ISSN 1975-8359(Print) / ISSN 2287-4364(Online)
The Transactions of the Korean Institute of Electrical Engineers Vol. 66, No. 11, pp. 1612~1619, 2017
http://doi.org/10.5370/KIEE.2017.66.11.1612

단일입력 불확실 비선형 시스템에 대한 Utkin 정리의 증명

A Poof of Utkin's Theorem for SI Uncertain Nonlinear Systems

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Abstract - In this note, a complete proof of Utkin's theorem is presented for SI(single input) uncertain nonlinear systems. The invariance theorem with respect to the two nonlinear transformation methods so called the two diagonalization methods is proved clearly, comparatively, and completely for SI uncertain nonlinear systems. With respect to the sliding surface and control input transformations, the equation of the sliding mode i.e., the sliding surface is invariant, which is proved completely. Through an illustrative example and simulation study, the usefulness of the main results is verified. By means of the two nonlinear transformation methods, the same results can be obtained.

Key Words : Variable structure system, Sliding mode control, Proof of Ukin's Theorem, Diagonalization methods, Transformation methods

1. Introduction

The variable structure system(VSS) with the sliding mode control(SMC) can provide the effective means to the control of uncertain nonlinear dynamical systems under parameter variations and external disturbances[1-4]. One of its essential advantages is the robustness of the controlled system to matched parameter uncertainties and external disturbances in the sliding mode on the predetermined sliding surface, $s=0$ [5-7]. To take the advantages of the sliding mode on the predetermined sliding surface, the precise existence condition of the sliding mode, $s \cdot \dot{s} < 0$ for the SI case as well as $s_i \cdot \dot{s}_i < 0,\ i=1,2,...m$ for the MI(Multi Input) case should be satisfied[8]. Therefore the precise existence condition of the sliding mode must be proved completely for linear plants moreover for nonlinear plants. Utkin in [4] presented the two nonlinear methodologies without the complete proofs in order to prove the precise existence condition of the sliding mode on the pre-selected sliding surface. It is so called the invariance theorem, that is the equation of the sliding mode is invariant with respect to the two nonlinear transformations. Those are the control input transformation and sliding surface transformation, so called the two diagonalization methods. The essential feature of both nonlinear transformation methods is the conversion of a multi-input design problem into m single-input design problems[4]. Those were only reviewed in [5]. DeCarlo, Zak, and Matthews tried to prove Utkin's invariance theorem. But, the proof also is incomplete, and only the same as those of Utkin. In [9], Su, Drakunov, and Ozguner mentioned the sliding surface transformation, which would diagonalize the control coefficient matrix to the dynamics for the sliding surface s. But they did not completely prove the precise existence condition of the sliding mode on the predetermined sliding surface. Even for a SI linear case, that proof is hardly reported. For MI linear plants, instead of proving the precise existence condition of the sliding mode, some design methods were studied, those are, including the two diagonalization methods[4,5], the hierarchical control methodology[4, 6], simplex algorithm[14], Lyapunov approach[1, 9, 18, 22], and so on. Until now in MIMO(Multi input multi output) VSSs, it is difficult to prove the precise existence condition of the sliding mode on the predetermined sliding surface theoretically, but in [9, 18, 22], only the result that the derivative of the Lyapunov candidate function is negative, i.e. $\dot{V} < 0$ is obtained when the Lyapunov candidate function is taken as $V=1/2s^T s$. In SI systems, both the VSS existence condition of the sliding mode and the Lyapunov stability are the same when the Lyapunov candidate function is taken as $V=1/2s^T s$. However, the VSS existence condition of the sliding mode in multi input systems is the more strict condition than the Lyapunov stability because if the VSS existence condition of the sliding mode is satisfied, then the Lyapunov stability is did but the reverse argument does not hold generally. In

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Received : June 16, 2017; Accepted : September 27, 2017

[23], for MI uncertain linear plants, the proof of Utkin's theorem is given comparatively, the precise existence condition of the sliding mode is proved completely, and the complete formulation of the multivariable VSS is possible from the design of the sliding surface to the proof of the precise existence of the sliding mode and the proof of the stability of the closed loop system.

Until now, for SI uncertain nonlinear systems, a rigorous proof of Utkin's theorem is hardly reported, while the proof of Utkin invariant theorem for MI linear plants is given. For SI uncertain nonlinear plants, the proof of Utkin theorem is necessary.

In this note, a complete proof of Utkin's theorem is presented for SI uncertain nonlinear plants. The invariance theorem with respect to the two nonlinear transformation methods so called the two diagonalization methods is proved clearly and comparatively. The complete formulation from the formulation of the objective plants to the proof of the existence condition of the sliding mode and the stability of the closed loop system is possible. If the control input matrix $g_0(x,t)$ in the model of nonlinear plants is constant(not function of x or t) i.e. $g_0(x,t)=B_0$, then both transformation (diagonalization) methods of Utkin's theorem can be used to, otherwise, only the control input transformation can be applied to the proof of the existence condition of the sliding mode on the predetermined sliding surface in the nonlinear VSS. A design example and simulation study shows the usefulness of the main results.

2. Main Results of Proof of Utkin's Theorem

The invariant theorem of Utkin is as follows[4, 5]:

Theorem 1: The equation of the sliding mode is invariant with respect to the two nonlinear transformations, i.e. the control input transformation and sliding surface transformation:

$$u^*(x)=H_u(x,t)\cdot u(x) \qquad (1)$$
$$s^*(x)=H_s(x,t)\cdot s(x)$$

where $H_u(x,t)$ and $H_s(x,t)$ are the nonlinear transformation matrices for $\det H_u\neq 0$ and $\det H_s\neq 0$.

To prove this theorem, consider a SI affine uncertain nonlinear system

$$\dot{x}=f(x,t)+g(x,t)u+d'(x,t) \quad x(0) \qquad (2)$$

where $x\in R^n$ is the state vector, $x(0)$ is its initial state, $u\in R^1$ is the control input, $f(x,t)\in C^k$ and $g(x,t)\in C^k$, $k\geq 1$, $g(x,t)\neq 0$, for $all\,x\in R^n$ and for $all\,t\geq 0$ are of suitable dimensions, and $d'(x,t)$ implies bounded matched external disturbances.

Assumption

A1: $f(x,t)$ is continuously differentiable.

Then, the uncertain nonlinear system (1) can be represented in the more affine nonlinear system of the modified state dependent coefficient form

$$\begin{aligned}\dot{x}&=[f_0(x,t)+\triangle f_1(x,t)]x+\triangle f_2(x,t)\\&\quad+[g_0(x,t)+\triangle g(x,t)]u+d'(x,t)\\&=f_0(x,t)x+g_0(x,t)u+d(x,t)\end{aligned} \qquad (3)$$

where $f_0(x,t)$ and $g_0(x,t)$ is each nominal value such that

$$f(x.t)=[f_0(x,t)+\triangle f_1(x,t)]x+\triangle f_2(x,t) \qquad (4a)$$
$$g(x.t)=[g_0(x,t)+\triangle g(x,t)] \qquad (4b)$$

respectively, $\triangle f_1(x,t)$ and $\triangle g(x,t)$ are mismatched uncertainties, $\triangle f_2(x,t)$ is matched uncertainties, $d'(x,t)$ is matched external disturbance, and $d(x,t)$ is the totally mismatched lumped uncertainties, respectively.

Assumption:

A2: The pair $(f_0(x,t), g_0(x,t))$ is controllable for $all\,x\in R^n$ and for $all\,t\geq 0$

A3: The lumped uncertainties $d(x,t)$ is bounded

The conventional typical sliding surface is a linear combination of the full state variable as

$$s=C^T\cdot x=\sum_{i=1}^{n}c_ix_i \quad c_n=1 \qquad (5)$$

where C is a non zero element constant coefficient column vector for the sliding surface.

A4: $C^Tg(x,t)$ and $C^Tg_0(x,t)$ have the full rank and invertible

A5: $C^T\triangle g(x,t)\left[C^Tg_0(x,t)\right]^{-1}=\triangle I$ and $|\triangle I|\leq\delta<1$.

The VSS control input is as follows:

$$u_1=-K(x)\cdot x-\Delta K\cdot x-G\cdot sign(s) \qquad (6)$$

where $K(x)$ is a static gain, ΔK is a state dependent switching gain, and G is a switching gain.

2.1 Control input transformation[4, 5]

$$\begin{aligned} u^* &= \left[C^T g_0(x,t)\right]^{-1} u_1, \quad H_u = \left[C^T g_0(x,t)\right]^{-1} \\ &= \left[C^T g_0(x,t)\right]^{-1}\left[-K(x)x - \Delta K x - G sign(s)\right] \end{aligned} \tag{7}$$

where the nonlinear control input transformation matrix is selected as $H_u = (C^T g_0(x,t))^{-1}$ for SI uncertain nonlinear plants. In [4] and [5], the proof for the nonlinear control input transformation is not complete. The real dynamics of s, i.e. the time derivative of s is as follows:

$$\begin{aligned} \dot{s} &= C^T \dot{x} \\ &= C^T(f_0(x,t) + \Delta f_1(x,t))x + C^T \triangle f_2(x,t) + C^T(g_0(x,t) \\ &\quad + \Delta g(x,t))u^* + C^T d'(x,t) \\ &= C^T(f_0(x,t) + \Delta f_1(x,t))x + C^T \triangle f_2(x,t) + (I + \Delta I)u_1 \\ &\quad + C^T d'(x,t) \\ &= C^T(f_0(x,t) + \Delta f_1(x,t))x + C^T \triangle f_2(x,t) + (I + \Delta I) \\ &\quad (-K(x)x - \Delta K x - G sign(s)) + C^T d'(x,t) \\ &= C^T f_0(x,t)x - K(x)x + C^T \Delta f_1(x,t)x - \Delta I K(x)x \\ &\quad - (I + \Delta I)\Delta K x + C^T \triangle f_2(x,t) + C^T d'(x,t) \\ &\quad - (I + \Delta I) G sign(s) \end{aligned} \tag{8}$$

If C is a function of the state vector x i.e. $C(x,t)$, then the formulation of (8) can not be obtained easily. By letting the static gain

$$K(x) = C^T f_0(x,t) \tag{9}$$

which is proposed in this paper. Then the real dynamics of s becomes

$$\begin{aligned} \dot{s} = \; & C^T \Delta f_1(x,t)x - \Delta I K(x)x - (I + \Delta I)\Delta K x \\ & + C^T \triangle f_2(x,t) + C^T d'(x,t) \\ & - (I + \Delta I) G sign(s) \end{aligned} \tag{10}$$

If one takes the switching gains as the design parameters

$$\triangle k_j = \begin{cases} \geq \dfrac{\max\left\{C^T \triangle f_1(x,t) - \triangle I C^T f_0(x,t)\right\}_j}{\min\{I + \triangle I\}_j} & sign(s x_j) > 0 \\ \leq \dfrac{\min\left\{C^T \triangle f_1(x,t) - \triangle I C^T f_0(x,t)\right\}_j}{\min\{I + \triangle I\}_j} & sign(s x_j) < 0 \end{cases}$$

$$j = 1, 2, \ldots, n \tag{11}$$

$$G = \begin{cases} \geq \dfrac{\max\left\{C^T \triangle f_2(x,t) + C^T d'(x,t)\right\}}{\min\{I + \triangle I\}} & sign(s) > 0 \\ \leq \dfrac{\min\left\{C^T \triangle f_2(x,t) + C^T d'(x,t)\right\}}{\min\{I + \triangle I\}} & sign(s) < 0 \end{cases} \tag{12}$$

then one can obtain the following equation

$$s \cdot \dot{s} < 0 \tag{13}$$

The existence condition of the sliding mode is proved for SI uncertain nonlinear systems. The equation of the sliding mode, i.e. the sliding surface is invariant to the nonlinear control input transformation. One takes the Lyapunov candidate function as $V(s) = 1/2 s^T s > 0$, then the time derivative of the Lyapunov candidate function is negative from (13), that is $\dot{V}(s) = s \cdot \dot{s} < 0$. Therefore the asymptotic stability of the closed loop system is satisfied in the sense of Lyapunov.

2.2 Sliding surface transformation[4, 5, 9]

$$s^* = \left[C^T g_0(x,t)\right]^{-1} \cdot s, \; H_s(x,t) = \left[C^T g_0(x,t)\right]^{-1} \tag{14}$$

The sliding surface transformation matrix is selected as $H_s(x,t) = \left[C^T g_0(x,t)\right]^{-1}$. In [5], the proof is not sufficient. Now, the VSS control input for the new sliding surface is taken as follows:

$$u_2 = -K(x) \cdot x - \Delta K \cdot x - G \cdot sign(s^*) \tag{15}$$

The real dynamics of the sliding surface, i.e. the time derivative of s^* becomes

$$\begin{aligned} \dot{s}^* &= \left[C^T \dot{g}_0(x,t)\right]^{-1} s + \left[C^T g_0(x,t)\right]^{-1} \dot{s} \\ &= \frac{\partial}{\partial x}\left[C^T g_0(x,t)\right]^{-1} \dot{x} C^T x + \left[C^T g_0(x,t)\right]^{-1} C^T \dot{x} \end{aligned} \tag{16}$$

Since $g_0(x,t)$ is a function of the state vector x, the further formulation of (16) is difficult. In [3] and [4], the proofs of the sliding surface transformation are the same and stopped here. Only an example to show the proof is given. However if $g_0(x,t)$ is constant, i.e. $g_0(x,t) = B_0$ ($H_s(x,t) = \left[C^T B_0\right]^{-1}$), then it is possible to go the further steps of the formulation with an assumption

A6: $\left[C^T B_0\right]^{-1} C^T \triangle g(x,t) = \triangle I$ and $|\triangle I| \leq \gamma < 1$.

Even though the objective plant is nonlinear, it is natural and convenient that the modelling of $g_0(x,t)$ is to be constant and the maximum bound of the parameter variation is found and used in most of the VSS controller design for nonlinear plants. Therefore the first term of the right hand side of (16) equation is zero, the real dynamics of the sliding surface leads to

$$\begin{aligned}\dot{s}^* &= \left[C^T B_0\right]^{-1} C^T \dot{x} \\ &= \left[C^T B_0\right]^{-1} C^T (f_0(x,t) + \Delta f_1(x,t))x \\ &\quad + \left[C^T B_0\right]^{-1} C^T \triangle f_2(x,t) \\ &\quad + \left[C^T B_0\right]^{-1} C^T (B_0 + \Delta g(x,t))u_2 \\ &\quad + \left[C^T B_0\right]^{-1} C^T d'(x,t) \\ &= \left[C^T B_0\right]^{-1} C^T (f_0(x,t) + \Delta f_1(x,t))x \\ &\quad + \left[C^T B_0\right]^{-1} C^T \triangle f_2(x,t) \\ &\quad + (I + \Delta I')u_2 + \left[C^T B_0\right]^{-1} C^T d'(x,t) \\ &= \left[C^T B_0\right]^{-1} C^T (f_0(x,t) + \Delta f_1(x,t))x \\ &\quad + \left[C^T B_0\right]^{-1} C^T \triangle f_2(x,t) \\ &\quad + (I + \Delta I')(-K(x)x - \Delta Kx - Gsign(s^*)) \\ &\quad + \left[C^T B_0\right]^{-1} C^T d'(x,t)\end{aligned} \tag{17}$$

By letting the static gain as

$$K(x) = \left[C^T B_0\right]^{-1} C^T f_0(x,t) \tag{18}$$

which is proposed in this paper. Then the real dynamics of s^* becomes

$$\begin{aligned}\dot{s}^* &= [(C^T B_0)^{-1} C^T \Delta f_1(x,t) - \Delta I' K(x)]x \\ &\quad - (I + \Delta I')\Delta Kx \\ &\quad + (C^T B_0)^{-1} C^T \triangle f_2(x,t) + (C^T B_0)^{-1} C^T d'(x,t) \\ &\quad - (I + \Delta I') Gsign(s^*)\end{aligned} \tag{19}$$

In [9], without uncertainty and disturbance, it is mentioned that the sliding surface transformation would diagonalize the control coefficient matrix to the dynamics for s and the $\dot{V}(x) < 0$ is proved when $V(x) = x^T P x > 0$. In [23], for MI linear plants, the proof of the sliding surface transformation is given.

If one takes the switching gains as follows:

$$\triangle k_j = \begin{cases} \geq \dfrac{\max\left\{(C^T B_0)^{-1} C^T \triangle f_1(x,t) - \triangle I'(C^T B_0)^{-1} C^T f_0(x,t)\right\}_j}{\min\{I + \triangle I'\}_j} & sign(s^* x_j) > 0 \\ \leq \dfrac{\min\left\{(C^T B_0)^{-1} C^T \triangle f_1(x,t) - \triangle I'(C^T B_0)^{-1} C^T f_0(x,t)\right\}_j}{\min\{I + \triangle I'\}_j} & sign(s^* x_j) < 0 \end{cases}$$

$$j = 1, 2, \ldots, n \tag{20}$$

$$G = \begin{cases} \geq \dfrac{\max\left\{(C^T B_0)^{-1} C^T \left(\triangle f_2(x,t) + d'(x,t)\right)\right\}}{\min\{I + \triangle I'\}} & sign(s^*) > 0 \\ \leq \dfrac{\min\left\{(C^T B_0)^{-1} C^T \left(\triangle f_2(x,t) + d'(x,t)\right)\right\}}{\min\{I + \triangle I'\}} & sign(s^*) < 0 \end{cases} \tag{21}$$

then

$$s^* \cdot \dot{s}^* < 0. \tag{22}$$

The existence condition of the sliding mode is proved. One takes the Lyapunov candidate function as $V(s^*) = 1/2 s^{*T} s^* > 0$, then the time derivative of the Lyapunov candidate function is negative from (22), that is $\dot{V}(s^*) = s^* \cdot \dot{s}^* < 0$. Therefore the asymptotic stability of the closed loop system is satisfied in the sense of Lyapunov. If the sliding mode equation $s^* = 0$, then $s = 0$ since $CB_0 \neq 0$ and invertible. The inverse augment also holds, therefore the two sliding surfaces both are equal i.e. $s = 0 = s^*$, which completes the proof of Theorem 1.

The sliding mode equation i.e. the sliding surface $s = 0$ is the same as that of $s^* = 0$. To compare the control inputs, u_1 and u_2, the form is the same but the gains of u_1 are multiplied by $\left[C^T g_0(x,t)\right]$. To compare the control input, u^* and u_2, the form and the gain is the same. The two transformation methods equivalently diagonalize the system, so those are called the two diagonalization methods.

3. Design Examples and Simulation Studies

Consider a second order affine uncertain nonlinear system with mismatched uncertainties and matched disturbance

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 \sin^2(x_1) + x_2 + 0.02\sin(x_1)u \\ \dot{x}_2 &= 0.7\sin(x_1) + x_2 - 0.8\sin(x_2) + 0.2(x_1^2 + x_2^2) \\ &\quad + x_2 \sin^2(x_2) + (2 + 0.3\sin(2t))u + 2\sin(5t)\end{aligned} \tag{23}$$

Since (23) satisfy the Assumption A1, (23) is represented in the state dependent coefficient form as

$$\begin{aligned}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-1 + \sin^2(x_1) & 1 \\ 0 & 1 + \sin^2(x_2)\end{bmatrix} x \\ &\quad + \begin{bmatrix}0 \\ 0.7\sin(x_1) - 0.8\sin(x_2) + 0.2(x_1^2 + x_2^2)\end{bmatrix} \\ &\quad + \begin{bmatrix}0.02\sin(x_1) \\ 2 + 0.3\sin(2t)\end{bmatrix} u + \begin{bmatrix}0 \\ 2\sin(5t)\end{bmatrix}\end{aligned} \tag{24}$$

where the nominal parameter $f_0(x,t)$ and $g_0(x,t)$ and mismatched uncertainties $\triangle f_1(x,t)$ and $\triangle g(x,t)$, and matched uncertainty $\triangle f_2(x,t)$ are

$$f_0(x,t) = \begin{bmatrix}-1 & 1 \\ 0 & 1\end{bmatrix}, \; g_0(x,t) = \begin{bmatrix}0 \\ 2\end{bmatrix}$$

$$\triangle f_1(x,t)=\begin{bmatrix}\sin^2(x_1) & 0\\ 0 & \sin^2(x_2)\end{bmatrix},$$
$$\triangle f_2(x,t)=\begin{bmatrix}0\\ 0.7\sin(x_1)-0.8\sin(x_2)+0.2(x_1^2+x_2^2)\end{bmatrix}$$
$$\triangle g(x,t)=\begin{bmatrix}0.02\sin(x_1)\\ 0.3\sin(2t)\end{bmatrix} \tag{25}$$
$$d'(x,t)=\begin{bmatrix}0\\ 2\sin(5t)\end{bmatrix}$$

The coefficient of the linear sliding surface is determined as

$$C^{T=}[10\ \ 1] \tag{26}$$

3.1 Control input transformation

$$u^*=(C^Tg_0(x,t))^{-1}u_1, \tag{27}$$
$$H_u=(C^Tg_0(x,t))^{-1}=[10\ \ 1]\begin{bmatrix}0\\2\end{bmatrix}=2^{-1}$$
$$=2^{-1}[-K(x)x-\Delta Kx-Gsign(s)]$$

Then, the real dynamics of s, i.e. the time derivative of s is as follows:

$$\dot{s}=C^T\Delta f_1(x,t)x-\Delta IK(x)x-(I+\Delta I)\Delta Kx$$
$$+C^T\triangle f_2(x,t)+C^Td'(x,t)-(I+\Delta I)Gsign(s) \tag{28}$$

where

$$\triangle I=C^T\triangle g(x,t)\left[C^Tg_0(x,t)\right]=[10\ \ \ 1]\begin{bmatrix}0.02\sin(x_1)\\ 0.3\sin(2t)\end{bmatrix}2^{-1} \tag{29}$$
$$=0.1\sin(x_1)+0.15\sin(2t)\le 0.25<1$$

By letting the constant gain

$$K(x)=C^Tf_0(x,t)=[10\ \ \ 1]\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix} \tag{30}$$
$$=[-10\ \ \ 11]$$

If one takes the switching gain as design parameters

$$\Delta k_1=\begin{cases}16.7 & \text{if } sx_1>0\\ -16.7 & \text{if } sx_1<0\end{cases}$$
$$\Delta k_2=\begin{cases}18.4 & \text{if } sx_2>0\\ -18.4 & \text{if } sx_2<0\end{cases}$$
$$G=\begin{cases}4.7+1.27(x_1^2+x_2^2) & \text{if } s>0\\ -4.7-1.27(x_1^2+x_2^2) & \text{if } s<0\end{cases} \tag{31}$$

then one can obtain the following equation

$$s\cdot\dot{s}<0 \tag{32}$$

The existence condition of the sliding mode is proved precisely. The equation of the sliding mode, i.e the sliding surface is invariant to the control input transformation.

The simulation is carried out under 0.1[msec] sampling time and with $x(0)=[10\ \ 2]^T$ initial state. Fig. 1 shows the two output responses, x_1 and x_2 by u^* with s. The

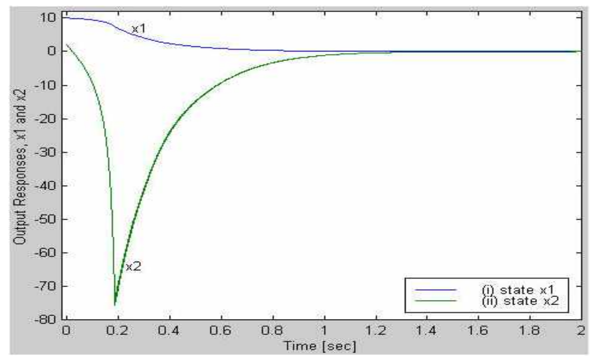

그림 1 제어입력 변환에 의한 두 출력 응답

Fig. 1 Two output responses, x_1 and x_2 by control input transformation u^* with s

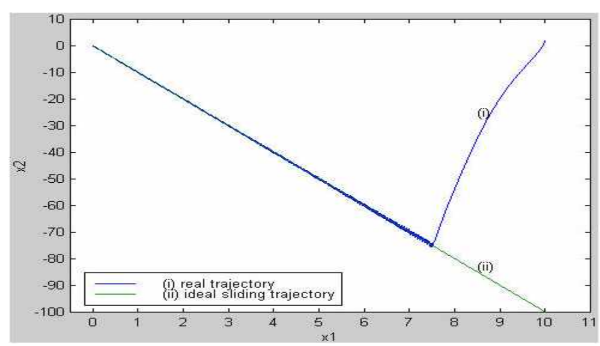

그림 2 실제 상 궤적(i)과 이상 슬라이딩 궤적(ii)

Fig. 2 Real phase trajectory(i) and ideal sliding trajectory(ii)

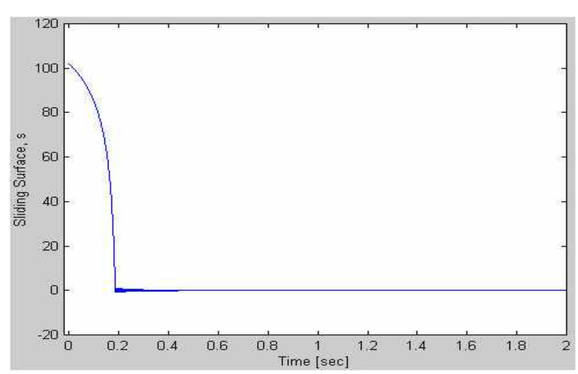

그림 3 슬라이딩 면

Fig. 3 Sliding surface $s(t)$

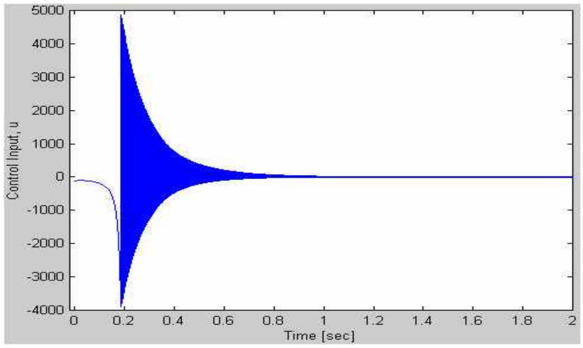

그림 4 제어입력

Fig. 4 Control input $u^*(t)$

real phase trajectory(i) and ideal sliding trajectory(ii) are depicted in Fig. 2. The sliding surface $s(t)$ is shown in Fig. 3. The control input $u^*(t)$ is depicted in Fig. 4.

3.2 Sliding surface transformation

$$s^* = (C^T B_0)^{-1} \cdot s, \quad H_s(x,t) = (C^T B_0)^{-1} = 2^{-1} \tag{33}$$

Now, the VSS control input is taken as follows:

$$u_2 = -K(x) \cdot x - \Delta K \cdot x - G \cdot sign(s^*) \tag{34}$$

The real dynamics of the sliding surface, i.e. the time derivative of s^* becomes

$$\dot{s}^* = [(C^T B_0)^{-1} C^T \Delta f_1(x,t) - \Delta I K(x)]x - (I + \Delta I)\Delta K x + (C^T B_0)^{-1} C^T \Delta f_2(x,t) + (C^T B_0)^{-1} C^T d'(x,t) - (I + \Delta I) G sign(s^*) \tag{35}$$

where

$$\Delta I = \left[C^T B_0\right]^{-1} C^T \Delta g(x,t) = 2^{-1}[10 \quad 1]\begin{bmatrix} 0.02\sin(x_1) \\ 0.3\sin(2t) \end{bmatrix} = 0.1\sin(x_1) + 0.15\sin(2t) \leq 0.25 < 1 \tag{36}$$

By letting gain

$$K(x) = \left[C^T B_0\right]^{-1} C^T f_0(x,t) = 2^{-2}[-10 \quad 11] = [-5.0 \quad 5.5] \tag{37}$$

If one takes the switching gains as follows:

$$\Delta k_1 = \begin{cases} 8.35 & \text{if } s^* x_1 > 0 \\ -8.35 & \text{if } s^* x_1 < 0 \end{cases}$$

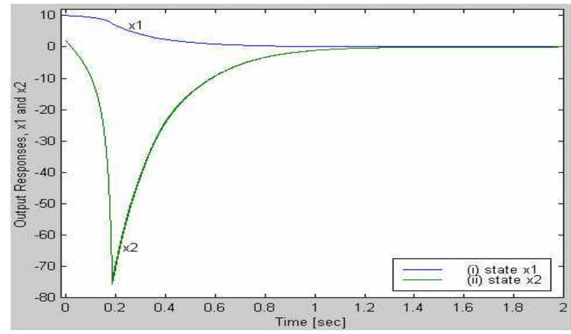

그림 5 슬라이딩 면 변환에 의한 두 출력 응답

Fig. 5 Two output responses, x_1 and x_2 by sliding surface transformation u_2 with s^*

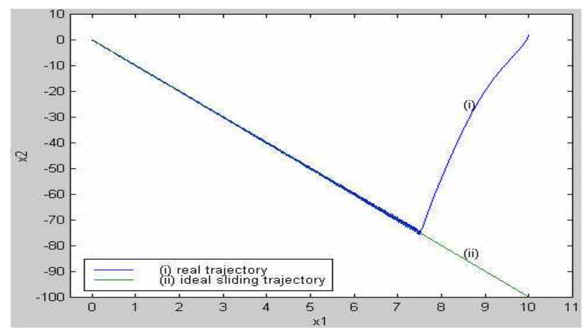

그림 6 실제 상 궤적(i)과 이상 슬라이딩 궤적(ii)

Fig. 6 Real phase trajectory(i) and ideal sliding trajectory(ii)

$$\Delta k_2 = \begin{cases} 9.2 & \text{if } s^* x_2 > 0 \\ -9.2 & \text{if } s^* x_2 < 0 \end{cases}$$

$$G = \begin{cases} 2.35 + 0.635(x_1^2 + x_2^2) & \text{if } s^* > 0 \\ -2.35 - 0.635(x_1^2 + x_2^2) & \text{if } s^* < 0 \end{cases} \tag{38}$$

then

$$s^* \cdot \dot{s}^* < 0 \tag{39}$$

if $s^* = 0$, then $s = 0$. The inverse augment also holds. The switching gains in (38) can be obtained also from (31) by multiplying $\left[C^T B_0\right]^{-1} = 2^{-1}$. Fig. 5 shows the two output responses, x_1 and x_2 by u_2 with s^*. Fig. 5 is almost identical to Fig. 1 because the sliding surfaces $s = 0 = s^*$ are equal and the continuous gains and discontinuous gains of the two controls, u^* and u_2, both are equal. The real phase trajectory(i) and ideal sliding trajectory(ii) are depicted in Fig. 6. In Fig. 7, the sliding surface $s^*(t)$ which is a $\left[C^T B_0\right]^{-1} = 2^{-1}$ multiplied value of $s(t)$ in Fig. 3 is shown. The control input $u_2(t)$ is depicted in Fig.

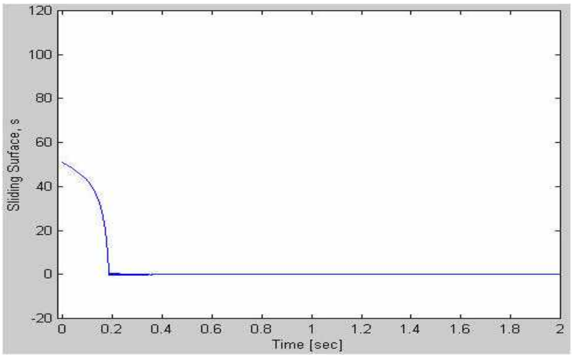

그림 7 슬라이딩 면

Fig. 7 Sliding surface $s^*(t)$

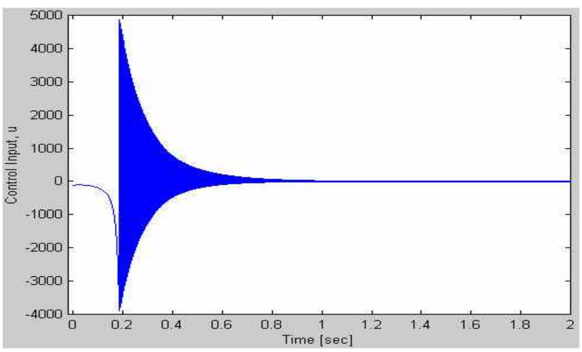

그림 8 제어입력

Fig. 8 Control input $u_2(t)$

8 which is the same as $u^*(t)$ in Fig. 4.

4. Conclusions

In this note, the invariant theorem of Utkin is rigorously, precisely, and completely proved for SI uncertain nonlinear systems. During the proof, the precise existence condition of the sliding mode on the pre-selected sliding surface is completely proved for SI uncertain nonlinear plants. The invariance theorem of the two diagonal(transformation) methods i.e., the control input transformation and sliding surface transformation is proved clearly, comparatively, and completely. Therefore, the equation of the sliding mode, i.e., the sliding surface is invariant with respect to the two diagonalization methods. These two methods diagonalize the input system of the real dynamics of the sliding surface s or $s*$ so that the existence condition of the sliding mode on the predetermined sliding surface is easily proved. During the proof of Utkin's theorem, the conventional linear sliding surface is applied to and the gain design rules for the two control inputs are proposed. Through an illustrative example and simulation study, the effectiveness of the proposed main results is verified. The same results in the outputs by the two diagonalization methods are obtained. The equation of the sliding mode, i.e., the sliding surface is invariant with respect to the two diagonalization methods. If the control input matrix $g_0(x,t)$ in the model of nonlinear plants is constant(not function of x or t) i.e. $g_0(x,t)=B_0$, then both transformation(diagonalization) methods of Utkin's theorem can be used, otherwise, only the control input transformation can be applied to the proof of the existence condition of the sliding mode on the predetermined sliding surface in the nonlinear VSS. It is possible to formulate completely the equation of the VSS controller design for SI uncertain nonlinear systems from the formulation of the objective plants to the complete proof of the existence condition of the sliding mode and the proof of the asymptotic stability of the closed loop system

Acknowledgement

This work was supported by the fund of research promotion program, Gyeongsang National University, 2016.

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