For any $X, Y \in \Im_0^1(M)$, $f \in \Im_0^0(M)$ and ∇^C is the complete lift of the affine connection ∇ to $T(M)$* [15].

(a) $\nabla^C_{X^V} Y^V = 0$,
(b) $\nabla^C_{X^V} Y^C = (\nabla_X Y)^V$,
(c) $\nabla^C_{X^C} Y^V = (\nabla_X Y)^V$,
(d) $\nabla^C_{X^C} Y^C = (\nabla_X Y)^C$.

Theorem 4.3. *For ∇_X the operator covariant derivation with respect to vector field X, $J \in \Im_1^1(\Im(M))$ defined by (3.5) and $\eta_p(Y) = 0$ we have*

$$(\nabla^C_{X^V} J)Y^V = 0, \tag{4.4}$$

$$(\nabla^C_{X^V} J)Y^C = ((\nabla_X F)Y)^V + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} ((\nabla_X \eta_p)Y)^V \xi_p^C, \tag{4.5}$$

$$(\nabla^C_{X^C} J)Y^V = ((\nabla_X F)Y)^V + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} ((\nabla_X \eta_p)Y)^V \xi_p^C, \tag{4.6}$$

$$\begin{aligned}(\nabla^C_{X^C} J)Y^C = {} & ((\nabla_X F)Y)^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \eta_p^V (\nabla_X Y)^V \xi_p^V \\ & - \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} (\eta_p(\nabla_X Y))^C \xi_p^C.\end{aligned} \tag{4.7}$$

Proof. At first, we will show (4.4).

$$\begin{aligned}(\nabla^C_{X^V} J)Y^V = {} & \nabla^C_{X^V}[F^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \{\xi_p^V \otimes \eta_p^V + \xi_p^C \otimes \eta_p^C\}]Y^V \\ & - [F^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \{\xi_p^V \otimes \eta_p^V + \xi_p^C \otimes \eta_p^C\}]\nabla^C_{X^V} Y^V\end{aligned}$$

$$= (\nabla^C_{X^V} F^C Y^V + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \nabla^C_{X^V} (\xi^V_p \otimes \eta^V_p) Y^V$$

$$+ \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \nabla^C_{X^V} (\xi^C_p \otimes \eta^C_p) Y^V,$$

as $\nabla^C_{X^V} Y^V = 0$. Therefore

$$(\nabla^C_{X^V} J) Y^V = 0, \ as \ \eta_p(Y) = 0, (\nabla^C_{X^V} ((\eta_p Y)^V) = 0.$$

Thus (4.4) is proved.

Second, we will show (4.5).

$$(\nabla^C_{X^V} J) Y^C = \nabla^C_{X^V} [F^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \{\xi^V_p \otimes \eta^V_p + \xi^C_p \otimes \eta^C_p\}] Y^C$$

$$- [F^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \{\xi^V_p \otimes \eta^V_p + \xi^C_p \otimes \eta^C_p\}] \nabla^C_{X^V} Y^C$$

$$= (\nabla^C_{X^V} F^C Y^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \nabla^C_{X^V} (\eta^V_p (Y)^C) \xi^V_p$$

$$+ \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \nabla^C_{X^V} (\eta^C_p (Y)^C) \xi^C_p - F^C \nabla^C_{X^V} Y^C$$

$$- \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \eta^V_p (\nabla^C_{X^V} Y^C) \xi^V_p - \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \eta^C_p (\nabla^C_{X^V} Y^C) \xi^C_p.$$

Therefore

$$(\nabla^C_{X^V} J) Y^C = ((\nabla_X F) Y)^V + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} ((\nabla_X \eta_p) Y)^V \xi^C_p.$$

Thus (4.5) is proved.

Third, we will show (4.6).

$$(\nabla^C_{X^C} J) Y^V = \nabla^C_{X^C} [F^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \{\xi^V_p \otimes \eta^V_p + \xi^C_p \otimes \eta^C_p\}] Y^V$$

$$- [F^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \{\xi^V_p \otimes \eta^V_p + \xi^C_p \otimes \eta^C_p\}] \nabla^C_{X^C} Y^V$$

$$
\begin{aligned}
&= (\nabla^C_{X^C} F^C Y^V + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \nabla^C_{X^C} (\eta^V_p (Y)^V) \xi^V_p \\
&\quad + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \nabla^C_{X^C} (\eta^C_p (Y)^V) \xi^C_p - F^C \nabla^C_{X^C} Y^V \\
&\quad - \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \eta^V_p (\nabla^C_{X^C} Y^V) \xi^V_p - \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \eta^C_p (\nabla^C_{X^C} Y^V) \xi^C_p \\
&= (\nabla^C_{X^C} F^C) Y^V + F^C (\nabla^C_{X^C} Y^V) - F^C \nabla^C_{X^C} Y^V \\
&\quad + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \nabla^C_{X^C} (\eta_p (Y))^V \xi^V_p + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \nabla^C_{X^C} (\eta_p (Y))^V \xi^C_p \\
&\quad - \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \eta^V_p (\nabla_X Y)^V \xi^V_p - \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} ((\nabla_X \eta_p) Y)^V \xi^C_p .
\end{aligned}
$$

Therefore

$$
(\nabla^C_{X^V} J) Y^V = ((\nabla_X F) Y)^V + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} ((\nabla_X \eta_p) Y)^V \xi^C_p .
$$

Thus (4.6) is proved.

Finally, we will show (4.6).

$$
\begin{aligned}
(\nabla^C_{X^C} J) Y^C &= \nabla^C_{X^C} [F^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \{\xi^V_p \otimes \eta^V_p + \xi^C_p \otimes \eta^C_p\}] Y^C \\
&\quad - [F^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \{\xi^V_p \otimes \eta^V_p + \xi^C_p \otimes \eta^C_p\}] \nabla^C_{X^C} Y^C \\
&= (\nabla^C_{X^C} F^C) Y^C + F^C (\nabla^C_{X^C} Y^C) - F^C \nabla^C_{X^C} Y^C \\
&\quad + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \nabla_X (\eta_p (Y))^V \xi^V_p + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} ((\nabla_X \eta_p) Y)^C \xi^C_p \\
&\quad - \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \eta^V_p (\nabla_X Y)^V \xi^V_p - \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} (\eta_p (\nabla_X Y)^C \xi^C_p .
\end{aligned}
$$

Therefore

$$(\nabla^C_{X^C} J)Y^C = ((\nabla_X F)Y)^C + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} \eta_p^V (\nabla_X Y)^V \xi_p^V$$
$$- \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} (\eta_p (\nabla_X Y)^C \xi_p^C.$$

Thus (4.7) is proved. □

Corollary 4.4. *If we put* $Y = \xi_p$, *that is,* $\eta_p^C(\xi_q^C) = \eta_p^V(\xi_q^V) = 0$, *then* $\eta_p^C(\xi_q^V) = \eta_p^V(\xi_q^C) = -\frac{a^r}{\varepsilon}\delta_{pq}$ *has condition (3.5), then we get different results such as*

(a) $(\nabla^C_{X^V} J)\xi_p^V = -a^{r/2} \sum_{p=1}^{r} (\nabla_X \xi_p)^V$,

(b) $(\nabla^C_{X^V} J)\xi_p^C = a^{r/2}((\nabla_X F)\xi_p)^V$,

(c) $(\nabla^C_{X^C} J)\xi_p^V$
$= ((\nabla_X F)\xi_p)^V + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} ((\nabla_X \eta_p)\xi_p)^V \xi_p^C - a^{r/2} \sum_{p=1}^{r} (\nabla^C_{X^C} \xi_p^C)$,

(d) $(\nabla^C_{X^C} J)\xi_p^C =$
$((\nabla_X F)\xi_p)^C - a^{r/2} \sum_{p=1}^{r} (\nabla_X \xi_p)^V + \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} ((\nabla_X \eta_p)\xi_p)^V \xi_p^V$
$$- \frac{\varepsilon}{a^{r/2}} \sum_{p=1}^{r} ((\nabla_X \eta_p)\xi_p)^C \xi_p^C.$$

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