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THE LAWS OF THE ITERATED LOGARITHM FOR THE TENT MAP

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Abstract. This paper considers the asymptotic behaviors of the processes generated by the classical ergodic tent map that is defined on the unit interval. We develop a sequential empirical process and get the uniform version of law of iterated logarithm for the tent map by using the bracketing entropy method.

1. Introduction and the main results

The tent map is an iterated function forming a discrete-time dynamical system. The tent map demonstrates a chaotic dynamical behavior. In Bae et al. [1], we have developed the uniform laws of large numbers for the tent map. In Bae et al. [2], we have developed the uniform central limit theorem for the tent map.

The aim of our work is to develop the law of the iterated logarithm (LIL) and the uniform LIL of Strassen type, see for example Kuelbs and Dudley [7], for the process generated by the tent map by employing Ossiander [8]'s idea of the bracketing entropy method.

We begin with illustrating the tent map. Let $\Omega = [0,1]$ be the sample space, $\mathcal{A}$ be the Borel sets and P be the Lebesgue measure. The tent map on the unit interval is defined by

$$\varphi(y) = \begin{cases} 2y & \text{for } 0 \le y < \frac{1}{2} \\ 2(1-y) & \text{for } \frac{1}{2} \le y \le 1. \end{cases}$$

The tent map is an iterated function, in the shape of a tent. More specifically, if you plot $\varphi(y)$ versus y, it has two linear sections which rise to meet at $[1/2, 1]$. It looks like a tent.

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The tent map φ preserves Lebesgue measure and is equivalent to a shift and flip map τ on $\{0,1\}^{\{0,1,2,\ldots\}}$:

$$\tau(\omega_0,\omega_1,\omega_2,\ldots) = \begin{cases} (\omega_1,\omega_2,\ldots) & \text{if } \omega_0 = 0 \\ (1-\omega_1, 1-\omega_2,\ldots) & \text{if } \omega_0 = 1. \end{cases}$$

We can think of $(\omega_0,\omega_1,\omega_2,\ldots) \in \{0,1\}^{\{0,1,2,\ldots\}}$ as a point y in the unit interval $[0,1]$ by putting $y = \sum_{i=0}^{\infty} \frac{\omega_i}{2^{i+1}}$. It is known that the map φ is ergodic. See Durrett [4].

We now consider a series of stationary processes generated by the tent map φ.

First, we start with $f_{\frac{1}{2}}(y) = 1_{[0,\frac{1}{2})}(y)$. Then $\{f_{\frac{1}{2}}(\varphi^{m-1}(y)) : m \geq 1\}$ are identically distributed random variables which have uniform distribution with

$$\begin{aligned} P(f_{\frac{1}{2}}(\varphi^{m-1}(y)) = 0) &= \frac{1}{2}, \\ P(f_{\frac{1}{2}}(\varphi^{m-1}(y)) = 1) &= \frac{1}{2}. \end{aligned}$$

Therefore $\{f_{\frac{1}{2}}(\varphi^{m-1}(y)) : m \geq 1\}$ is a sequence of stationary random variables. Observe that $Ef_{\frac{1}{2}}(y) = \frac{1}{2}$ and $Var(f_{\frac{1}{2}}(y)) = \frac{1}{4}$. Define

$$T_n(1,1) = n^{-1/2} \sum_{m=1}^{n} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right).$$

Then, by the LIL for stationary process (see Heyde [6]), the set of the limit points of $(2\ln\ln n)^{-1/2} T_n(1,1)$ is $[-1,1]$.

Second, for fixed $j \in \mathbb{N}$ and for fixed $i = 1,2,\ldots,2^j$, we look at $f_{i,j}(y) = 1_{[\frac{i-1}{2^j},\frac{i}{2^j})}(y)$. Then $\{f_{i,j}(\varphi^{m-1}(y)) : m \geq 1\}$ are identically distributed random variables with

$$\begin{aligned} P(f_{i,j}(\varphi^{m-1}(y)) = 0) &= 1 - 2^{-j}, \\ P(f_{i,j}(\varphi^{m-1}(y)) = 1) &= 2^{-j}. \end{aligned}$$

Observe that $Ef_{i,j}(y) = \frac{1}{2^j}$ and $Var(f_{i,j}(y)) = \frac{1}{2^j}(1 - \frac{1}{2^j})$. Define

$$T_n(i,j) = n^{-1/2} \sum_{m=1}^{n} \frac{f_{i,j}(\varphi^{m-1}(y)) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}} \text{ for given } i \text{ and } j.$$

Then, similar as above, the set of the limit points of $(2\ln\ln n)^{-1/2}\ T_n(i,j)$ is $[-1,1]$.

Third, for each fixed $j \in \mathbb{N}$, we consider the sum

$$f_j(y) = \sum_{i=1}^{2^j} \frac{f_{i,j}(y) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}$$

of the random variables

$$\frac{f_{1,j}(y)-2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}, \frac{f_{2,j}(y)-2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}, \ldots, \frac{f_{2^j,j}(y)-2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}.$$

Then, for fixed $j \in \mathbb{N}$, being a sequence of identically distributed random variables, $\{f_j(\varphi^{m-1}(y)) : m \geq 1\}$ is stationary and ergodic process. Consider the equation

$$\sum_{i=1}^{2^j} \frac{f_{i,j}(y)-2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}} - \sum_{i=1}^{2^j-1} \frac{f_{i,j}(y)-2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}} = \frac{f_{2^j,j}(y)-2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}.$$

We simply denote

$$d^j(y) := \frac{f_{2^j,j}(y)-2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}.$$

Observe that $d^j(y)$ is uniformly bounded in the sense that $\sup_{j\in\mathbb{N}} |d^j(y)| < \infty$. Observe also that for $j, k \in \mathbb{N}$,

$$Cov(d^j(y), d^k(y)) = \frac{2^{-j} \wedge 2^{-k} - 2^{-j} \cdot 2^{-k}}{\{2^{-j}(1-2^{-j})\}^{1/2}\{2^{-k}(1-2^{-k})\}^{1/2}},$$

where $x \wedge y$ denotes the minimum of x and y.

Recall that $\Omega = [0,1]$ is the sample space, $\mathcal{A}$ is the Borel sets and P is the Lebesgue measure. Then $\varphi : \Omega \to \Omega$ is a P-preserving measurable transformation. Assume that $\mathcal{F}_0 := \{\emptyset, \Omega\}$ is the φ-invariant σ-field (i.e., $\varphi^{-1}\mathcal{F}_0 \subset \mathcal{F}_0$), set $\mathcal{F}_n = \varphi^{-n}\mathcal{F}_0$, and denote by E_n the conditional expectation operator with respect to the σ-algebra $\mathcal{F}_n$. Then, for fixed $j \in \mathbb{N}$, the sequence $\{d^j(\varphi^{m-1}(y)) : m \geq 1\}$ is an ergodic, stationary sequence of martingale-differences. See Bae et al. [2].

From stationarity, using the Kolmogorov consistency theorem, we may assume that the process, for fixed $j \in \mathbb{N}$, $\{d^j(\varphi^{m-1}(y)) : m \in \mathbb{Z}\}$ is double sided. Define a $(j,s) \in \mathbb{N} \otimes [0,1]$ indexed sequential empirical process

$$T_n(j,s) := n^{-1/2} \sum_{m=1}^{[ns]} d^j(\varphi^{m-1}(y)) \text{ for } (j,s) \in \mathbb{N} \otimes [0,1], \tag{1}$$

where $[x]$ denotes the integer part of x.

Let X be a *uniformly bounded* random variable defined on $([0,1], \mathcal{B}[0,1]$, P = Lebesgue measure) whose distribution function is F. Consider a sequence $\{X_i : i \geq 1\}$ of independent copies of X. Given a Borel measurable function $f : \mathbb{R} \to [-1,1]$, we see that $\{f(X_i) : i \geq 1\}$ forms a sequence of IID random variables that are more flexible in applications than the sequence $\{X_i : i \geq 1\}$. Consider a class $\mathcal{F}$ of real-valued Borel measurable functions defined on $\mathbb{R}$. Introduce the usual empirical distribution function F_n defined by $F_n(x) = n^{-1}\sum_{i=1}^n 1_{\{X_i \leq x\}}$ for $x \in \mathbb{R}$. Define a function indexed sequential integral process S_n by $S_n(f,s) = n^{1/2}\frac{[ns]}{n}\int f(x)d(F_{[ns]}-F)(x)$ for $f \in \mathcal{F}$ and $s \in [0,1]$.

Given a class $\mathcal{F}$ of functions, establishing a uniform LIL includes the following two steps. First, establish a uniform central limit theorem (CLT) for the sequential integral process. Second, try to find an almost sure representation of the uniform CLT for the sequential integral process that involves the order of $(\ln\ln n)^{-1/2}$.

Establishing a uniform CLT for the sequential integral process means showing that $\mathcal{L}(S_n(f,s) : (f,s) \in \mathcal{F}\times[0,1]]) \to \mathcal{L}(Z(f,s) : (f,s) \in \mathcal{F}\times[0,1])$, where the processes are considered as random elements of the Banach space,

$$B\left(\mathcal{F}\times[0,1]\right) := \left\{z : \mathcal{F}\times[0,1] \to \mathbb{R} : ||z||_{\mathcal{F}} < \infty\right\},$$

the space of the bounded real-valued functions on $\mathcal{F}\times[0,1]$, taken with the sup norm. The limiting process $Z = (Z(f,s) : (f,s) \in \mathcal{F}\times[0,1])$ is a Gaussian process whose sample paths are contained in

$$U_B(\mathcal{F}\times[0,1],\rho) := \{z \in B(\mathcal{F}\times[0,1]) : z \text{ is uniformly continuous in } \rho\}.$$

Notice that $\left(B\left(\mathcal{F}\times[0,1]\right), ||\cdot||_{\mathcal{F}\times[0,1]}\right)$ is a Banach space and $U_B(\mathcal{F}\times[0,1],\rho)$ is a closed subspace which is separable if and only if $(\mathcal{F}\times[0,1],\rho)$ is totally bounded. We equip the space $\mathcal{F}$ with the L^2 metric d. Consider $\rho((f,s),(g,t))$ $= d(f,g) + |s-t|$ so that $(\mathcal{F}\times[0,1],\rho)$ is totally bounded.

We use the following weak convergence. See Van der Vaart and Wellner [11].

Definition 1. A sequence of $B\left(\mathcal{F}\times[0,1]\right)$-valued random functions $\{Y_n : n \geq 1\}$ converges in law to a $B\left(\mathcal{F}\times[0,1]\right)$-valued Borel measurable random function Y whose law concentrates on a separable subset of $B\left(\mathcal{F}\times[0,1]\right)$, denoted by $Y_n \Rightarrow Y$, if $Eg\left(Y\right) = \lim_{n\to\infty} E^{*}g\left(Y_n\right)$ for all $g \in C(B\left(\mathcal{F}\times[0,1]\right), ||\cdot||_{\mathcal{F}\times[0,1]})$, where $C(B\left(\mathcal{F}\times[0,1]\right),\ ||\cdot||_{\mathcal{F}\times[0,1]})$ is the set of real bounded, continuous functions. Here E^{*} denotes upper expectation.

In [8], Ossiander obtained the uniform LIL for the sequence of IID random variables under bracketing entropy. Ossiander's result states that if $\mathcal{F}$ has a bracketing entropy then $\{(2\ln\ln n)^{-1/2}S_n(f,1) : f \in \mathcal{F}\}$ is relatively compact.

Identify the points $j \in \mathbb{N}$ with the class of indicator functions $\{1_{[0,2^{-j}]} : j \in \mathbb{N}\}$.

We firstly state the following LIL for the sequence of random variables generated by the tent map.

Notice that $\{T_n(j,1) : j \geq 1\}$ is a sequence of stochastic processes indexed by $j \in \mathbb{N}$, whereas by fixing j, $\{T_n(j,1)\}$ is a sequence of random variables. The LIL is one dimensional version of the uniform LIL given in Theorem 2.

Theorem 1. *For each fixed $j \in \mathbb{N}$, the set of the limit points of the sequence $\left\{(2\ln\ln n)^{-1/2}T_n(j,1) : n \geq 3\right\}$ is the closed interval $[-1,1]$.*

Proof. The result directly follows from Theorem 2 below by fixing $j \in \mathbb{N}$. □

Recall that $d^j(y) = \frac{f_{2^j,j}(y)-2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}$ for $j \in \mathbb{N}$. We define $\mathcal{M} = \{d^j(y) \in l^2(\mathbb{N}) : Ed^j(y) = 0\}$. It is easy to see that $\mathcal{M}$ is a closed subspace of the Hilbert space $l^2(\mathbb{N})$, and hence $\mathcal{M}$ is also a Hilbert space. Let $\mathcal{U}$ be the unit ball of $\mathcal{M}$: $\mathcal{U} = \left\{d^j(y) \in \mathcal{M} : E|d^j(y)|^2 \leq 1\right\}$. Then $\mathcal{U}$ defines a set $\mathcal{U}(\mathbb{N})$ of function on $\mathbb{N}$:

$$\mathcal{U}(\mathbb{N}) = \left\{d^j(y) \mapsto E(d^j(y) \cdot d^k(y)) : j \in \mathbb{N}, d^k(y) \in \mathcal{U}\right\},$$

where

$$E(d^j(y) \cdot d^k(y)) = \frac{2^{-j} \wedge 2^{-k} - 2^{-j} \cdot 2^{-k}}{\{2^{-j}(1-2^{-j})\}^{1/2}\{2^{-k}(1-2^{-k})\}^{1/2}}.$$

We secondly state the uniform LIL for the process generated by the tent map.

Theorem 2. $\left\{(2\ln\ln n)^{-1/2} T_n(j,1) : j \in \mathbb{N}, n \geq 3\right\}$ *is relatively compact with respect to* $l^2(\mathbb{N})$ *norm almost surely, and the set of its limit points is* $\mathcal{U}(\mathbb{N})$.

Remark 1. Consider the example in Pollard [10, p. 2]: Goodness of fit test statistics can be expressed as functionals on a suitably standardized empirical distribution function. Consider the basic case of an independent sample $\xi_1, \ldots, \xi_n$ from the Uniform$(0,1)$ distribution. Define the uniform empirical process U_n by

$$U_n(t) = n^{-1/2} \sum_{i=1}^{n} \left(1_{\{\xi_i \leq t\}} - t\right) \text{ for } 0 \leq t \leq 1.$$

Then, they discussed the motivation of the studying the uniform CLT of $U_n \Rightarrow U$ where U is the mean zero Gaussian process with $Cov(U(t_i), U(t_j)) = t_i \wedge t_j - t_i t_j$. One can consider the problem of LIL and uniform LIL for the process U_n. Our process T_n is, in some sense, non-independent stationary martingale-difference version of the process U_n.

In Section 3, we restate the main results in a more general setting of function indexed processes. Finally, in Section 4, we provide the proofs.

2. The sequential empirical process for martingale differences

We use the following setup to state problem in a concrete fashion. From stationarity, using the Kolmogorov consistency theorem, we may assume that the process, for fixed $j \in \mathbb{N}$, $\{d^j(\varphi^{m-1}(y))\}$ is double sided. We choose $(\Omega = [0,1]^{\mathbb{Z}}, \mathcal{T} = (\mathcal{B}[0,1])^{\mathbb{Z}}, P)$. We know that the Lebesgue measure P is invariant under φ, that is, $P\varphi^{-1} = P$. We also know that φ is an ergodic map. Define for $m \in \mathbb{Z}$ a σ-field $M_{m-1} = \sigma(\varphi^n(y) : n \leq m-1)$ and $H_{m-1} = \{f : \Omega \to [-1,1] : f \text{ is } M_{m-1} \text{ measurable and } f \in L^2(\Omega)\}$. For each $f \in L^2(\Omega)$ we simply denote $E_{m-1}f$ to mean $E(f|M_{m-1})$ and $H_0 \ominus H_{-1} = \{f \in H_0 : E(fg) = 0 \text{ for each } g \in H_{-1}\}$. On $L^2(\Omega)$ we define a metric d by $d(f,g) = [E(f-g)^2]^{1/2}$. Let $\mathcal{F} \subseteq H_0 \ominus H_{-1}$. Consider the function indexed sequential empirical process defined by

$$T_n(f,s) = n^{-1/2} \sum_{m=1}^{[ns]} f(\varphi^{m-1}(y)), \ f \in \mathcal{F} \text{ and } s \in [0,1],$$

where $Ef(y) = 0$ and $Var(f(y)) = 1$ for $f \in \mathcal{F}$.

We introduce the empirical distribution function F_n for the random variables $y, \varphi(y), \varphi^2(y), \ldots, \varphi^{n-1}(y)$ defined by

$$F_n(x) = n^{-1} \sum_{m=1}^{n} 1_{\{\varphi^{m-1}(y) \leq x\}}$$

for $x \in \mathbb{R}$. Then the function indexed processes $T_n(f, s)$ can be represented as the following integral forms:

$$T_n(f, s) = \sqrt{n}\frac{[ns]}{n} \int f(x) dF_{[ns]}(x) \text{ for } f \in \mathcal{F} \text{ and } s \in [0, 1]. \tag{2}$$

We name T_n in (2) as the sequential integral process.

We firstly state the following LIL for the sequence of the random variables.

Notice that $\{T_n(f, 1) : f \in \mathcal{F}\}$ is a sequence of stochastic processes indexed by $f \in \mathcal{F}$, whereas by fixing $f \in \mathcal{F}$, $\{T_n(f, 1)\}$ is a sequence of random variables. The LIL is one dimensional version of the uniform LIL given in Theorem 2.

Theorem 3. *For each fixed $f \in \mathcal{F}$, as $n \to \infty$, the set of the limit points of the sequence $\left\{(2 \ln \ln n)^{-1/2} T_n(f, 1) : n \geq 3\right\}$ is the closed interval $[-1, 1]$.*

Proof. The result directly follows from Theorem 4 below by fixing $f \in \mathcal{F}$. □

We define $\mathcal{M} = \{f \in L^2(\Omega) : Ef = 0\}$. Then, as before, $\mathcal{M}$ is a closed subspace of the Hilbert space $L^2(\Omega)$, and $\mathcal{M}$ is a Hilbert space. Let $\mathcal{U}$ be the unit ball of $\mathcal{M}$. Then $\mathcal{U}$ defines a set $\mathcal{U}(\mathcal{F})$ of function on $\mathcal{F}$: $\mathcal{U}(\mathcal{F}) = \{f \mapsto E(f \cdot g) : f \in \mathcal{F}, g \in \mathcal{U}\}$.

In order to measure the size of the function space, we define the following version of metric entropy with bracketing. See Van der Vaart and Wellner [11].

Definition 2. Given two functions l and u, the bracket $[l, u]$ is the set of all functions f with $l \leq f \leq u$. An ϵ-bracket is a bracket $[l, u]$ with $||u - l|| < \epsilon$. The bracketing number $N_{[\,]}(\epsilon) := N_{[\,]}(\epsilon, \mathcal{F}, d)$ is the minimum number of ϵ-brackets needed to cover $\mathcal{F}$. We say that $\mathcal{F}$ has a bracketing entropy if $\int_0^1 [\ln N_{[\,]}(\epsilon, \mathcal{F}, d)]^{1/2} d\epsilon < \infty$.

Remark 2. It is known that $\mathcal{F} = \{1_{[0, 2^{-j}]} : j \in \mathbb{N}\}$ has a bracketing entropy. This is possible because the cardinality of $\mathcal{F}$ is the same as that of $\mathbb{N}$. See Van der Vaart and Wellner [11].

Define $\sigma_n^2(f, g) := n^{-1} \sum_{m=1}^{n} E_{m-2}[f(\varphi^{m-1}(y)) - g(\varphi^{m-1}(y))]^2$ for $f, g \in \mathcal{F}$.

We secondly state the following uniform LIL for the process.

Theorem 4. *Suppose that $\mathcal{F}$ has a bracketing entropy. Assume that there exists a constant L such that*

$$P^* \left\{ \sup_{f, g \in \mathcal{F}} \frac{\sigma_n^2(f, g)}{d^2(f, g)} \geq L \right\} \to 0 \text{ as } n \to \infty. \tag{3}$$

Then $\left\{(2\ln\ln n)^{-1/2}T_n(f,1) : f \in \mathcal{F}, n \geq 3\right\}$ *is relatively compact with respect to* $||\cdot||_{\mathcal{F}}$ *almost surely, and the set of its limit points is* $\mathcal{U}(\mathcal{F})$.

3. The proofs

In this section we make our effort to prove Theorem 4 and Theorem 2. The following Proposition 1 whose proof depends on a chaining argument with stratification appears in Bae and Levental [3].

Proposition 1. *Suppose that* $\mathcal{F}$ *has the bracketing entropy. Assume that there exists a constant* L *such that*

$$P^*\left\{\sup_{f,g\in\mathcal{F}} \frac{\sigma_n^2(f,g)}{d^2(f,g)} \geq L\right\} \to 0 \ \text{as } n \to \infty.$$

Then there exist a Gaussian process $\{Z(f,s) : (f,s) \in \mathcal{F}\times[0,1]\}$ *with bounded and continuous sample paths such that* $T_n \Rightarrow Z$, *as random elements of* $B(\mathcal{F}\times[0,1])$. *The Gaussian process has the mean* $EZ(f,s) = 0$ *and covariance structure* $EZ(f,s)Z(g,t) = (s \wedge t)E(fg)$.

We need the following definition.

Definition 3. A sequence of $B(\mathcal{F}\times[0,1])$-valued random functions $\{Y_n : n \geq 1\}$ converges in probability to 0, denoted $Y_n \to^P 0$, if $\lim_{n\to\infty} P^*\{|Y_n| > \epsilon\} \to 0$ for every $\epsilon > 0$. Here P^* denotes outer probability.

We will use the following restatement of Proposition 2 in the proof of Theorem 4. See Theorem 1.3 of Dudley and Philipp [5].

Proposition 2. *Suppose that* $\mathcal{F}$ *has the bracketing entropy. Assume that there exists a constant* L *such that*

$$P^*\left\{\sup_{f,g\in\mathcal{F}} \frac{\sigma_n^2(f,g)}{d^2(f,g)} \geq L\right\} \to 0 \ \text{as } n \to \infty.$$

Then there exist a sequence $\{Z_n : n \geq 1\}$ *of IID copies of Gaussian process* $\{Z(f) : f \in \mathcal{F}\}$, *defined on* $(\Omega, \mathcal{T}, P)$, *with bounded and continuous sample paths on* $\mathcal{F}$ *with the mean* $EZ(f) = 0$ *and the covariance structure* $EZ(f)Z(g) = E(fg)$ *such that*

$$n^{-1/2} \max_{1\leq k\leq n} \sup_{f\in\mathcal{F}} \Big|\sum_{m=1}^{k} (f(\varphi^{m-1})(y) - Z_m(f))\Big| \to^P 0.$$

The Z_m*'s can be chosen such that, almost surely for some measurable* U_n

$$\sup_{f\in\mathcal{F}} \Big|n^{-1/2} \sum_{m=1}^{k} (f(\varphi^{m-1})(y) - Z_m(f))\Big| \leq U_n = o((\ln\ln n)^{-1/2}).$$

The following Corollary 1 follows easily from Proposition 2.

Corollary 1. *Suppose that $\mathcal{F}$ has the bracketing entropy. Assume that there exists a constant L such that*

$$P^*\left\{\sup_{f,g\in\mathcal{F}}\frac{\sigma_n^2(f,g)}{d^2(f,g)}\geq L\right\}\to 0 \text{ as } n\to\infty.$$

Then there exist a sequence $\{\tilde{Z}_n : n\geq 1\}$, with bounded and continuous sample paths, of copies of a Gaussian process $\{Z(f) : f\in\mathcal{F}\}$, defined on $(\Omega,\mathcal{T},P)$ such that $||T_n(\cdot,1)-\tilde{Z}_n||\to^P 0$. The $\tilde{Z}_m$'s can be chosen such that, almost surely for some measurable U_n $||T_n(\cdot,1)-\tilde{Z}_n||\leq U_n=o((\ln\ln n)^{-1/2})$.

Proof. Let $\{Z_m\}$ be as in Proposition 2. Set $\tilde{Z}_n=n^{-1/2}\sum_{m=1}^n Z_m$. Observe that

$$\begin{aligned}||T_n(\cdot,1)-\tilde{Z}_n||\leq\ &\sup_{f\in\mathcal{F}}|n^{-1/2}\sum_{m=1}^n(f(\varphi^{m-1}(y))-Z_m(f))|\\ &+\sup_{f\in\mathcal{F}}|n^{-1/2}\sum_{m=1}^n(Z_m(f)-\tilde{Z}_m(f))|.\end{aligned}$$

Since the Gaussian processes Z and $\tilde{Z}$ have the same mean and the same covariance structure, they have the same distribution. That is, $P(||T_n(\cdot,1)-\tilde{Z}_n||=0)=1$. Proposition 2 implies the result. □

Proposition 3 (Theorem 4.3 of Pisier [9])**.** *Suppose that $\mathcal{F}$ has the bracketing entropy. Let $\{Z_i : i\geq 1\}$ be a sequence of IID copies of a Gaussian process $\{Z(f) : f\in\mathcal{F}\}$ defined on $(\Omega,\mathcal{T},P)$. Suppose $\{Z(f) : f\in\mathcal{F}\}$ has bounded and continuous sample paths with the mean $EZ(f)=0$ and $E||Z||<\infty$. Then, as $n\to\infty$, $\left\{(2n\ln\ln n)^{-1/2}\sum_{m=1}^n Z_m(f) : f\in\mathcal{F}, n\geq 3\right\}$ is relatively compact with respect to $||\cdot||_{\mathcal{F}}$ almost surely, and the set of its limit points is $\mathcal{U}(\mathcal{F})$ where $\mathcal{U}=\{h\in L^2(\Omega,\mathcal{T},P) : EZ^2(h)\leq 1\}$.*

Remark 3. Z takes values in $C(\mathcal{F})$, the bounded and continuous functions from $\mathcal{F}$ to $\mathbb{R}$, forms a separable Banach space with sup-norm $||\cdot||_{\mathcal{F}}$.

We are ready to prove Theorem 4.

Proof of Theorem 4. By Proposition 1, there exists a Gaussian process $\{Z(f,1) : f\in\mathcal{F}\}$ with bounded and continuous sample paths whose mean is zero and covariance structure is

$$EZ(f,1)Z(g,1)=E(gh). \tag{4}$$

Apply Corollary 1 to choose a sequence $\{\tilde{Z}_n : n\geq 1\}$ of Gaussian process such that

$$||(2\ln\ln n)^{-1/2}(T_n(\cdot,1)-\tilde{Z}_n)||\leq Y_n=o(1), \tag{5}$$

almost surely for some sequence of measurable Y_n's. By Proposition 3,

$$\left\{(2\ln\ln n)^{-1/2}\tilde{Z}_n(f) : f\in\mathcal{F}, n\geq 3\right\}$$

is relatively compact with respect to $||\cdot||_{\mathcal{F}}$ almost surely, and the set of its limit points is $\mathcal{U}(\mathcal{F})$ where $\mathcal{U} = \{h \in L^2(\Omega, \mathcal{T}, P) : EZ^2(h) \leq 1\}$. This, together with (4) and (5), complete the proof of Theorem 4. $\square$

Finally, we finish the proof of Theorem 2.

Proof of Theorem 2. By Remark 2, we see that $\mathcal{F} = \{1_{[0,2^{-j}]} : j \in \mathbb{N}\}$ has a bracketing entropy. We need to verify the assumption (3) for $\mathcal{F} = \{1_{[0,2^{-j}]} : j \in \mathbb{N}\}$. Notice that $d^2(f,g) = E[1_{[0,2^{-j}]} - 1_{[0,2^{-k}]}]^2 = |2^{-j} - 2^{-k}|$ and $E_{m-2}[f(y_{m-1}) - g(y_{m-1})]^2 = 1_{[2^{-j},2^{-k}]}(y_{m-1})$, where $y_{m-1} := \varphi^{m-1}(y)$. Then the assumption (3) boils down to the following: There exists a constant L such that

$$P\left(\sup_{j\in\mathbb{N}} \sum_{i=1}^{n} \frac{1_{[0,(2^j-1)2^{-j})}(y_i)}{n(2^j-1)2^{-j}} > L\right) \to 0 \text{ as } n \to \infty. \tag{6}$$

For the proof of (6), see the proof of Lemma 1 in [2]. Now apply Theorem 4 to finish the proof of Theorem 2. $\square$

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