

## CERTAIN CURVATURE CONDITIONS IN KENMOTSU MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION

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ABSTRACT. The conharmonic curvature tensor under certain conditions has been studied for Kenmotsu manifolds with respect to the semi-symmetric metric connection.

### 1. Introduction

In 1969, S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension [17]. For such a manifold, the sectional curvature of plane sections containing  $\xi$  is a constant, say  $c$ . He showed that they can be divided into three classes: (1) homogeneous normal contact Riemannian manifolds with  $c > 0$ , (2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if  $c = 0$  and (3) a warped product space  $R \times_f C$  if  $c > 0$ . It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. K. Kenmotsu [13] characterized the differential geometric properties of the manifolds of class (3); the structure obtained in this way is now known as Kenmotsu structure. In general, these structures are not Sasakian [13]. Kenmotsu manifolds have been studied by various authors such as N. Asghari and A. Taleshian [3], A. Barman and U. C. De [5], U. C. De and G. Pathak [7], A. Yildiz and U. C. De [20] and many others.

In 1924, the idea of semi-symmetric linear connection on a differentiable manifold was introduced by A. Friedmann and J. A. Schouten [8]. Then in 1932, H. A. Hayden [10] introduced semi-symmetric metric connection in Riemannian manifolds and this was studied systematically by K. Yano [18].

The paper is organized as follows: Section 2 is equipped with some prerequisites about Kenmotsu manifolds. In Section 3, we give the relation between the curvature tensor of Kenmotsu manifolds and related results with respect to the semi-symmetric metric connection and the Levi-Civita connection. In Section

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4, we showed that a conharmonically flat  $\eta$ -Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection is of quasi-constant curvature. Section 5 deals with the study of  $\phi$ -conharmonically semi-symmetric  $\eta$ -Einstein Kenmotsu manifolds with respect to the semi-symmetric metric connection. In Section 6, we study conharmonically flat Kenmotsu manifolds satisfying the curvature condition  $\bar{R}(X, Y) \cdot \bar{S} = 0$ . In the last Section 7, we obtain the non-existence of Kenmotsu manifolds whose curvature tensor of manifold is covariantly constant with respect to the semi-symmetric metric connection and the manifold is recurrent with respect to the Levi-Civita connection.

## 2. Kenmotsu manifolds

A smooth manifold  $(M^n, g)$  ( $n = 2m + 1$ ) is said to be an almost contact metric manifold [13] if it admits a  $(1, 1)$ -tensor field  $\phi$ , a vector field  $\xi$ , a 1-form  $\eta$  and a Riemannian metric  $g$  which satisfy

$$(1) \quad \phi^2 X = -X + \eta(X)\xi,$$

$$(2) \quad g(X, \xi) = \eta(X), \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$

$$(3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields  $X, Y$  on  $M$ .

An almost contact metric manifold  $M^n(\phi, \xi, \eta, g)$  is said to be a Kenmotsu manifold if the following conditions hold:

$$(4) \quad (\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X,$$

$$(5) \quad \nabla_X \xi = X - \eta(X)\xi,$$

where  $\nabla$  is the Levi-Civita connection.

In a Kenmotsu manifold the following relations hold [12, 13, 15]:

$$(6) \quad (\nabla_X \eta)Y = g(X, Y)\xi - \eta(X)\eta(Y),$$

$$(7) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$(8) \quad R(\xi, X)Y = -R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi,$$

$$(9) \quad \eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X),$$

$$(10) \quad S(X, \xi) = -(n-1)\eta(X), \quad Q\xi = -(n-1)\xi$$

for arbitrary vector fields  $X, Y, Z$  on  $M$  and  $R$  is the Riemannian curvature tensor and  $S$  is the Ricci tensor of type  $(0, 2)$  such that  $g(QX, Y) = S(X, Y)$ .

A linear connection  $\bar{\nabla}$  in a Riemannian manifold  $M$  is said to be a semi-symmetric connection [9, 14] if its torsion tensor  $T$  of the connection  $\bar{\nabla}$

$$(11) \quad T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]$$

satisfies

$$(12) \quad T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where  $\eta$  is a 1-form and  $\xi$  is a vector field given by

$$(13) \quad g(X, \xi) = \eta(X)$$

for all vector fields  $X, Y \in \chi(M)$ . Here  $\chi(M)$  is the set of all differentiable vector fields on  $M$ .

A semi-symmetric connection  $\bar{\nabla}$  is called a semi-symmetric metric connection [6, 8] if it further satisfies

$$(14) \quad \bar{\nabla}g = 0.$$

A relation between the semi-symmetric metric connection  $\bar{\nabla}$  and the Levi-Civita connection  $\nabla$  on  $M$  has been obtained by K. Yano [18], which is given by

$$(15) \quad \bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi,$$

where  $\eta(Y) = g(Y, \xi)$ .

### 3. Curvature tensor of a Kenmotsu manifold with respect to the semi-symmetric metric connection

Let  $R$  and  $\bar{R}$ , respectively, be the curvature tensors of the Levi-Civita connection  $\nabla$  and the semi-symmetric metric connection  $\bar{\nabla}$  in a Kenmotsu manifold  $M$ . Then we have [4]

$$(16) \quad \begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z - 3g(Y, Z)X + 3g(X, Z)Y + 2\eta(Y)\eta(Z)X \\ &\quad - 2\eta(X)\eta(Z)Y + 2g(Y, Z)\eta(X)\xi - 2g(X, Z)\eta(Y)\xi, \end{aligned}$$

$$(17) \quad \eta(\bar{R}(X, Y)Z) = 2g(X, Z)\eta(Y) - 2g(Y, Z)\eta(X),$$

$$(18) \quad \bar{R}(X, Y)\xi = 2\eta(X)Y - 2\eta(Y)X,$$

$$(19) \quad \bar{R}(X, \xi)Y = 2g(X, Y)\xi - 2\eta(Y)X,$$

$$(20) \quad \bar{R}(\xi, X)\xi = 2X - 2\eta(X)\xi,$$

$$(21) \quad \bar{S}(Y, Z) = S(Y, Z) - (3n - 5)g(Y, Z) + 2(n - 2)\eta(Y)\eta(Z),$$

$$(22) \quad \bar{S}(\phi Y, \phi Z) = \bar{S}(Y, Z) + 2(n - 1)\eta(Y)\eta(Z),$$

$$(23) \quad \bar{S}(Y, \xi) = -2(n - 1)\eta(Y), \quad \bar{S}(\xi, \xi) = -2(n - 1),$$

$$(24) \quad \bar{Q}Y = QY - (3n - 5)Y + 2(n - 2)\eta(Y)\xi,$$

$$(25) \quad \bar{Q}\xi = -2(n - 1)\xi,$$

$$(26) \quad \bar{r} = r - 3n^2 + 7n - 4,$$

$$(27) \quad \bar{\nabla}_X \xi = 2X - 2\eta(X)\xi$$

for any vector fields  $X, Y, Z$  on  $M$ .

**Definition.** A Kenmotsu manifold  $M$  is called a manifold of quasi-constant curvature with respect to the semi-symmetric metric connection if the curvature tensor  $\bar{R}$  of type  $(0, 4)$  satisfies the condition

$$(28) \quad \begin{aligned} \bar{R}(X, Y, Z, W) = & a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & + b[g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W) \\ & + g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)], \end{aligned}$$

where  $\bar{R}(X, Y, Z, W) = g(\bar{R}(X, Y)Z, W)$ ,  $\bar{R}$  is the curvature tensor of type  $(1, 3)$ ;  $a, b$  are the scalar functions and  $\rho$  is a unit vector field defined by

$$(29) \quad g(X, \rho) = T(X)$$

for any vector fields  $X, Y, Z, W$  on  $M$ .

**Definition.** A Kenmotsu manifold  $M$  is said to be an  $\eta$ -Einstein manifold with respect to the semi-symmetric metric connection if its Ricci tensor  $\bar{S}$  of type  $(0, 2)$  satisfies

$$(30) \quad \bar{S}(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y),$$

where  $\alpha$  and  $\beta$  are smooth functions on  $M$ . In particular, if  $\beta = 0$ , then an  $\eta$ -Einstein manifold is an Einstein manifold.

Contracting (30), we have

$$(31) \quad \bar{r} = n\alpha + \beta.$$

On the other hand, putting  $X = Y = \xi$  in (30) and using (23), we also have

$$(32) \quad -2(n-1) = \alpha + \beta.$$

Hence it follows from (31) and (32) that

$$\alpha = 2 + \frac{\bar{r}}{n-1}, \quad \beta = -2n - \frac{\bar{r}}{n-1}.$$

So the Ricci tensor  $\bar{S}$  of an  $\eta$ -Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection is given by

$$(33) \quad \bar{S}(X, Y) = \left(2 + \frac{\bar{r}}{n-1}\right)g(X, Y) - \left(2n + \frac{\bar{r}}{n-1}\right)\eta(X)\eta(Y),$$

from which we have

$$(34) \quad \bar{Q}X = \left(2 + \frac{\bar{r}}{n-1}\right)X - \left(2n + \frac{\bar{r}}{n-1}\right)\eta(X)\xi.$$

Now, we give an example of Kenmotsu manifold which is an  $\eta$ -Einstein manifold with respect to the semi-symmetric metric connection.

**Example.** We consider a 7-dimensional Kenmotsu manifold with respect to the semi-symmetric metric connection  $M^7 = \{(x_1, x_2, x_3, y_1, y_2, y_3, z) \in \mathbb{R}^7 : z > 0\}$ , where  $(x_1, x_2, x_3, y_1, y_2, y_3, z)$  are the standard coordinates in  $\mathbb{R}^7$ . We choose the vector fields

$$e_1 = e^{-z} \frac{\partial}{\partial x_1}, \quad e_2 = e^{-z} \frac{\partial}{\partial x_2}, \quad e_3 = e^{-z} \frac{\partial}{\partial x_3}, \quad e_4 = e^{-z} \frac{\partial}{\partial y_1},$$

$$e_5 = e^{-z} \frac{\partial}{\partial y_2}, \quad e_6 = e^{-z} \frac{\partial}{\partial y_3}, \quad e_7 = \frac{\partial}{\partial z}$$

which are linearly independent at each point of  $M^7$ . Let  $g$  be the Riemannian metric on  $M^7$  defined by

$$g = \sum_{i=1}^3 e^{2z} (dx_i \otimes dx_i + dy_i \otimes dy_i) + dz \otimes dz.$$

Here it is clear that  $g(e_i, e_i) = 1$  and  $g(e_i, e_j) = 0$  for all  $i \neq j$ , where  $i, j = 1, 2, 3, 4, 5, 6, 7$ . Hence  $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$  is an orthonormal base field on  $M^7$ .

Let  $X = \sum_{i=1}^3 (X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + Z \frac{\partial}{\partial z}$  be a vector field on  $M^7$ . We define a  $(1, 1)$  tensor field  $\phi$  and 1-form  $\eta$  as

$$\phi \left( \sum_{i=1}^3 (X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + Z \frac{\partial}{\partial z} \right) = \sum_{i=1}^3 (Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i})$$

and  $\eta(X) = g(X, e_7) = dz(X)$ . Thus we have

$$\phi(e_1) = e_4, \quad \phi(e_2) = e_5, \quad \phi(e_3) = e_6, \quad \phi(e_4) = -e_1,$$

$$\phi(e_5) = -e_2, \quad \phi(e_6) = -e_3, \quad \phi(e_7) = 0.$$

The linearity property of  $\phi$  and  $g$  yields that

$$\eta(e_7) = g(e_7, e_7) = 1, \quad \phi^2 X = -X + \eta(X)e_7,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields  $X, Y$  on  $M^7$ . Let us denote  $e_7$  by  $\xi$ , then  $M^7(\phi, \xi, \eta, g)$  defines an almost contact metric manifold.

The Koszul's formula for the Riemannian connection  $\nabla$  of the Riemannian metric  $g$  is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$

$$+ g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y)$$

for any vector fields  $X, Y, Z$  on  $M^7$ .

Using the Koszul's formula, for any vector fields  $X, Y, Z$  on  $M^7$ , we have

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$$

which proves that  $M^7(\phi, \xi, \eta, g)$  is a Kenmotsu manifold. Moreover, we have

$$[e_i, \xi] = e_i, \quad [e_i, e_j] = 0, \quad i \neq j, \quad i, j = 1, 2, 3, 4, 5, 6.$$

Using the Koszul's formula, we also obtain

$$(35) \quad \begin{aligned} \nabla_{e_i} e_i &= -\xi, \quad \nabla_{e_i} e_j = 0 \text{ for } i \neq j, \\ \nabla_{e_i} \xi &= e_i, \quad \nabla_{\xi} e_i = 0, \quad i, j = 1, 2, 3, 4, 5, 6. \end{aligned}$$

Therefore the semi-symmetric metric connection  $\bar{\nabla}$  on  $M^7$  is given by

$$(36) \quad \begin{aligned} \bar{\nabla}_{e_i} e_i &= -2\xi, \quad \bar{\nabla}_{e_i} e_j = 0 \text{ for } i \neq j, \\ \bar{\nabla}_{e_i} \xi &= 2e_i, \quad \bar{\nabla}_{\xi} e_i = 0, \quad i, j = 1, 2, 3, 4, 5, 6. \end{aligned}$$

With help of results given in (35), it can be easily verified that

$$(37) \quad \begin{aligned} R(e_i, e_j)e_k &= 0 \text{ for } i \neq j \neq k, \quad R(e_i, e_j)e_i = e_j \text{ for } i \neq j, \\ R(e_i, e_j)e_j &= -e_i \text{ for } i \neq j, \quad R(e_i, e_i)e_j = 0 \text{ for } i \neq j, \\ R(e_i, \xi)e_j &= 0 \text{ for } i \neq j, \quad R(\xi, e_j)\xi = e_j, \quad R(e_i, \xi)e_i = \xi, \quad R(\xi, e_i)e_i = -\xi, \\ R(e_i, \xi)\xi &= -e_i, \quad R(e_i, e_j)\xi = 0 \text{ for } i \neq j, \quad i, j, k = 1, 2, 3, 4, 5, 6. \end{aligned}$$

From above results in (37) and the equations (16), (18)-(20), we have

$$(38) \quad \begin{aligned} \bar{R}(e_i, e_j)e_k &= 0 \text{ for } i \neq j \neq k, \quad \bar{R}(e_i, e_j)e_i = 4e_j \text{ for } i \neq j, \\ \bar{R}(e_i, e_j)e_j &= -4e_i \text{ for } i \neq j, \quad \bar{R}(e_i, e_i)e_j = 0 \text{ for } i \neq j, \\ \bar{R}(e_i, \xi)e_j &= 0 \text{ for } i \neq j, \quad \bar{R}(\xi, e_i)\xi = 2e_i, \quad \bar{R}(e_i, \xi)e_i = 2\xi, \\ \bar{R}(\xi, e_i)e_i &= -2\xi, \quad \bar{R}(e_i, \xi)\xi = -2e_i, \\ \bar{R}(e_i, e_j)\xi &= 0 \text{ for } i \neq j, \quad i, j, k = 1, 2, 3, 4, 5, 6. \end{aligned}$$

We know that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Let

$$X = \sum_{i=1}^7 X_i e_i, \quad Y = \sum_{i=1}^7 Y_i e_i, \quad \text{and} \quad Z = \sum_{i=1}^7 Z_i e_i, \quad \text{where } e_7 = \xi.$$

Then using (37), we get

$$(39) \quad R(X, Y)Z = -[g(Y, Z)X - g(X, Z)Y].$$

So,  $M^7$  is a manifold of constant curvature  $-1$ . Contracting  $X$  in (39), we get

$$(40) \quad S(Y, Z) = -(n - 1)g(Y, Z).$$

For the manifold  $M^7$ , we have

$$(41) \quad S(Y, Z) = -6g(Y, Z)$$

which yields

$$(42) \quad QY = -6Y.$$

For the manifold  $M^7$  with respect to the semi-symmetric metric connection, we also have

$$(43) \quad \begin{aligned} \bar{R}(X, Y)Z &= -4[g(Y, Z)X - g(X, Z)Y] + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y \\ &\quad + 2g(Y, Z)\eta(X)\xi - 2g(X, Z)\eta(Y)\xi \end{aligned}$$

and

$$(44) \quad \bar{S}(Y, Z) = -22g(Y, Z) + 10\eta(Y)\eta(Z).$$

Thus the manifold is an  $\eta$ -Einstein manifold with respect to the semi-symmetric metric connection.

**4. Conharmonically flat  $\eta$ -Einstein Kenmotsu manifolds with respect to the semi-symmetric metric connection**

**Definition.** The conharmonic curvature tensor  $C$  of type  $(1, 3)$  in a Riemannian manifold  $M$  of dimension  $n$  is defined by [2, 11]

$$(45) \quad C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]$$

for any vector fields  $X, Y, Z \in \chi(M)$ . In the differential geometry of certain  $F$ -structures (for example, complex, almost complex, Kahler, almost Kahler, Hermitian, almost Hermitian structures, etc.), the importance of conharmonic curvature tensor is very well known [19]. While the relativistic significance of this tensor has been explored by Z. Ahsan [1] and S. A. Siddiqui and Z. Ahsan [16].

Analogous to the above definition, we define the conharmonic curvature tensor  $\bar{C}$  in a Kenmotsu manifold  $M$  with respect to the semi-symmetric metric connection  $\bar{\nabla}$  by

$$(46) \quad \bar{C}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{(n-2)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y],$$

where  $\bar{R}, \bar{S}$  are the curvature tensor, the Ricci tensor respectively on  $M$  with respect to the semi-symmetric metric connection and  $\bar{Q}$  is the Ricci operator with respect to the semi-symmetric metric connection and is related by  $g(\bar{Q}X, Y) = \bar{S}(X, Y)$ . If  $\bar{C}$  vanishes identically on  $M$ , then we say that the manifold is conharmonically flat.

Now, we consider that the manifold  $M$  with respect to the semi-symmetric metric connection is conharmonically flat, that is,  $\bar{C} = 0$ . Then from (46), we have

$$(47) \quad \bar{R}(X, Y)Z = \frac{1}{(n-2)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y].$$

Taking inner product of (47) with  $W$ , we have

$$(48) \quad g(\bar{R}(X, Y)Z, W) = \frac{1}{(n-2)}[\bar{S}(Y, Z)g(X, W) - \bar{S}(X, Z)g(Y, W) + g(Y, Z)\bar{S}(X, W) - g(X, Z)\bar{S}(Y, W)].$$

By using (33), (48) becomes

$$\begin{aligned}
 g(\bar{R}(X, Y)Z, W) &= \frac{1}{(n-2)} \left[ \left(4 + \frac{2\bar{r}}{n-1}\right) g(Y, Z)g(X, W) \right. \\
 &\quad - \left(2n + \frac{\bar{r}}{n-1}\right) \eta(Y)\eta(Z)g(X, W) \\
 &\quad - \left(4 + \frac{2\bar{r}}{n-1}\right) g(X, Z)g(Y, W) \\
 &\quad + \left(2n + \frac{\bar{r}}{n-1}\right) \eta(X)\eta(Z)g(Y, W) \\
 &\quad - \left(2n + \frac{\bar{r}}{n-1}\right) \eta(X)\eta(W)g(Y, Z) \\
 &\quad \left. + \left(2n + \frac{\bar{r}}{n-1}\right) \eta(Y)\eta(W)g(X, Z) \right]
 \end{aligned}
 \tag{49}$$

or,

$$\begin{aligned}
 g(\bar{R}(X, Y)Z, W) &= \frac{1}{(n-2)} \left(4 + \frac{2\bar{r}}{n-1}\right) [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
 &\quad - \frac{1}{(n-2)} \left(2n + \frac{\bar{r}}{n-1}\right) [g(X, W)\eta(Y)\eta(Z) \\
 &\quad - g(X, Z)\eta(Y)\eta(W) + g(Y, Z)\eta(X)\eta(W) \\
 &\quad - g(Y, W)\eta(X)\eta(Z)].
 \end{aligned}
 \tag{50}$$

Thus we can state the following theorem:

**Theorem 4.1.** *An  $n$ -dimensional conharmonically flat  $\eta$ -Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection is of quasi-constant curvature.*

Next, putting  $X = \xi$  in (48) and using (2) and (33), we have

$$g(\bar{R}(\xi, Y)Z, W) = \frac{1}{(n-2)} \left(4 + \frac{\bar{r}}{n-1} - 2n\right) [g(Y, Z)\eta(W) - g(Y, W)\eta(Z)]
 \tag{51}$$

which in view of (19) reduces to

$$\frac{\bar{r}}{n-1} [g(Y, Z)\eta(W) - g(Y, W)\eta(Z)] = 0.
 \tag{52}$$

This shows that either  $\bar{r} = 0$  or,

$$\begin{aligned}
 &g(Y, Z)\eta(W) = g(Y, W)\eta(Z) \\
 \implies &g(Y, Z)\xi = \eta(Z)Y \quad \text{for all } Y, Z \in \chi(M)
 \end{aligned}
 \tag{53}$$

which is not possible. Thus we can state the following theorem:

**Theorem 4.2.** *In an  $n$ -dimensional conharmonically flat  $\eta$ -Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection the scalar curvature vanishes.*



**5.  $\phi$ -conharmonically semisymmetric  $\eta$ -Einstein Kenmotsu manifolds with respect to the semi-symmetric metric connection**

**Definition.** An  $\eta$ -Einstein Kenmotsu manifold  $(M^n, g)$ ,  $n > 1$  is said to be  $\phi$ -conharmonically semisymmetric with respect to the semi-symmetric metric connection if

$$(54) \quad \bar{C}(X, Y) \cdot \phi = 0$$

for all  $X, Y \in \chi(M)$ .

Now, let  $M$  be an  $n$ -dimensional  $\phi$ -conharmonically semisymmetric  $\eta$ -Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection. Therefore  $\bar{C}(X, Y) \cdot \phi = 0$  turns into

$$(55) \quad (\bar{C}(X, Y) \cdot \phi)Z = \bar{C}(X, Y)\phi Z - \phi\bar{C}(X, Y)Z = 0$$

for any vector fields  $X, Y, Z$  on  $M$ . From (46), we have

$$(56) \quad \begin{aligned} \bar{C}(X, Y)\phi Z &= \bar{R}(X, Y)\phi Z - \frac{1}{(n-2)}[\bar{S}(Y, \phi Z)X - \bar{S}(X, \phi Z)Y \\ &\quad + g(Y, \phi Z)\bar{Q}X - g(X, \phi Z)\bar{Q}Y]. \end{aligned}$$

By using (16), (33) and (34), (56) takes the form

$$(57) \quad \begin{aligned} \bar{C}(X, Y)\phi Z &= R(X, Y)\phi Z - 3g(Y, \phi Z)X + 3g(X, \phi Z)Y \\ &\quad + 2g(Y, \phi Z)\eta(X)\xi - 2g(X, \phi Z)\eta(Y)\xi \\ &\quad - \frac{1}{(n-2)}[(4 + \frac{2\bar{r}}{n-1})g(Y, \phi Z)X - (4 + \frac{2\bar{r}}{n-1})g(X, \phi Z)Y \\ &\quad - (2n + \frac{\bar{r}}{n-1})g(Y, \phi Z)\eta(X)\xi + (2n + \frac{\bar{r}}{n-1})g(X, \phi Z)\eta(Y)\xi]. \end{aligned}$$

We also have

$$(58) \quad \begin{aligned} \phi\bar{C}(X, Y)Z &= \phi R(X, Y)Z - 3g(Y, Z)\phi X + 3g(X, Z)\phi Y \\ &\quad + 2\eta(Y)\eta(Z)\phi X - 2\eta(X)\eta(Z)\phi Y \\ &\quad - \frac{1}{(n-2)}[(4 + \frac{2\bar{r}}{n-1})g(Y, Z)\phi X - (4 + \frac{2\bar{r}}{n-1})g(X, Z)\phi Y \\ &\quad + (2n + \frac{\bar{r}}{n-1})\eta(X)\eta(Z)\phi Y - (2n + \frac{\bar{r}}{n-1})\eta(Y)\eta(Z)\phi X]. \end{aligned}$$

From (55), (57) and (58), we have

$$(59) \quad \begin{aligned} &(\frac{2\bar{r}}{n-1} + 4n - 4)g(X, \phi Z)Y - (\frac{2\bar{r}}{n-1} + 4n - 4)g(Y, \phi Z)X \\ &+ (\frac{\bar{r}}{n-1} + 4n - 4)g(Y, \phi Z)\eta(X)\xi - (\frac{\bar{r}}{n-1} + 4n - 4)g(X, \phi Z)\eta(Y)\xi \\ &- (\frac{2\bar{r}}{n-1} + 4n - 4)g(X, Z)\phi Y + (\frac{2\bar{r}}{n-1} + 4n - 4)g(Y, Z)\phi X \\ &- (\frac{\bar{r}}{n-1} + 4n - 4)\eta(Y)\eta(Z)\phi X + (\frac{\bar{r}}{n-1} + 4n - 4)\eta(X)\eta(Z)\phi Y = 0 \end{aligned}$$

which by taking  $Y = \xi$  and then using (2) reduces to

$$(60) \quad \frac{\bar{r}}{n-1}(g(X, \phi Z)\xi + \eta(Z)\phi X) = 0.$$

By operating  $\phi$  on (60) and using (2), we get

$$\frac{\bar{r}}{n-1}\phi^2 X = 0.$$

Since  $\phi^2 X = 0$  is not possible for all  $X$ . So we get  $\bar{r} = 0$ . Thus we can state the following theorem:

**Theorem 5.1.** *In an  $n$ -dimensional  $\phi$ -conharmonically semisymmetric  $\eta$ -Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection, the scalar curvature vanishes.*

### 6. Conharmonically flat Kenmotsu manifolds with respect to the semi-symmetric metric connection satisfying the curvature condition $\bar{R} \cdot \bar{S} = 0$

In this section we consider the conharmonically flat Kenmotsu manifolds with respect to the semi-symmetric metric connection  $\bar{\nabla}$  satisfying the curvature condition

$$(61) \quad \bar{R}(X, Y) \cdot \bar{S} = 0.$$

Then we have

$$(62) \quad \bar{S}(\bar{R}(X, Y)Z, W) + \bar{S}(Z, \bar{R}(X, Y)W) = 0$$

for all vector fields  $X, Y, Z, W \in \chi(M)$ .

Using (47) in (62), we have

$$(63) \quad \begin{aligned} &g(Y, Z)\bar{S}(\bar{Q}X, W) - g(X, Z)\bar{S}(\bar{Q}Y, W) \\ &+ g(Y, W)\bar{S}(\bar{Q}X, Z) - g(X, W)\bar{S}(\bar{Q}Y, Z) = 0 \end{aligned}$$

which by replacing  $Y = Z = \xi$  and then using (2), (23) and (25) takes the form

$$(64) \quad \bar{S}(\bar{Q}X, W) - 4(n-1)^2\eta(X)\eta(W) + \eta(W)\bar{S}(\bar{Q}X, \xi) - 4(n-1)^2g(X, W) = 0.$$

Let  $\lambda$  be the eigenvalue of the endomorphism  $\bar{Q}$  corresponding to an eigenvector  $X$ . Then

$$(65) \quad \bar{Q}X = \lambda X.$$

Using (65) in (64), we have

$$(66) \quad \lambda\bar{S}(X, W) - [4(n-1)^2 + 2(n-1)\lambda]\eta(X)\eta(W) - 4(n-1)^2g(X, W) = 0.$$

By putting  $W = \xi$  in (66), we get

$$(67) \quad [\lambda^2 - 2(n-1)\lambda - 8(n-1)^2]\eta(X) = 0.$$

This gives

$$(68) \quad \lambda^2 - 2(n-1)\lambda - 8(n-1)^2 = 0 \quad \text{as } \eta(X) \neq 0.$$

Thus we can state the following theorem:

**Theorem 6.1.** *If an  $n$ -dimensional ( $n \geq 3$ ) conharmonically flat Kenmotsu manifold with respect to the semi-symmetric metric connection with non zero Ricci tensor  $\bar{S}$  satisfying the curvature condition  $\bar{R}(X, Y) \cdot \bar{S} = 0$ , then the symmetric endomorphism  $\bar{Q}$  of the tangent space corresponding to  $\bar{S}$  has two different non-zero eigenvalues, namely,  $4(n - 1)$  and  $-2(n - 1)$ .*

**7. Non-existence of Kenmotsu manifolds whose curvature tensor of manifold is covariantly constant with respect to the semi-symmetric metric connection and  $M$  is recurrent with respect to the Levi-Civita connection**

**Definition.** A Kenmotsu manifold  $M$  with respect to the Levi-Civita connection is called the recurrent, if its curvature tensor  $R$  satisfies the condition

$$(69) \quad (\nabla_W R)(X, Y)Z = A(W)R(X, Y)Z,$$

where  $A$  is the 1-form.

From (15), we can write

$$(70) \quad \begin{aligned} (\bar{\nabla}_W R)(X, Y)Z &= \bar{\nabla}_W R(X, Y)Z - R(\bar{\nabla}_W X, Y)Z - R(X, \bar{\nabla}_W Y)Z, \\ -R(X, Y)\bar{\nabla}_W Z &= (\nabla_W R)(X, Y)Z + \eta(R(X, Y)Z)W \\ &\quad - g(W, R(X, Y)Z)\xi - \eta(X)R(W, Y)Z \\ &\quad - \eta(Y)R(X, W)Z - \eta(Z)R(X, Y)W \\ &\quad + g(X, W)R(\xi, Y)Z + g(Y, W)R(X, \xi)Z \\ &\quad + R(W, Z)R(X, Y)\xi \end{aligned}$$

which on using (7)-(9) reduces to

$$(71) \quad \begin{aligned} (\bar{\nabla}_W R)(X, Y)Z &= (\nabla_W R)(X, Y)Z + \eta(X)[g(W, Y)Z - g(Z, Y)W \\ &\quad - g(W, Z)Y] + \eta(Y)[g(W, Z)X - g(W, X)Z + g(Z, X)W]. \end{aligned}$$

Let  $(\bar{\nabla}_W R)(X, Y)Z = 0$ , then from (71), it follows that

$$(72) \quad \begin{aligned} (\nabla_W R)(X, Y)Z + \eta(X)[g(W, Y)Z - g(Z, Y)W - g(W, Z)Y] \\ + \eta(Y)[g(W, Z)X - g(W, X)Z + g(Z, X)W] = 0. \end{aligned}$$

Now using (69) in (72), we have

$$(73) \quad \begin{aligned} A(W)R(X, Y)Z + \eta(X)[g(W, Y)Z - g(Z, Y)W - g(W, Z)Y] \\ + \eta(Y)[g(W, Z)X - g(W, X)Z + g(Z, X)W] = 0. \end{aligned}$$

Contracting  $X$  in (73), we get

$$(74) \quad A(W)S(Y, Z) + g(W, Y)\eta(Z) - g(Z, Y)\eta(W) + (n - 1)g(W, Z)\eta(Y) = 0.$$

Now considering  $Y = \xi$  in (74) and using (2), we get

$$(75) \quad A(W)\eta(Z) - g(W, Z) = 0.$$

Suppose the associated 1-form  $A$  is equal to the associated 1-form  $\eta$ , then from (75), we have

$$(76) \quad g(W, Z) = \eta(W)\eta(Z).$$

Therefore in view of (3), we get  $g(\phi W, \phi Z) = 0$ , which is not possible.

Thus we can state the following theorem:

**Theorem 7.1.** *There is no Kenmotsu manifold whose curvature tensor of manifold is covariantly constant with respect to the semi-symmetric metric connection and the manifold is recurrent with respect to the Levi-Civita connection and the associated 1-form  $A$  is equal to the associated 1-form  $\eta$ .*

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