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NOTES ON THE EVENTUAL SHADOWING PROPERTY OF A CONTINUOUS MAP

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Abstract. Let (X, d) be a compact metric space with metric d and let $f : X \to X$ be a continuous map. In this paper, we consider that for a subset Λ, a map f has the eventual shadowing property if and only if f has the eventual shadowing property on Λ. Moreover, a map f has the eventual shadowing property if and only if f has the eventual shadowing property in Λ.

1. Introduction.

Let (X, d) be a compact metric space with metric d, and let $f : X \to X$ be a continuous map. On the most of dynamical systems, many important research topics are related to the orbit structure (transitive, mixing, chaotic, etc). It is very closely related to the shadowing theory. For any $\delta > 0$, a sequence $\{x_i\}_{i=0}^{\infty}$ is said to be a δ *pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \geq 0$. We say that f has the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ pseudo orbit $\{x_i\}_{i=0}^{\infty}$ there is $y \in X$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \geq 0$. The shadowing theory is an useful notion to investigate for complex dynamical systems. For instance, if a chain transitive system has shadowing property then either it is sensitive or all points are equicontinuous (see [2]). Moreover, Kulczycki and Oprocha [5] proved that if a map f is c expansive and has the shadowing property then it is mixing. Recently, Li *et al.* [9] proved that if a Devaney chaotic system has the shadowing property then it is distributionally chaotic.

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2. Eventual shadowing property

Let (X, d) be as before, and let $f : X \to X$ be a continuous map. Let Λ be a closed f-invariant subset of X. We say that f has the *shadowing property on* Λ if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ pseudo orbit $\{x_i\}_{i=0}^{\infty} \subset \Lambda$ there is $y \in X$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \geq 0$. If $\Lambda = M$ then we say that f has the shadowing property.

REMARK 2.1. Let Λ be a closed f-invariant set of X. It is known that if a continuous map $f : X \to X$ has the shadowing property then f has the shadowing property on Λ.

We introduce the eventual shadowing property which is a general notion of the shadowing property (see [4]). We say that f has the *eventual shadowing property* on Λ if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ there are $N > 0$ and $z \in X$ such that $d(f^i(z), x_i) < \epsilon$ for all $i \geq N$. If $\Lambda = M$ then we say that f has the eventual shadowing property.

REMARK 2.2. It can easily show that f has the shadowing, or eventual shadowing property if and only if f^k has the shadowing, eventual shadowing property for all $k \in \mathbb{Z} \setminus \{0\}$ (see [1]).

The following is for the eventual shadowing property.

THEOREM 2.3. *Let Λ be a closed f-invariant set of X. If a continuous map f has the eventual shadowing property then f has the eventual shadowing property on Λ.*

Proof. Suppose that f has the eventual shadowing property. Let $\epsilon > 0$ and choose $\delta > 0$ such that every δ pseudo orbit in X is ϵ eventual shadowed. Let $\xi = \{x_0, x_1, \cdots, x_n, \cdots\} \subset \Lambda$ be a δ pseudo orbit in Λ. Then it is clear that ξ is a δ pseudo orbit in X. Sine f has the eventual shadowing property, there are a point $y \in M$ and $N > 0$ such that $d(f^i(y), x_i) < \epsilon$ for $i \geq N$. This means that f has the eventual shadowing property on Λ. $\square$

In Remark 2.1, the converse is not true. In fact, if the converse is true then an f-invariant subset Λ should be dense in X (see [3, Lemma 3.1]). From the result, we prove the following.

THEOREM 2.4. *Let Λ be a f-invariant dense subset of X. If a continuous map f has the eventual shadowing property on Λ then f has the eventual shadowing property.*

Proof. Suppose that f has the eventual shadowing property on Λ. Let $\epsilon > 0$ and $\delta > 0$ be given by the eventual shadowing property on Λ for $\epsilon/2$. Let $\xi = \{x_0, x_1, \ldots, x_n, \ldots\}$ be a $\delta/3(< \epsilon/2)$ pseudo orbit in X. Since f is continuous and X is compact, there is $r > 0$ with $r < \delta/3$ such that for any $x, y \in X$, if $d(x, y) < r$ then $d(f(x), f(y)) < \delta/3$. Since Λ is dense in X, we know $B(x_i, r) \cap \Lambda \neq \emptyset$ for each $i \geq 0$. Let $y_i \in B(x_i, r) \cap \Lambda$. Since f is uniformly continuous, we have $d(f(x_i), f(y_i)) < \delta/3$ for each $i \geq 0$. Then we have

$$\begin{aligned} d(f(y_i), y_{i+1}) &< d(f(y_i), f(x_i)) + d(f(x_i), x_{i+1}) + d(x_{i+1}, y_{i+1}) \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta. \end{aligned}$$

Thus $\eta = \{y_0, y_i, \ldots, y_n, \ldots\}$ is a δ pseudo orbit in Λ. Since f has the eventual shadowing property on Λ, there are $N > 0$ and a point $z \in X$ such that $d(f^i(z), y_i) < \epsilon/2$ for $i \geq N$. Then we have

$$d(f^i(z), x_i) < d(f^i(z), y_i) + d(y_i, x_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus the point z eventual shadows η, and so, f has the eventual shadowing property. □

Now we consider that another case of the shadowing property which is stronger than original shadowing. We say that f has the *shadowing property in* Λ if for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ pseudo orbit $\{x_i\}_{i=0}^{\infty} \subset \Lambda$ there is $y \in \Lambda$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \geq 0$. Similarly, we say that f has the *eventual shadowing property in* Λ if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ pseudo orbit $\{x_i\}_{i=0}^{\infty} \subset \Lambda$ there are $N > 0$ and $y \in \Lambda$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \geq N$.

In [3, Lemma 3.1], the authors proved that if f has the shadowing property in Λ then f has the shadowing property, where Λ is a dense invariant subset of X. As the result, we prove the following.

Theorem 2.5. *Let Λ be a closed f-invariant set of X. A continuous map f has the eventual shadowing property if and only if f has then eventual shadowing property in Λ.*

Proof. The proof is similar to Theorem 2.3. □

Theorem 2.6. *Let Λ be a f-invariant dense subset of X. If a continuous map f has the eventual shadowing property in Λ then f has the eventual shadowing property.*

Proof. The proof is almost similar to Theorem 2.4. For convenience, we give a sketch in this proof. Suppose that f has the eventual shadowing property in Λ. Let $\epsilon > 0$ and $\delta > 0$ be given by the eventual shadowing property on Λ for $\epsilon/2$. Let $\xi = \{x_0, x_1, \ldots, x_n, \ldots\}$ be a $\delta/3$ pseudo orbit in X. As in the proof of Theorem 2.4, we have a δ pseudo orbit $\eta = \{y_0, y_i, \ldots, y_n, \ldots\}$ in Λ. Since f has the eventual shadowing property in Λ, there are $N > 0$ and $z \in \Lambda$ such that $d(f^i(z), x_i) < \epsilon$ for all $i \geq N$. Clearly, $z \in X$, and so, f has the eventual shadowing property. □

The following (Remark 2.7) is similar to the shadowing property (see [10]). The results of shadowing property can be applied various shadowing property (see [6, 7, 8]).

Remark 2.7. Consider a dynamical system is generated by homeomorphisms of a compact metric space (X, d). Then we have the followings.

(a) For any dynamical systems $f, g : X \to X$, there is a homeomorphism $h : X \to X$ such that $f \circ h = h \circ g$. Then if f has the eventual shadowing property then g has the eventual shadowing property.

(b) For a linear dynamical system $f(x) = Ax$ of $\mathbb{C}^n$, the matrix is hyperbolic if and only if f has the eventual shadowing property, where A is hyperbolic if the spectrum does not intersect the unit circle, that is, $\{\lambda : |\lambda| \neq 1\}$.

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