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GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER MANIFOLD WITH A NON-METRIC ϕ-SYMMETRIC CONNECTION

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Abstract. The notion of a *non-metric ϕ-symmetric connection* on semi-Riemannian manifolds was introduced by Jin [6, 7]. The object of study in this paper is generic lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$ with a non-metric ϕ-symmetric connection. First, we provide several new results for such generic lightlike submanifolds. Next, we investigate generic lightlike submanifolds of an indefinite complex space form $\bar{M}(c)$ with a non-metric ϕ-symmetric connection.

1. Introduction

A lightlike submanifold M of an indefinite almost complex manifold $\bar{M}$ equipped with an almost complex structure J or an indefinite almost contact manifold $\bar{M}$ equipped with an almost contact structure J is called a *generic lightlike submanifold* if there exists a screen distribution $S(TM)$ such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{1.1}$$

where $S(TM)^{\perp}$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\bar{M}$ of $\bar{M}$, *i.e.*, $T\bar{M} = S(TM) \oplus_{orth} S(TM)^{\perp}$. The notion of generic lightlike submanifolds was introduced by Jin-Lee [8] in 2011 and later, studied by Duggal-Jin [2], Jin [3, 4] and Jin-Lee [9]. Every lightlike hypersurface and every half lightlike submanifold of codimension 2 of an indefinite almost complex manifold are examples of generic lightlike submanifolds. The geometry of generic lightlike submanifolds is an generalization of that of lightlike hypersurfaces and half lightlike submanifolds. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds.

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *non-metric ϕ-symmetric connection* if $\bar{\nabla}$ and its torsion tensor $\bar{T}$ satisfy

$$(\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y}, \bar{Z}) = -\theta(\bar{Y})\phi(\bar{X}, \bar{Z}) - \theta(\bar{Z})\phi(\bar{X}, \bar{Y}), \tag{1.2}$$

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$$\bar{T}(\bar{X},\bar{Y}) = \theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}, \tag{1.3}$$

where ϕ and J are tensor fields of types $(0,2)$ and $(1,1)$ respectively, and θ is a 1-form associated with a smooth vector field ζ by $\theta(\bar{X}) = \bar{g}(\bar{X},\zeta)$. Throughout this paper, denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$. The notion of non-metric ϕ-symmetric connections on an indefinite almost complex or almost contact manifolds was introduced by Jin [6, 7].

In this paper, we study generic lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M} = (\bar{M}, \bar{g}, J)$ with a non-metric ϕ-symmetric connection, in which the tensor field J in (1.3) is identical with the indefinite almost complex structure J of $\bar{M}$ and the tensor field ϕ in (1.2) is identical with the fundamental 2-form associated with the indefinite almost complex structure J, that is,

$$\phi(\bar{X},\bar{Y}) = \bar{g}(J\bar{X},\bar{Y}). \tag{1.4}$$

Remark 1.1. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of an indefinite Kaehler manifold $\bar{M}$ with respect to the metric $\bar{g}$. It is known [6] that *a linear connection $\bar{\nabla}$ on $\bar{M}$ is a non-metric ϕ-symmetric connection if and only if $\bar{\nabla}$ satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})J\bar{X}. \tag{1.5}$$

2. Structure equations

Let $\bar{M} = (\bar{M}, \bar{g}, J)$ be an indedinite Kaeler manifold, where $\bar{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure such that

$$J^2\bar{X} = -\bar{X}, \qquad \bar{g}(J\bar{X},J\bar{Y}) = \bar{g}(\bar{X},\bar{Y}), \qquad (\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = 0. \tag{2.1}$$

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the non-metric ϕ-symmetric connection $\bar{\nabla}$, the third equation of three equations in (2.1) is reduced to

$$(\bar{\nabla}_{\bar{X}}J)\bar{Y} = \theta(\bar{Y})\bar{X} + \theta(J\bar{Y})J\bar{X}. \tag{2.2}$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of dimension $(m+n)$. Then the radical distribution $Rad(TM)$ of M, defined by $Rad(TM) = TM \cap TM^{\perp}$, is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r\,(1 \le r \le \min\{m, n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, respectively, which are called the *screen distribution* and the *co-screen distribution* of M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use the same notations for any others. Let X, Y, Z and W be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i, j, k, \ldots \in \{1, \ldots, r\}, \qquad a, b, c, \ldots \in \{r+1, \ldots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^\perp$ in $S(TM)^\perp$ respectively and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \qquad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^\perp). \end{aligned}$$

We say that a lightlike submanifold $M = (M, g, S(TM), S(TM^\perp))$ of $\bar{M}$ is

(1) *r-lightlike submanifold* if $1 \le r < \min\{m, n\}$;
(2) *co-isotropic submanifold* if $1 \le r = n < m$;
(3) *isotropic submanifold* if $1 \le r = m < n$;
(4) *totally lightlike submanifold* if $1 \le r = m = n$.

The above three classes (2)$\sim$(4) are particular cases of the class (1) as follows:

$$S(TM^\perp) = \{0\}, \qquad S(TM) = \{0\}, \qquad S(TM) = S(TM^\perp) = \{0\},$$

respectively. Thus the geometry of r-lightlike submanifolds is more general than that of the other three types. For this reason, we consider only r-lightlike submanifolds M with following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \ldots, \xi_r, N_1, \ldots, N_r, F_{r+1}, \ldots, F_m, E_{r+1}, \ldots, E_n\},$$

where $\{F_{r+1}, \ldots, F_m\}$ and $\{E_{r+1}, \ldots, E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^\perp)$, respectively. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a \delta_{ab} = \bar{g}(E_a, E_b)$.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingarten formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^r h_i^\ell(X,Y) N_i + \sum_{a=r+1}^n h_a^s(X,Y) E_a, \tag{2.3}$$

$$\bar{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^r \tau_{ij}(X) N_j + \sum_{a=r+1}^n \rho_{ia}(X) E_a, \tag{2.4}$$

$$\bar{\nabla}_X E_a = -A_{E_a} X + \sum_{i=1}^r \lambda_{ai}(X) N_i + \sum_{b=r+1}^n \mu_{ab}(X) E_b; \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^r h_i^*(X, PY)\xi_i, \tag{2.6}$$

$$\nabla_X \xi_i = -A^*_{\xi_i} X - \sum_{j=1}^r \sigma_{ji}(X)\xi_j, \tag{2.7}$$

where ∇ and ∇^* are induced linear connections on M and $S(TM)$ respectively, h_i^ℓ and h_a^s are called the *local second fundamental forms* on M, h_i^* are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A^*_{\xi_i}$ are called the *shape operators*, and τ_{ij}, ρ_{ia}, λ_{ai}, μ_{ab} and σ_{ji} are 1-forms on TM.

Let M be a generic lightlike submanifold of $\bar{M}$. From (1.1) we show that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are vector subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions H_o and H with respect to J, *i.e.*, $J(H_o) = H_o$ and $J(H) = H$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o,$$
$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

In this case, the tangent bundle TM of M is decomposed as follow:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})). \tag{2.8}$$

Consider $2r$-th local null vector fields U_i and V_i, $(n-r)$-th local non-null unit vector fields W_a, and their 1-forms u_i, v_i and w_a defined by

$$U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_a = -JE_a, \tag{2.9}$$
$$u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a). \tag{2.10}$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a. \tag{2.11}$$

Let η_i be the 1-forms such that $\eta_i(X) = \bar{g}(X, N_i)$. From (2.11), we see that

$$\begin{aligned} &u_i(FX) = 0, \qquad w_a(FX) = 0, \qquad v_i(FX) = -\eta_i(X), \\ &FU_i = 0, \qquad FW_a = 0, \qquad FV_i = \xi_i. \end{aligned} \tag{2.12}$$

Applying J to (2.11) and using $(2.1)_1$, (2.9) and (2.12), we have

$$F^2X = -X + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a. \tag{2.13}$$

Substituting (2.11) into $(2.1)_2$ and using (2.9) and (2.10), we have

$$\begin{aligned} g(FX, FY) = g(X, Y) &- \sum_{i=1}^{r}\{u_i(X)v_i(Y) + u_i(Y)v_i(X)\} \\ &- \sum_{a=r+1}^{n} \epsilon_a w_a(X)w_a(Y). \end{aligned} \tag{2.14}$$

In the following, we say that F is the *structure tensor field* of M.

3. Equations related to non-metric ϕ-symmetric connection

Denote by α_i, β_i and γ_a the smooth functions on $\bar{M}$ given by

$$\alpha_i = \theta(\xi_i), \qquad \beta_i = \theta(N_i), \qquad \gamma_a = \theta(E_a).$$

Using (1.2), (1.3), (1.4), (2.3), (2.9) and (2.11), we see that

$$(\nabla_X g)(Y,Z) = \sum_{i=1}^{r}\{h_i^\ell(X,Y)\eta_i(Z) + h_i^\ell(X,Z)\eta_i(Y)\} - \theta(Y)\phi(X,Z) - \theta(Z)\phi(X,Y), \tag{3.1}$$

$$T(X,Y) = \theta(Y)FX - \theta(X)FY, \tag{3.2}$$

$$h_i^\ell(X,Y) - h_i^\ell(Y,X) = \theta(Y)u_i(X) - \theta(X)u_i(Y), \tag{3.3}$$

$$h_a^s(X,Y) - h_a^s(Y,X) = \theta(Y)w_a(X) - \theta(X)w_a(Y), \tag{3.4}$$

$$\phi(X,Y) = g(FX,Y) + \sum_{i=1}^{r} u_i(X)\eta_i(Y), \tag{3.5}$$

$$\begin{aligned} &\phi(X,\xi_i) = u_i(X), \quad \phi(X,N_i) = v_i(X), \qquad \phi(X,E_a) = \epsilon_a w_a(X),\\ &\phi(X,V_i) = 0, \qquad\quad \phi(X,U_i) = -\eta_i(X), \qquad \phi(X,W_a) = 0. \end{aligned} \tag{3.6}$$

From the facts that $h_i^\ell(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^s(X,Y) = \bar{g}(\bar{\nabla}_X Y, E_a)$, we know that h_i^ℓ and h_a^s are independent of the choice of $S(TM)$. The above local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X,Y) + u_i(X)\theta(Y) + \alpha_i g(FX,Y) - \sum_{k=1}^{r}\{h_k^\ell(X,\xi_i) - \alpha_i u_k(X)\}\eta_k(Y), \tag{3.7}$$

$$\epsilon_a h_a^s(X,Y) = g(A_{E_a}X,Y) + \epsilon_a w_a(X)\theta(Y) + \gamma_a g(FX,Y) - \sum_{k=1}^{r}\{\lambda_{ak}(X) - \gamma_a u_k(X)\}\eta_k(Y), \tag{3.8}$$

$$h_i^*(X,PY) = g(A_{N_i}X,PY) + v_i(X)\theta(PY) + \beta_i g(FX,PY). \tag{3.9}$$

Applying $\bar{\nabla}_X$ to $g(\xi_i,\xi_j) = 0$, $\bar{g}(\xi_i,E_a) = 0$, $\bar{g}(N_i,N_j) = 0$, $\bar{g}(N_i,E_a) = 0$, $\bar{g}(E_a,E_b) = \epsilon_a\delta_{ab}$ and $\bar{g}(N_i,\xi_j) = \delta_{ij}$ by turns, we obtain

$$\begin{aligned} &h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = \alpha_i u_j(X) + \alpha_j u_i(X),\\ &h_a^s(X,\xi_i) = -\epsilon_a\{\lambda_{ai}(X) - \gamma_a u_i(X)\} + \alpha_i w_a(X),\\ &\eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = -\beta_i v_j(X) - \beta_j v_i(X),\\ &\eta_i(A_{E_a}X) = \epsilon_a\{\rho_{ia}(X) - \beta_i w_a(X)\} - \gamma_a v_i(X),\\ &\epsilon_b\{\mu_{ab}(X) - \gamma_a w_b(X)\} + \epsilon_a\{\mu_{ba}(X) - \gamma_b w_a(X)\} = 0,\\ &\tau_{ij}(X) = \sigma_{ij}(X) + \alpha_j v_i(X) + \beta_i u_j(X). \end{aligned} \tag{3.10}$$

As a consequence of (3.3) and $(3.10)_1$, it follows that

$$h_i^\ell(X, \xi_i) = \alpha_i u_i(X), \qquad h_i^\ell(\xi_i, X) = 0. \tag{3.11}$$

By using (3.3), (3.7), $(3.10)_1$ and $(3.11)_2$, we deduce

$$h_i^\ell(\xi_j, \xi_k) = 0, \qquad A^*_{\xi_i}\xi_i = \alpha_i V_i. \tag{3.12}$$

Applying $\bar{\nabla}_X$ to $(2.9)_{1,2,3}$ and (2.11) by turns and using (2.2), (2.3) ∼ (2.7), (2.9) ∼ (2.11) and (3.7) ∼ (3.9), we have

$$\begin{aligned}
h_j^\ell(X, U_i) &= u_j(A_{N_i}X) + \theta(U_i)u_j(X) \\
&= h_i^*(X, V_j) + \theta(U_i)u_j(X) - \theta(V_j)v_i(X), \\
h_a^s(X, U_i) &= w_a(A_{N_i}X) + \theta(U_i)w_a(X) \\
&= \epsilon_a\{h_i^*(X, W_a) - \theta(W_a)v_i(X)\} + \theta(U_i)w_a(X), \\
h_j^\ell(X, V_i) &= u_j(A^*_{\xi_i}X) + \theta(V_i)u_j(X) \\
&= h_i^\ell(X, V_j) + \theta(V_i)u_j(X) - \theta(V_j)u_i(X), \\
h_a^s(X, V_i) &= w_a(A^*_{\xi_i}X) + \theta(V_i)w_a(X) \\
&= \epsilon_a\{h_i^\ell(X, W_a) - \theta(W_a)u_i(X)\} + \theta(V_i)w_a(X), \\
\epsilon_b\{h_b^s(X, W_a) - \theta(W_a)w_b(X)\} &= \epsilon_a\{h_a^s(X, W_b) - \theta(W_b)w_a(X)\},
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\nabla_X U_i \;=\; & F(A_{N_i}X) + \sum_{j=1}^r \tau_{ij}(X)U_j + \sum_{a=r+1}^n \rho_{ia}(X)W_a \\
& - \beta_i X + \theta(U_i)FX,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\nabla_X V_i \;=\; & F(A^*_{\xi_i}X) - \sum_{j=1}^r \sigma_{ji}(X)V_j + \sum_{j=1}^r h_j^\ell(X, \xi_i)U_j \\
& + \sum_{a=r+1}^n h_a^s(X, \xi_i)W_a - \alpha_i X + \theta(V_i)FX,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\nabla_X W_a \;=\; & F(A_{E_a}X) + \sum_{i=1}^r \lambda_{ai}(X)U_i + \sum_{b=r+1}^n \mu_{ab}(X)W_b \\
& - \gamma_a X + \theta(W_a)FX,
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
(\nabla_X F)(Y) \;=\; & \sum_{i=1}^r u_i(Y)A_{N_i}X + \sum_{a=r+1}^n w_a(Y)A_{E_a}X \\
& - \sum_{i=1}^r h_i^\ell(X, Y)U_i - \sum_{a=r+1}^n h_a^s(X, Y)W_a \\
& + \theta(Y)X + \theta(JY)FX.
\end{aligned} \tag{3.17}$$

Definition. We say that a lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is *irrotational* [10] if $\bar{\nabla}_X \xi_i \in \Gamma(TM)$ for all $i \in \{1, \ldots, r\}$.

Remark 3.1. From (2.3), the above definition is equivalent to

$$h_j^\ell(X,\xi_i)=0, \qquad h_a^s(X,\xi_i)=0. \tag{3.18}$$

From $(3.10)_{1,\,2}$, we see that if M is irrotational, then we have

$$\alpha_i=0, \qquad \lambda_{ai}(X)=\gamma_a u_i(X). \tag{3.19}$$

4. Recurrent and Lie recurrent generic lightlike submanifolds

Definition. The structure tensor field F of M is said to be *recurrent* [5] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y=\varpi(X)FY.$$

A lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is called *recurrent* if it admits a recurrent structure tensor field F.

Theorem 4.1. *Let M be a recurrent generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a non-metric ϕ-symmetric connection. Then*

(1) *F is parallel with respect to the induced connection ∇ on M,*
(2) *M is irrotational,*
(3) *the 1-form θ vanishes, i.e., $\theta=0$, on M. Thus the indeced connection ∇ on M is a torsion-free non-metric connection,*
(4) *H, $J(ltr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on M,*
(5) *M is locally a product manifold*

$$M=M^r\times M^{n-r}\times M^{m-n},$$

where M^r, M^{n-r} and M^{m-n} are the leaves of the parallel distributions $J(ltr(TM))$, $J(S(TM^\perp))$ and H respectively.

Proof. (1) From the above definition and (3.17), we get

$$\begin{aligned}\varpi(X)FY={}&\sum_{i=1}^r u_i(Y)A_{N_i}X+\sum_{a=r+1}^n w_a(Y)A_{E_a}X\\&-\sum_{i=1}^r h_i^\ell(X,Y)U_i-\sum_{a=r+1}^n h_a^s(X,Y)W_a\\&+\theta(Y)X+\theta(JY)FX.\end{aligned} \tag{4.1}$$

Replacing Y by ξ_j and using the fact that $F\xi_j=-V_j$, we obtain

$$\varpi(X)V_j=\sum_{i=1}^r h_i^\ell(X,\xi_j)U_i+\sum_{a=r+1}^n h_a^s(X,\xi_j)W_a-\ \alpha_j X+\theta(V_j)FX. \tag{4.2}$$

Taking the scalar product with N_i to this equation, we obtain

$$\alpha_j\eta_i(X)-\theta(V_j)v_i(X)=0.$$

Taking $X=\xi_j$ and $X=V_j$ by turns to this equation, we have

$$\alpha_i=0, \qquad \theta(V_i)=0, \qquad \forall\, i. \tag{4.3}$$

Taking the scalar product with U_j to (4.2), we get $\varpi = 0$. It follows that $\nabla_X F = 0$. Therefore, F is parallel with respect to the connection ∇.

(2) Taking the scalar product with V_i and W_a to (4.2) by turns and using (4.3), we get two equations in (3.18). Thus M is irrotational.

(3) Replacing Y by V_j to (4.1) and using (3.3), (3.4) and (4.3), we have

$$h_i^\ell(X, V_j) = h_i^\ell(V_j, X) = 0, \qquad h_a^s(X, V_j) = h_a^s(V_j, X) = 0. \tag{4.4}$$

Taking $Y = U_i$ and $Y = W_a$ to (4.1) such that $\varpi = 0$ by turns, we have

$$A_{N_i} X = \sum_{k=1}^{r} h_k^\ell(X, U_i)U_k + \sum_{a=r+1}^{n} h_a^s(X, U_i)W_a - \theta(U_i)X - \beta_i FX, \tag{4.5}$$

$$A_{E_a} X = \sum_{i=1}^{r} h_i^\ell(X, W_a)U_i + \sum_{b=r+1}^{n} h_b^s(X, W_a)W_b - \theta(W_a)X - \gamma_a FX. \tag{4.6}$$

Taking the scalar product with N_j and U_j to (4.5) by turns, we obtain

$$\eta_j(A_{N_i} X) = -\theta(U_i)\eta_j(X) - \beta_i v_j(X),$$
$$g(A_{N_i} X, U_j) = -\theta(U_i)v_j(X) + \beta_i \eta_j(X).$$

Taking $i = j$ to the first equation and using $(3.10)_3$, we have $\theta(U_i)\eta_i(X) = 0$. It follows that $\theta(U_i) = 0$. Replacing PY by U_j to (3.9) and using the second equation, we have $h_i^*(X, U_j) = 0$. Therefore

$$\theta(U_i) = 0, \qquad \eta_j(A_{N_i} X) = -\beta_i v_j(X), \qquad h_i^*(X, U_j) = 0. \tag{4.7}$$

Taking the scalar product with N_i and U_i to (4.6) by turns, we obtain

$$\eta_i(A_{E_a} X) = -\theta(W_a)\eta_i(X) - \gamma_a v_i(X),$$
$$g(A_{E_a} X, U_i) = -\theta(W_a)v_i(X) + \gamma_a \eta_i(X).$$

By using (3.8), $(3.10)_4$ and $(4.7)_1$, the last two equations reduce to

$$\rho_{ia}(X) = \beta_i w_a(X) - \epsilon_a \theta(W_a)\eta_i(X),$$
$$h_a^s(X, U_i) = -\epsilon_a \theta(W_a)v_i(X).$$

Let $X = V_i$ to the second equation and using $(4.4)_2$, we get $\theta(W_a) = 0$. Thus

$$\begin{aligned} &\theta(W_a) = 0, \qquad &&\eta_i(A_{E_a} X) = -\gamma_a v_i(X), \\ &\rho_{ia}(X) = \beta_i w_a(X), \qquad &&h_a^s(X, U_i) = 0. \end{aligned} \tag{4.8}$$

Taking the scalar product with N_i to (4.1) and using $(4.7)_2$ and $(4.8)_2$, we get

$$\theta(Y)\eta_i(X) + \{\theta(JY) - \sum_{k=1}^{r} \beta_k u_k(Y) - \sum_{a=r+1}^{n} \gamma_a w_a(Y)\}v_i(X) = 0.$$

Replacing X by ξ_i and V_i to this equation by turns, we obtain

$$\theta(X) = 0, \qquad \theta(JY) = \sum_{k=1}^{r} \beta_k u_k(Y) + \sum_{a=r+1}^{n} \gamma_a w_a(Y). \tag{4.9}$$

As $\theta = 0$ on M, from (3.1) and (3.2), ∇ is a torsion-free non-metric connection.

(4) Applying F to (4.5) and (4.6) and using (2.13), $(4.7)_1$ and $(4.8)_1$, we get

$$F(A_{N_i}X) - \beta_i X = -\sum_{j=1}^{r} \beta_i u_j(X) U_j - \sum_{a=r+1}^{n} \beta_i w_a(X) W_a,$$
$$F(A_{E_a}X) - \gamma_a X = -\sum_{i=1}^{r} \gamma_a u_i(X) U_i - \sum_{b=r+1}^{n} \gamma_a w_b(X) W_b.$$

Using these equations, $(3.19)_2$ and $(4.8)_3$, (3.14) and (3.16) are reduced to

$$\nabla_X U_i = \sum_{j=1}^{r} \{\tau_{ij}(X) - \beta_i u_j(X)\} U_j, \tag{4.10}$$

$$\nabla_X W_a = \sum_{b=r+1}^{n} \{\mu_{ab}(X) - \gamma_a w_b(X)\} W_b. \tag{4.11}$$

It follows from (4.10) and (4.11) that $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are parallel distributions on M with respect to the connection ∇ on M, that is,

$$\nabla_X U_i \in \Gamma(J(ltr(TM))), \qquad \nabla_X W_a \in \Gamma(J(S(TM^{\perp}))).$$

On the other hand, from $(3.13)_4$, $(4.4)_2$ and $(4.9)_1$, we see that

$$h_i^{\ell}(X, W_a) = 0. \tag{4.12}$$

Taking $Y = FZ$ to (4.1) and using $(4.9)_{1,2}$ and $u(FZ) = w(FZ) = 0$, we get

$$h_i^{\ell}(X, FZ) = 0, \qquad h_a^{s}(X, FZ) = 0. \tag{4.13}$$

For any $X \in \Gamma(TM)$ and $Z \in \Gamma(H_o)$, by using (2.7), (2.14), (3.1), (3.5), $(3.6)_4$, (3.7), (4.3), $(4.4)_1$, $(4.9)_1$, (4.12) and (4.13), we derive

$$\begin{aligned}
&g(\nabla_X \xi_j, V_i) = -h_j^{\ell}(X, V_i) + \theta(V_i) u_j(X) = 0,\\
&g(\nabla_X \xi_j, W_a) = -h_j^{\ell}(X, W_a) + \theta(W_a) u_j(X) = 0,\\
&g(\nabla_X V_j, V_i) = h_i^{\ell}(X, \xi_j) - \alpha_j u_i(X) = 0,\\
&g(\nabla_X V_j, W_a) = h_a^{s}(X, \xi_j) - \epsilon_a \alpha_j w_a(X) = 0,\\
&g(\nabla_X Z, V_i) = h_i^{\ell}(X, FZ) - \theta(FZ) u_i(X) = 0,\\
&g(\nabla_X Z, W_a) = h_a^{s}(X, FZ) - \epsilon_a \theta(FZ) w_a(X) = 0.
\end{aligned}$$

It follows that H is also a parallel distribution on M, that is,

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H).$$

(5) As $J(ltr(TM))$, $J(S(TM^{\perp}))$ and H are parallel distributions and satisfed (2.8), by the decomposition theorem of de Rham [1], M is locally a product manifold $M = M^r \times M^{n-r} \times M^{m-n}$, where M^r, M^{n-r} and M^{m-n} are the leaves of $J(ltr(TM))$, $J(S(TM^{\perp}))$ and H respectively. $\square$

Definition. The structure tensor field F of M is said to be *Lie recurrent* [5] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X. In case $\vartheta = 0$, we say that F is *Lie parallel*. A lightlike submanifold M is called *Lie recurrent* if it admits a Lie recurrent structure tensor field F.

Theorem 4.2. *Let M be a Lie recurrent generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a non-metric ϕ-symmetric connection. Then*

(1) *F is Lie parallel,*

(2) *τ_{ij} and ρ_{ia} are satisfied $\tau_{ij} \circ F = 0$ and $\rho_{ia} \circ F = 0$. Moreover,*

$$\tau_{ij}(X) = -\sum_{k=1}^{r} \eta_k(A_{N_i} V_j)u_k(X),$$

$$\rho_{ia}(X) = \sum_{k=1}^{r} \rho_{ia}(U_k)u_k(X) + \sum_{b=r+1}^{n} \rho_{ia}(W_b)w_b(X),$$

(3) *$\alpha_i = 0$ for all i, and the shape operators $A^*_{\xi_i}$ for all i satisfy*

$$A^*_{\xi_i} U_j = 0, \qquad A^*_{\xi_i} V_j = -F(A^*_{\xi_i} \xi_j). \tag{4.14}$$

Proof. (1) As $(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]$, we obtain

$$\begin{aligned} \vartheta(X)FY = & -\nabla_{FY} X + F\nabla_Y X \\ & + \sum_{i=1}^{r} u_i(Y)A_{N_i} X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a} X \\ & - \sum_{i=1}^{r} \{h_i^\ell(X, Y) - \theta(Y)u_i(X)\}U_i \\ & - \sum_{a=r+1}^{n} \{h_a^s(X, Y) - \theta(Y)w_a(X)\}W_a \\ & + \{\sum_{i=1}^{r} \beta_i u_i(Y) + \sum_{a=r+1}^{n} \gamma_a w_a(Y)\}FX, \end{aligned} \tag{4.15}$$

by (2.13), (3.2) and (3.17). Taking $Y = \xi_j$ and $Y = V_j$ by turns, we have

$$\begin{aligned} -\vartheta(X)V_j = & \nabla_{V_j} X + F\nabla_{\xi_j} X \\ & - \sum_{i=1}^{r} \{h_i^\ell(X, \xi_j) - \alpha_j u_i(X)\}U_i \\ & - \sum_{a=r+1}^{n} \{h_a^s(X, \xi_j) - \alpha_j w_a(X)\}W_a, \end{aligned} \tag{4.16}$$

$$\vartheta(X)\xi_j = -\nabla_{\xi_j} X + F\nabla_{V_j} X \tag{4.17}$$

$$
\begin{aligned}
&- \sum_{i=1}^{r} \{h_i^\ell(X, V_j) - \theta(V_j) u_i(X)\} U_i \\
&- \sum_{a=r+1}^{n} \{h_a^s(X, V_j) - \theta(V_j) w_a(X)\} W_a.
\end{aligned}
$$

Taking the scalar product with U_i to (4.16) and N_i to (4.17) by turns, we get

$$
\begin{aligned}
-\delta_{ij}\vartheta(X) &= g(\nabla_{V_j} X, U_i) - \bar{g}(\nabla_{\xi_j} X, N_i), \\
\delta_{ij}\vartheta(X) &= g(\nabla_{V_j} X, U_i) - \bar{g}(\nabla_{\xi_j} X, N_i).
\end{aligned}
$$

Comparing the last two equations, we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with N_i to (4.16) such that $X = W_a$ and using (3.4), (3.8), $(3.10)_4$ and (3.16), we get $h_a^s(U_i, V_j) = \rho_{ia}(\xi_j)$. On the other hand, taking the scalar product with W_a to (4.17) such that $X = U_i$ and using (3.14), we obtain $h_a^s(U_i, V_j) = -\rho_{ia}(\xi_j)$. Thus we have $\rho_{ia}(\xi_j) = 0$ and $h_a^s(U_i, V_j) = 0$.

Taking the scalar product with V_i to (4.16) such that $X = W_a$ and using (3.3), (3.4), $(3.10)_2$, $(3.13)_4$ and (3.16), we obtain $\lambda_{ai}(V_j) = -\lambda_{aj}(V_i)$. Next, taking the scalar product with W_a to (4.16) such that $X = V_i$ and using (3.15), we have $h_a^s(V_i, \xi_j) = h_a^s(V_j, \xi_i)$. As $h_a^s(V_j, \xi_i) = -\epsilon_a \lambda_{ai}(V_j)$ by $(3.10)_2$, we see that $\lambda_{ai}(V_j) = \lambda_{aj}(V_i)$. Thus we obtain $\lambda_{ai}(V_j) = 0$ and $h_a^s(V_i, \xi_j) = 0$.

Taking the scalar product with W_a to (4.16) such that $X = \xi_i$ and using (2.7), (3.4), (3.7) and $(3.10)_2$, we get $h_i^\ell(V_j, W_a) = \lambda_{ai}(\xi_j)$. Next, taking the scalar product with V_i to (4.17) such that $X = W_a$ and using (3.3) and (3.16), we get $h_i^\ell(V_j, W_a) = -\lambda_{ai}(\xi_j)$. Thus $\lambda_{ai}(\xi_j) = 0$ and $h_i^\ell(V_j, W_a) = 0$.

Taking the scalar product with U_i to (4.16) such that $X = W_a$ and using (3.4), (3.8), $(3.10)_{2,4}$ and (3.16), we have $\epsilon_a \rho_{ia}(V_j) = \lambda_{aj}(U_i) - \gamma_a \delta_{ij}$. Next, taking the scalar product with W_a to (4.16) such that $X = U_i$ and using $(3.10)_2$ and (3.14), we obtain $\epsilon_a \rho_{ia}(V_j) = -\lambda_{aj}(U_i) + \gamma_a \delta_{ij}$. Thus $\rho_{ia}(V_j) = 0$, $\lambda_{aj}(U_i) = \gamma_a \delta_{ij}$ and $h_a^s(U_i, \xi_j) = 0$. Summarizing the above results, we get

$$
\begin{aligned}
&\rho_{ia}(\xi_j) = 0, \qquad \rho_{ia}(V_j) = 0, \qquad \lambda_{ai}(V_j) = 0, \qquad \lambda_{ai}(\xi_j) = 0, \\
&\lambda_{ai}(U_j) = \gamma_a \delta_{ij}, \quad h_a^s(U_i, V_j) = 0, \quad h_a^s(U_i, \xi_j) = h_i^*(\xi_j, W_a) = 0, \\
&h_i^\ell(V_j, W_a) = h_a^s(V_j, V_i) = 0, \qquad h_a^s(V_i, \xi_j) = h_i^\ell(W_a, \xi_j) = 0.
\end{aligned} \tag{4.18}
$$

Taking the scalar product with N_i to (4.15) and using $(3.10)_4$, we have

$$
\begin{aligned}
&- \bar{g}(\nabla_{FY} X, N_i) + g(\nabla_Y X, U_i) \\
&+ \sum_{k=1}^{r} u_k(Y) \{\bar{g}(A_{N_k} X, N_i) + \beta_k v_i(X)\} \\
&+ \sum_{a=r+1}^{n} \epsilon_a w_a(Y) \{\rho_{ia}(X) - \beta_i w_a(X)\} = 0.
\end{aligned} \tag{4.19}
$$

Replacing X by ξ_j to (4.19) and using (2.7), (3.7), $(3.10)_6$ and $(4.18)_1$, we have

$$h^\ell_j(X, U_i) - \theta(U_i)u_j(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) + \tau_{ij}(FX). \tag{4.20}$$

Taking $X = U_k$ to this equation and using $(3.13)_1$, we have

$$h^*_i(U_k, V_j) = \bar{g}(A_{N_k}\xi_j, N_i). \tag{4.21}$$

Taking $X = U_i$ to (4.15) and using (2.13), (3.3), (3.4) and $(3.13)_{1,2}$, we get

$$\begin{aligned}\sum_{k=1}^{r} u_k(Y)A_{N_k}U_i + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}U_i - A_{N_i}Y\\ - F(A_{N_i}FY) - \sum_{j=1}^{r}\tau_{ij}(FY)U_j - \sum_{a=r+1}^{n}\rho_{ia}(FY)W_a = 0.\end{aligned} \tag{4.22}$$

Taking the scalar product with V_j to (4.22) and using (3.8), (3.9), $(3.10)_3$, $(3.13)_1$, $(4.18)_6$ and (4.21), we obtain

$$h^\ell_j(X, U_i) - \theta(U_i)u_j(X) = -\sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) - \tau_{ij}(FX).$$

Comparing this equation with (4.20), we obtain

$$h^\ell_j(X, U_i) = \theta(U_i)u_j(X), \quad \tau_{ij}(FX) + \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) = 0. \tag{4.23}$$

Replacing X by U_h to $(4.23)_2$, we have $\bar{g}(A_{N_k}\xi_j, N_i) = 0$. Thus we see that

$$\tau_{ij}(FX) = 0.$$

From $(4.23)_1$, we have $h^\ell_j(FX, U_i) = 0$. Replacing X by V_j to (4.19) and using (3.7), $(3.10)_{3,6}$, (3.15) and $(4.18)_2$, we obtain

$$\begin{aligned}\tau_{ij}(X) &= \sum_{k=1}^{r}\{\eta_i(A_{N_k}V_j) + \beta_k\delta_{ij} + \beta_i\delta_{kj}\}u_k(X)\\ &= -\sum_{k=1}^{r}\eta_k(A_{N_i}V_j)u_k(X).\end{aligned}$$

Replacing Y by W_a to (4.22), we obtain $A_{E_a}U_i = A_{N_i}W_a$. Taking the scalar product with U_j to this and using (3.4), (3.8), (3.9) and $(3.13)_2$, we have

$$h^*_i(W_a, U_j) = \epsilon_a h^s_a(U_i, U_j) = \epsilon_a h^s_a(U_j, U_i) = h^*_i(U_j, W_a). \tag{4.24}$$

Taking the scalar product with W_a to (4.22) and using (3.8) and (3.9), we get

$$\begin{aligned}\epsilon_a\rho_{ia}(FY) = &- h^*_i(Y, W_a) + \theta(W_a)v_i(Y)\\ &+ \sum_{k=1}^{r} u_k(Y)h^*_k(U_i, W_a) + \sum_{b=r+1}^{n}\epsilon_b w_b(Y)h^s_b(U_i, W_a).\end{aligned}$$

Taking the scalar product with U_i to (4.15) and then, taking $X = W_a$ and using (3.4), (3.8), (3.9), $(3.10)_4$, $(3.13)_2$, (3.16) and (4.24), we obtain

$$\begin{aligned}\epsilon_a \rho_{ia}(FY) = {} & h_i^*(Y, W_a) - \theta(W_a) v_i(Y) \\ & - \sum_{k=1}^{r} u_k(Y) h_k^*(U_i, W_a) - \sum_{b=r+1}^{n} \epsilon_b w_b(Y) h_b^s(U_i, W_a).\end{aligned}$$

Comparing the last two equations, we obtain

$$\rho_{ia}(FY) = 0.$$

Replacing X by W_a to (4.19) and using (3.8), $(3.10)_4$ and (3.16), we have

$$\begin{aligned}\epsilon_a \rho_{ia}(X) = {} & -\epsilon_a h_a^s(FX, U_i) \\ & + \sum_{k=1}^{r} u_k(X) \bar{g}(A_{N_k} W_a, N_i) + \sum_{b=r+1}^{n} \epsilon_b w_b(X) \rho_{ib}(W_a).\end{aligned}$$

Taking $X = U_k$, $X = W_b$ and $X = FY$ to this equation by turns, we have

$$\epsilon_a \rho_{ia}(U_k) = \bar{g}(A_{N_k} W_a, N_i), \qquad \epsilon_a \rho_{ia}(W_b) = \epsilon_b \rho_{ib}(W_a),$$

$$h_a^s(Y, U_i) = \sum_{j=1}^{r} u_j(Y) h_a^s(U_j, U_i) + \sum_{a=r+1}^{n} w_a(Y) h_a^s(W_a, U_i).$$

Replacing Y by FX to the last equation, we have $h_a^s(FX, U_i) = 0$. Thus

$$\rho_{ia}(X) = \sum_{k=1}^{r} \rho_{ia}(U_k) u_k(X) + \sum_{b=r+1}^{n} \rho_{ia}(W_b) w_b(X).$$

(4) Replacing Y by U_j to (3.3) and using $(4.23)_1$, we obtain

$$h_i^\ell(U_j, X) = \theta(X)\delta_{ij}, \qquad h_i^\ell(U_j, \xi_k) = 0. \tag{4.25}$$

Taking $X = U_j$ to (3.7) and using $(4.25)_{1,2}$, we get $g(A^*_{\xi_i} U_j, X) = -\alpha_i \eta_j(X)$. Replacing X by ξ_k to this equation, we have

$$\alpha_i = 0, \qquad \forall\, i. \tag{4.26}$$

Thus $g(A^*_{\xi_i} U_j, X) = 0$. As $S(TM)$ is non-degenerate, we get $A^*_{\xi_i} U_j = 0$.

Taking $X = \xi_i$ to (4.16) and using $(3.10)_{2,6}$, $(3.12)_1$, $(4.18)_{3,4}$ and the fact that $\tau_{ij}(\xi_k) = \tau_{ij}(V_j) = 0$, we get $A^*_{\xi_i} V_j = -F(A^*_{\xi_i} \xi_j)$. □

5. Generic submanifolds of an indefinite complex space form

Definition. An indefinite complex space form $\bar{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$\begin{aligned}\widetilde{R}(\bar{X}, \bar{Y})\bar{Z} = {} & \frac{c}{4}\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} \\ & + \bar{g}(J\bar{Y}, \bar{Z})J\bar{X} - \bar{g}(J\bar{X}, \bar{Z})J\bar{Y} + 2\bar{g}(\bar{X}, J\bar{Y})J\bar{Z}\},\end{aligned} \tag{5.1}$$

where $\widetilde{R}$ denote the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$.

Let $\bar{R}$ be the curvature tensor of the non-metric ϕ-symmetric connection $\bar{\nabla}$ on $\bar{M}$. By directed calculations from (1.3) and (1.5), we see that

$$\bar{R}(\bar{X},\bar{Y})\bar{Z} = \widetilde{R}(\bar{X},\bar{Y})\bar{Z} + (\bar{\nabla}_{\bar{X}}\theta)(\bar{Z})J\bar{Y} - (\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z})J\bar{X}. \tag{5.2}$$

Denote by R and R^* the curvature tensors of the induced connections ∇ and ∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulae, we have the Gauss equations for M and $S(TM)$ such that

$$\begin{aligned}
\bar{R}(X,Y)Z = R(X,Y)Z &+ \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} \\
&+ \sum_{a=r+1}^{n}\{h_a^s(X,Z)A_{E_a}Y - h_a^s(Y,Z)A_{E_a}X\} \\
&+ \sum_{i=1}^{r}\{(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\
&\quad + \sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)] \\
&\quad + \sum_{a=r+1}^{n}[\lambda_{ai}(X)h_a^s(Y,Z) - \lambda_{ai}(Y)h_a^s(X,Z)] \\
&\quad - \theta(X)h_i^\ell(FY,Z) + \theta(Y)h_i^\ell(FX,Z)\}N_i \\
&\quad + \sum_{a=r+1}^{n}\{(\nabla_X h_a^s)(Y,Z) - (\nabla_Y h_a^s)(X,Z) \\
&\quad + \sum_{i=1}^{r}[\rho_{ia}(X)h_i^\ell(Y,Z) - \rho_{ia}(Y)h_i^\ell(X,Z)] \\
&\quad + \sum_{b=r+1}^{n}[\mu_{ba}(X)h_b^s(Y,Z) - \mu_{ba}(Y)h_b^s(X,Z)] \\
&\quad - \theta(X)h_a^s(FY,Z) + \theta(Y)h_a^s(FX,Z)\}E_a,
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
R(X,Y)PZ = R^*(X,Y)PZ &+ \sum_{i=1}^{r}\{h_i^*(X,PZ)A^*_{\xi_i}Y - h_i^*(Y,PZ)A_{\xi_i}X\} \\
&+ \sum_{i=1}^{r}\{(\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ) \\
&\quad + \sum_{k=1}^{r}[\sigma_{ik}(Y)h_k^*(X,PZ) - \sigma_{ik}(X)h_k^*(Y,PZ)] \\
&\quad - \theta(X)h_i^*(FY,PZ) + \theta(Y)h_i^*(FX,PZ)\}\xi_i.
\end{aligned} \tag{5.4}$$

Comparing the tangential components of (5.2) and (5.3), we obtain

$$
\begin{aligned}
R(X,Y)Z = {} & \sum_{i=1}^{r}\{h_i^\ell(Y,Z)A_{N_i}X - h_i^\ell(X,Z)A_{N_i}Y\} \\
& + \sum_{a=r+1}^{n}\{h_a^s(Y,Z)A_{E_a}X - h_a^s(X,Z)A_{E_a}Y\} \\
& + (\bar\nabla_X\theta)(Z)FY - (\bar\nabla_Y\theta)(Z)FX \\
& + \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y \\
& + \bar g(JY,Z)FX - \bar g(JX,Z)FY + 2\bar g(X,JY)FZ\},
\end{aligned} \tag{5.5}
$$

due to (5.1). Taking the scalar product with N_i to (5.4) and then, substituting (5.5) into the resulting equation, we have

$$
\begin{aligned}
& (\nabla_X h_i^*)(Y,\, PZ) - (\nabla_Y h_i^*)(X,\, PZ) \\
& + \sum_{j=1}^{r}\{\sigma_{ij}(Y)h_j^*(X,PZ) - \sigma_{ij}(X)h_j^*(Y,PZ)\} \\
& + \sum_{j=1}^{r}\{h_j^\ell(X,PZ)\eta_i(A_{N_j}Y) - h_j^\ell(Y,PZ)\eta_i(A_{N_j}X)\} \\
& + \sum_{a=r+1}^{n}\{h_a^s(X,PZ)\eta_i(A_{E_a}Y) - h_a^s(Y,PZ)\eta_i(A_{E_a}X)\} \\
& - \theta(X)h_i^*(FY,PZ) + \theta(Y)h_i^*(FX,PZ) \\
& - (\bar\nabla_X\theta)(PZ)v_i(Y) + (\bar\nabla_Y\theta)(PZ)v_i(X) \\
= {} & \frac{c}{4}\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y) \\
& + v_i(X)\bar g(JY,PZ) - v_i(Y)\bar g(JX,PZ) + 2v_i(PZ)\bar g(X,JY)\}.
\end{aligned} \tag{5.6}
$$

Theorem 5.1. *Let M be a generic lightlike submanifold of an indefinite complex space form $\bar M(c)$ with a non-metric ϕ-symmetric connection $\bar\nabla$. If one of the following four statements*

(1) *M is recurrent,*

(2) *U_is are parallel with respect to the induced connection ∇, or*

(3) *V_is are parallel with respect to the induced connection ∇*

is satisfied, then $\bar M(c)$ is flat, i.e., $c = 0$.

Proof. (1) Applying $\bar\nabla_X$ to $(4.7)_1$ and using (2.3), $(4.8)_4$ and $(4.9)_1$, we get

$$(\bar\nabla_X\theta)(U_i) = -\sum_{k=1}^{r}\beta_k h_k^\ell(X,U_i). \tag{5.7}$$

Applying ∇_X to $(4.7)_3$: $h_i^*(Y,U_j) = 0$ and using (4.10), we obtain

$$(\nabla_X h_i^*)(Y,U_j) = 0.$$

Taking $PZ = U_j$ to (5.6) and using $(4.7)_2$, $(4.8)_4$ and (5.7), we get

$$\frac{c}{4}\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y) + v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we get $c = 0$.

(2) As $\nabla_X U_i = 0$, taking the scalar product with U_j to (3.14), we get

$$\eta_j(A_{N_i}X) + \beta_i v_j(X) + \theta(U_i)\eta_j(X) = 0.$$

Taking the skew-symmetric part with respect to i and j and using $(3.10)_3$, we have $\theta(U_i)\eta_j(X) + \theta(U_j)\eta_i(X) = 0$. Taking $X = \xi_j$ to this result, we obtain

$$\theta(U_i) = 0, \qquad \eta_j(A_{N_i}X) = -\beta_i v_j(X). \tag{5.8}$$

Applying $\bar{\nabla}_X$ to $\theta(U_i) = 0$ and using (2.3) and the fact that $\nabla_X U_i = 0$, we get

$$(\bar{\nabla}_X\theta)(U_i) = -\sum_{k=1}^{r}\beta_k h_k^\ell(X, U_i) - \sum_{a=r+1}^{n}\gamma_a h_a^s(X, U_i). \tag{5.9}$$

Taking the scalar product with W_a and N_j to (3.14) and using $(5.8)_1$, we have

$$\rho_{ia}(X) = \beta_i w_a(X), \qquad h_i^*(X, U_j) = 0. \tag{5.10}$$

From $(3.10)_4$ and $(5.10)_1$, we see that

$$\eta_i(A_{E_a}X) = -\gamma_a v_i(X). \tag{5.11}$$

Applying ∇_Y to $(5.10)_2$ and using the fact that $\nabla_X U_j = 0$, we obtain

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Replacing PZ by U_j to (5.6) and using $(5.8)_2$, (5.9), $(5.10)_2$, (5.11) and the last equation, we have

$$\frac{c}{4}\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y) + v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we have $c = 0$.

(3) As $\nabla_X V_i = 0$, taking the scalar product with V_j, W_a and N_j to (3.15) by turns and using (3.7) and $(3.13)_1$, we get

$$h_j^\ell(X, \xi_i) = \alpha_i u_j(X), \qquad h_a^s(X, \xi_i) = \alpha_i w_a(X), \qquad h_i^*(Y, V_j) = 0. \tag{5.12}$$

Taking $Y = \xi_i$ to (3.3) and (3.4) by turns and using $(5.12)_{1,2}$, we get

$$h_j^\ell(\xi_i, X) = 0, \qquad h_a^s(\xi_i, X) = 0. \tag{5.13}$$

Applying ∇_X to $(5.12)_3$ and using the fact that $\nabla_X V_j = 0$, we have

$$(\nabla_X h_i^*)(Y, V_j) = 0.$$

Taking $PZ = V_j$ to (5.6) and using $(5.12)_3$ and the last equation, we get

$$\sum_{j=1}^{r}\{h_k^\ell(X, V_j)\eta_i(A_{N_k}Y) - h_k^\ell(Y, V_j)\eta_i(A_{N_k}X)\}$$

$$
\begin{aligned}
&+\sum_{a=r+1}^{n}\{h_a^s(X,V_j)\eta_i(A_{E_a}Y)-h_a^s(Y,V_j)\eta_i(A_{E_a}X)\}\\
&-(\bar{\nabla}_X\theta)(V_j)v_i(Y)+(\bar{\nabla}_Y\theta)(V_j)v_i(X)\\
=&\ \frac{c}{4}\{u_j(Y)\eta_i(X)-u_j(X)\eta_i(Y)+2\delta_{ij}\bar{g}(X,JY)\}.
\end{aligned}
$$

Taking $X=\xi_i$ and $Y=U_j$ to this equation and using (5.13), we have

$$
-\sum_{k=1}^{r}h_k^\ell(U_j,V_j)\eta_i(A_{N_k}\xi_i)-\sum_{a=r+1}^{n}h_a^s(U_j,V_j)\eta_i(A_{E_a}\xi_i)=\frac{3}{4}c.
$$

By using (3.3), (3.4), $(3.10)_4$, $(3.13)_{1,4}$, and $(5.12)_3$, we see that

$$
\begin{aligned}
h_k^\ell(U_j,V_j)&=h_k^\ell(V_j,U_j)+\theta(V_k)=h_j^*(V_j,V_k)=0,\\
h_a^s(U_j,V_j)&=\epsilon_a\{h_j^\ell(U_j,W_a)-\theta(W_a)\}\\
&=\ \epsilon_a h_j^\ell(W_a,U_j)=\epsilon_a h_j^*(W_a,V_j)=0.
\end{aligned}
$$

From the last three equations, we see that $c=0$. □

Theorem 5.2. *Let M be a Lie recurrent generic lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with a non-metric ϕ-symmetric connection $\bar{\nabla}$. If each A_{N_i} is $S(TM)$-valued, then $\bar{M}(c)$ is flat, i.e., $c=0$.*

Proof. In general, using the Gauss-Weingarten formulae (2.6) and (2.7) for the screen distribution $S(TM)$, we have the Codazzi equation for $S(TM)$:

$$
\begin{aligned}
R(X,Y)\xi_i=&-\nabla_X^*(A_{\xi_i}^*Y)+\nabla_Y^*(A_{\xi_i}^*X)+A_{\xi_i}^*[X,\,Y]\\
&+\sum_{j=1}^{r}\{\sigma_{ji}(Y)A_{\xi_j}^*X-\sigma_{ji}(X)A_{\xi_j}^*Y\}\\
&+\sum_{j=1}^{r}\{h_j^*(Y,A_{\xi_i}^*X)-h_j^*(X,A_{\xi_i}^*Y)-2d\sigma_{ji}(X,Y)\\
&+\sum_{k=1}^{r}[\sigma_{jk}(X)\sigma_{ki}(Y)-\sigma_{jk}(Y)\sigma_{ki}(X)]\}\xi_j.
\end{aligned}
\tag{5.14}
$$

Assume that each A_{N_i} is $S(TM)$-valued. As $\eta_k(A_{N_i}X)=0$, from the first equation of (2) in Theorem 4.2 we have $\tau_{ij}=0$. From $(3.10)_3$, we obtain

$$
\beta_i v_j(X)+\beta_j v_i(X)=0.
$$

Taking $X=V_j$ to this equation, we obtain $\beta_i=0$. As $\tau_{ij}=\alpha_i=\beta_i=0$, from $(3.10)_6$, we see that $\sigma_{ij}=0$. Summing up these results

$$
\alpha_i=0,\qquad \beta_i=0,\qquad \tau_{ij}=0,\qquad \sigma_{ij}=0. \tag{5.15}
$$

Applying $\bar{\nabla}_X$ to $\theta(\xi_i) = 0$ and using (2.3), (3.7), $(3.10)_2$ and (5.15), we get

$$(\bar{\nabla}_X \theta)(\xi_i) = \theta(A^*_{\xi_i} X) + \sum_{a=r+1}^{n} \epsilon_a \gamma_a \{\lambda_{ai}(X) - \gamma_a u_i(X)\}. \tag{5.16}$$

Taking the scaler product with N_j to (5.5) with $Z = \xi_i$ and then, comparing this result with the radical component of (5.14) and using (5.16), we obtain

$$\begin{aligned} h^*_j(Y, A^*_{\xi_i} X) - h^*_j(X, A^*_{\xi_i} Y) &= \frac{c}{4}\{u_i(Y)v_j(X) - u_i(X)v_j(Y)\} \\ &\quad - \sum_{a=r+1}^{n} \{\lambda_{ai}(X) - \gamma_a u_i(X)\}\rho_{ja}(Y) \\ &\quad + \sum_{a=r+1}^{n} \{\lambda_{ai}(Y) - \gamma_a u_i(Y)\}\rho_{ja}(X) \\ &\quad + \theta(A^*_{\xi_i} X)v_j(Y) - \theta(A^*_{\xi_i} Y)v_j(X). \end{aligned}$$

Taking $X = U_i$ and $Y = V_j$ and using $(4.14)_1$ and $(4.18)_{2,3}$, we obtain

$$h^*_j(U_i, A^*_{\xi_i} V_j) = \frac{c}{4}. \tag{5.17}$$

Replacing X by ξ_j to (3.7) and using $(3.12)_1$, we have

$$h^\ell_i(\xi_j, X) = g(A^*_{\xi_i}\xi_j, X).$$

Taking $Y = \xi_j$ to (3.3), we obtain $h^\ell_i(X, \xi_j) = h^\ell_i(\xi_j, X)$ since $\alpha_i = 0$. Due to $(3.10)_1$, $h^\ell_i(X, \xi_j)$ are skew-symmetric. Thus $h^\ell_i(\xi_j, X)$ are also skew-symmetric with respect to i and j. It follow that $A^*_{\xi_i}\xi_j = -A^*_{\xi_j}\xi_i$, *i.e.*, $A^*_{\xi_i}\xi_j$ are skew-symmetric with respect to i and j. From this result and $(4.14)_2$, we see that $A^*_{\xi_i} V_j$ are skew-symmetric with respect to i and j. On the other hand, taking $Y = U_j$ to (4.18), we have $A_{N_i} U_j = A_{N_j} U_i$. Thus $A_{N_i} U_j$ are symmetric with respect to i and j. Therefore, we obtain

$$h^*_j(U_i, A^*_{\xi_i} V_j) = g(A_{N_j} U_i, A^*_{\xi_i} V_j) = 0. \tag{5.18}$$

From (5.17) and (5.18), we have $c = 0$. Thus we have our theorem. □

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