Commun. Korean Math. Soc. **32** (2017), No. 4, pp. 1009–1024
https://doi.org/10.4134/CKMS.c170031
pISSN: 1225-1763 / eISSN: 2234-3024

SOME RESULTS OF THE NEW ITERATIVE SCHEME IN HYPERBOLIC SPACE

Metin Başarir and Aynur Şahin

Abstract. In this paper, we consider the new faster iterative scheme due to Sintunavarat and Pitea ([32]) for further investigation and we prove its strong and $\triangle$-convergence theorems, data dependence and stability results in hyperbolic space. Our results extend, improve and generalize several recent results in CAT(0) space and uniformly convex Banach space.

1. Introduction

Let (X, d) be a metric space, and let $T : X \to X$ be a mapping. A point $x \in X$ is called a *fixed point* of T if $Tx = x$. Denote the set of fixed points of T by $F(T)$. The mapping T is

(i) an *a-contraction* if

$$d(Tx, Ty) \leq ad(x, y) \quad \text{for all } x, y \in X, \tag{1}$$

where $0 \leq a < 1$,

(ii) *Kannan mapping* [17] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \quad \text{for all } x, y \in X, \tag{2}$$

(iii) *Chatterjea mapping* [9] if there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \quad \text{for all } x, y \in X. \tag{3}$$

If the constant a in (1) is equal to 1, then T is called a *nonexpansive mapping.*

Combining the definitions (1)-(3), Zamfirescu [36] proved the following important result.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a Zamfirescu mapping, i.e., there exist the real numbers a, b and c satisfying $0 \leq a < 1$ and $b, c \in (0, \frac{1}{2})$ such that for each $x, y \in X$, at least one of the following conditions holds*:

(Z_1) $d(Tx, Ty) \leq ad(x, y)$;

Received February 2, 2017; Accepted May 11, 2017.
2010 *Mathematics Subject Classification.* Primary 47H09, 47H10, 49M05.
Key words and phrases. hyperbolic space, fixed point, iterative scheme, strong convergence, Δ-convergence, stability, data dependence.

(Z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;
(Z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.
Then T *has unique fixed point* p *and the Picard iterative sequence* $\{x_n\}$ *defined by* $x_0 \in X, x_{n+1} = Tx_n, n \geq 0$, *converges to* p.

Berinde [4] introduced a new class of mappings on a metric space (X, d) satisfying

$$d(Tx, Ty) \leq ad(x, y) + Ld(x, Tx) \quad \text{for all } x, y \in X, \tag{4}$$

where $0 \leq a < 1$ and $L \geq 0$. He also showed that the class of nonlinear mappings satisfying (4) is wider than the class of Zamfirescu mappings.

It is known, see Osilike [26], that the mappings satisfying (4) need not have a fixed point but, if $F(T) \neq \emptyset$, then $F(T)$ is a singleton.

Sintunavarat and Pitea [32] introduced a new iterative scheme in a Banach space as follows: For an arbitrary $x_0 \in K$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ z_n = (1 - \gamma_n)x_n + \gamma_n y_n, \\ x_{n+1} = (1 - \alpha_n)Tz_n + \alpha_n Ty_n, \quad \forall n \geq 0, \end{cases} \tag{5}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$. They proved that the iterative scheme (5) is faster than the iterative schemes of Mann [25], Ishikawa [16] and Agarwal et al. [1]. It is worth mentioning that this iterative scheme is reduced to the S-iterative scheme of Agarwal et al. [1] when $\gamma_n = 0$ for all $n \geq 0$.

In this paper, we prove the strong and Δ-convergence theorems of the iterative scheme (5) for a finite family of nonexpansive mappings in a uniformly convex hyperbolic space. Furthermore, we give the data dependence and stability results of the iterative scheme (5) for a mapping satisfying (4) in a hyperbolic space.

2. Preliminaries and lemmas

Throughout this paper, we study in the setting of hyperbolic space introduced by Kohlenbach [21], defined below, which is more restrictive than the hyperbolic type introduced in [11] and more general than the concept of hyperbolic space in [27].

A hyperbolic space is a triple (X, d, W) where (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ is a mapping such that
(W1) $d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y)$,
(W2) $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2| \, d(x, y)$,
(W3) $W(x, y, \lambda) = W(y, x, (1 - \lambda))$,
(W4) $d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(z, w)$
for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$.

If a space satisfies only (W1), it coincides with the convex metric space introduced by Takahashi [34]. A subset K of a hyperbolic space X is convex if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. The class of hyperbolic spaces contains all normed linear spaces and convex subsets thereof, $\mathbb{R}$-trees, the Hilbert ball with the hyperbolic metric (see [12]), Cartesian products of Hilbert balls, Hadamard manifolds and CAT(0) spaces (see [6]), as a special case.

A hyperbolic space (X, d, W) is said to be *uniformly convex* [31] if for all $u, x, y \in X, r > 0$ and $\varepsilon \in (0, 2]$, there exists $\delta \in (0, 1]$ such that

$$d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1 - \delta)r$$

whenever $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called *modulus of uniform convexity.* We call η *monotone* if it decreases with r (for a fixed ε).

Let K be a nonempty subset of a metric space X, and let $\{x_n\}$ be a bounded sequence in K. For $x \in X$, define $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The *asymptotic radius* of the sequence $\{x_n\}$ in K denoted by $r(K, \{x_n\})$ is defined by $r(K, \{x_n\}) = \inf \{r(x, \{x_n\}) : x \in K\}$. A point z is called an *asymptotic center* of the sequence $\{x_n\}$ in K if $r(z, \{x_n\}) = r(K, \{x_n\})$. The set of all asymptotic center of the sequence $\{x_n\}$ in K is denoted by $A(K, \{x_n\})$. The asymptotic radius and asymptotic center of the sequence $\{x_n\}$ with respect to whole space are denoted by $r(\{x_n\})$ and $A(\{x_n\})$, respectively.

It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that bounded sequences have unique asymptotic center with respect to closed convex subsets. In case of hyperbolic space, we have the following result.

Lemma 2.1 ([23, Proposition 3.3])**.** *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has unique asymptotic center with respect to any nonempty closed convex subset K of X.*

The concept of $\triangle$-convergence, introduced by Kuczumow [22] and Lim [24] independently several years ago, is shown in CAT(0) space to behave similarly as the weak convergence in Banach space.

Definition. A bounded sequence $\{x_n\}$ in X is said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $x_n \overset{\triangle}{\to} x$ and call x as $\triangle$-limit of $\{x_n\}$.

It is well known that one of the fundamental and celebrated results in the theory of nonexpansive mappings is Browder's demiclosedness principle [7] which states that if K is a nonempty closed convex subset of a uniformly convex Banach space X and $T : K \to X$ is a nonexpansive mapping, then $I - T$ is *demiclosed* at 0, that is, for any sequence $\{x_n\}$ in K if $x_n \to x$ weakly and $(I - T)x_n \to 0$ strongly, then $(I - T)x = 0$. Fukhar-ud-din and Khamsi [10] proved the demiclosedness principle for nonexpansive mappings in a hyperbolic space as follows.

Lemma 2.2 ([10, Lemma 4.1]). *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X. Let $T : K \to K$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in K such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. If $x \in K$ is the asymptotic center of $\{x_n\}$ with respect to K, then x is a fixed point of T. In particular, if $\{x_n\}$ is a sequence in K such that $x_n \overset{\Delta}{\to} x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x \in F(T)$.*

In the sequel, we shall need the following results.

Lemma 2.3 ([19, Lemma 2.5]). *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \le r, \quad \limsup_{n\to\infty} d(y_n, x) \le r, \quad \lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$$

for some $r \ge 0$, then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 2.4 ([33]). *Let $\{a_n\}$ be a non-negative sequence for which one assumes that there exists an $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$,*

$$a_{n+1} \le (1 - r_n)a_n + r_n t_n$$

is satisfied, where $r_n \in (0, 1)$ for all $n \in \mathbb{N}, \sum_{n=0}^{\infty} r_n = \infty$ and $t_n \ge 0, \forall n \in \mathbb{N}$. Then the following holds:

$$0 \le \limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} t_n.$$

3. Some strong and Δ-convergence theorems

In the sequel, we denote $\{1, 2, \ldots, N\}$ by I, and we assume that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. First, we define the iterative scheme (5) for a finite family of nonexpansive mappings in a hyperbolic space as follows.

$$\begin{cases} y_n = W(x_n, T_n x_n, \beta_n), \\ z_n = W(x_n, y_n, \gamma_n), \\ x_{n+1} = W(T_n z_n, T_n y_n, \alpha_n), \quad \forall n \ge 0, \end{cases} \tag{6}$$

where $T_n = T_{n(\mathrm{mod} N)}, \{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b]$ with $a, b \in (0, 1)$ and the sequence $\{\gamma_n\}$ is in $[0, 1]$.

We prove the Δ-convergence theorem of the iterative sequence $\{x_n\}$ defined by (6) for a finite family of nonexpansive mappings in a hyperbolic space.

Theorem 3.1. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with the modulus of uniform convexity η, and let $\{T_i : i \in I\}$ be a finite family of nonexpansive self mappings on K. Then the sequence $\{x_n\}$ defined by (6), is Δ-convergent to a point in F.*

Proof. We divide our proof into three steps.

Step 1. First we prove that for each $p \in F$,

$$\lim_{n\to\infty} d(x_n, p) \text{ exists.} \tag{7}$$

By using (6), we get

$$\begin{aligned} d(x_{n+1}, p) &= d(W(T_n z_n, T_n y_n, \alpha_n), p) \\ &\leq (1-\alpha_n) d(T_n z_n, p) + \alpha_n d(T_n y_n, p) \\ &\leq (1-\alpha_n) d(z_n, p) + \alpha_n d(y_n, p). \end{aligned} \tag{8}$$

Using (6) again, we obtain

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T_n x_n, \beta_n), p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n d(T_n x_n, p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p) \end{aligned} \tag{9}$$

and

$$\begin{aligned} d(z_n, p) &= d(W(x_n, y_n, \gamma_n), p) \\ &\leq (1-\gamma_n) d(x_n, p) + \gamma_n d(y_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{10}$$

From (8), (9) and (10), we have

$$d(x_{n+1}, p) \leq d(x_n, p). \tag{11}$$

This inequality guarantees that the sequence $\{d(x_n, p)\}$ is non-increasing and bounded below, and so $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F$.

Step 2. Next we prove that

$$\lim_{n\to\infty} d(x_n, T_l x_n) = 0 \quad \text{for each } l = 1, 2, \ldots, N. \tag{12}$$

It follows from (7) that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. We may assume that

$$\lim_{n\to\infty} d(x_n, p) = r \geq 0. \tag{13}$$

The case $r = 0$ is trivial. Next, we deal with the case $r > 0$. Taking lim sup on both sides of the inequalities (9) and (10) and using (13), we have

$$\limsup_{n\to\infty} d(y_n, p) \leq r \tag{14}$$

and

$$\limsup_{n\to\infty} d(z_n, p) \leq r,$$

respectively. Since T_n is nonexpansive for all $n = 1, 2, \ldots$, so

$$\limsup_{n\to\infty} d(T_n y_n, p) \le r \quad \text{and} \quad \limsup_{n\to\infty} d(T_n z_n, p) \le r.$$

Moreover,

$$r = \lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d(W(T_n z_n, T_n y_n, \alpha_n), p)$$

gives, by Lemma 2.3, that

$$\lim_{n\to\infty} d(T_n z_n, T_n y_n) = 0. \tag{15}$$

By (8) and (10), we have

$$\begin{aligned} d(x_{n+1}, p) &\le (1-\alpha_n) d(z_n, p) + \alpha_n d(y_n, p) \\ &\le (1-\alpha_n) d(x_n, p) + \alpha_n d(y_n, p). \end{aligned}$$

This gives that

$$\alpha_n d(x_n, p) \le d(x_n, p) + \alpha_n d(y_n, p) - d(x_{n+1}, p)$$

or

$$\begin{aligned} d(x_n, p) &\le d(y_n, p) + \frac{1}{\alpha_n} \left[d(x_n, p) - d(x_{n+1}, p) \right] \\ &\le d(y_n, p) + \frac{1}{a} \left[d(x_n, p) - d(x_{n+1}, p) \right]. \end{aligned}$$

This implies

$$r \le \liminf_{n\to\infty} d(y_n, p).$$

Reading it together with (14), we get

$$r = \lim_{n\to\infty} d(y_n, p) = \lim_{n\to\infty} d(W(x_n, T_n x_n, \beta_n), p). \tag{16}$$

Also

$$d(T_n x_n, p) \le d(x_n, p)$$

for all $n = 1, 2, \ldots$, so

$$\limsup_{n\to\infty} d(T_n x_n, p) \le r. \tag{17}$$

With the help of (13), (16), (17) and Lemma 2.3, we have

$$\lim_{n\to\infty} d(x_n, T_n x_n) = 0. \tag{18}$$

Now

$$\begin{aligned} d(y_n, x_n) &= d(W(x_n, T_n x_n, \beta_n), x_n) \\ &\le \beta_n d(x_n, T_n x_n) \end{aligned}$$

implies by (18) that

$$\lim_{n\to\infty} d(y_n, x_n) = 0. \tag{19}$$

Using (18) and (19), we obtain

$$\begin{aligned} d(T_n y_n, x_n) &\le d(T_n y_n, T_n x_n) + d(T_n x_n, x_n) \\ &\le d(y_n, x_n) + d(T_n x_n, x_n) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{20}$$

Next,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(W(T_n z_n, T_n y_n, \alpha_n), x_n) \\ &\le (1-\alpha_n) d(T_n z_n, x_n) + \alpha_n d(T_n y_n, x_n) \\ &\le (1-\alpha_n)[d(T_n z_n, T_n y_n) + d(T_n y_n, x_n)] + \alpha_n d(T_n y_n, x_n) \\ &= (1-\alpha_n) d(T_n z_n, T_n y_n) + d(T_n y_n, x_n) \end{aligned}$$

gives by (15) and (20) that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

This implies that

$$\lim_{n\to\infty} d(x_n, x_{n+l}) = 0 \quad \text{for each } l \in I.$$

Further, we observe that

$$\begin{aligned} d(x_n, T_{n+l} x_n) &\le d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}) + d(T_{n+l} x_{n+l}, T_{n+l} x_n) \\ &\le 2d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}). \end{aligned}$$

Therefore

$$\lim_{n\to\infty} d(x_n, T_{n+l} x_n) = 0 \quad \text{for each } l \in I.$$

This implies that $\lim_{n\to\infty} d(x_n, T_l x_n) = 0$ for each $l \in I$.

Step 3. Now we are in a position to prove the Δ-convergence case of $\{x_n\}$. It follows from (7) that $\{x_n\}$ is bounded. Therefore by Lemma 2.1, $\{x_n\}$ has unique asymptotic center $A(K, \{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ with $A(K, \{u_n\}) = \{u\}$. By (12), we have $\lim_{n\to\infty} d(u_n, T_l u_n) = 0$ for each $l = 1, 2, \ldots, N$. Then it follows from Lemma 2.2 that $u \in F$. By the uniqueness of asymptotic center, we get $x = u \in F$. It implies that the sequence $\{x_n\}$ is Δ-convergent to $x \in F$. The proof is completed. □

By taking $T_i = T$ for all $i \in I$ in Theorem 3.1, we get the following corollary, yet it is new in the literature.

Corollary 3.2. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with the modulus of uniform convexity η, and let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by* (5)*, is Δ-convergent to a fixed point of T.*

Recall that a sequence $\{x_n\}$ in a metric space X is said to be *Fejér monotone* with respect to K (a subset of X) if $d(x_{n+1}, p) \le d(x_n, p)$ for all $p \in K$ and $n \ge 1$. A mapping $T : K \to K$ is *semi-compact* if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a strongly convergent subsequence.

Khan et al. [19] defined the Condition (A) for a finite family of mappings as follows.

Let f be a non-decreasing self mapping on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$, and let $d(x, H) = \inf\{d(x, y) : y \in H\}$. Then a family $\{T_i : i \in I\}$ of self mappings on K with $F \neq \emptyset$, satisfies *Condition* (A) if

$$d(x, Tx) \geq f(d(x, F)) \quad \text{for all } x \in K,$$

holds for at least one $T \in \{T_i : i \in I\}$ or

$$\max_{i \in I} d(x, T_i x) \geq f(d(x, F)) \quad \text{for all } x \in K,$$

holds.

For further development, we need the following technical result.

Lemma 3.3 ([2])**.** *Let K be a nonempty closed subset of a complete metric space (X, d), and let $\{x_n\}$ be Fejér monotone with respect to K. Then $\{x_n\}$ converges strongly to some $p \in K$ if and only if* $\lim_{n\to\infty} d(x_n, K) = 0$.

Lemma 3.4. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with the modulus of uniform convexity η, and let $\{T_i : i \in I\}$ be a finite family of nonexpansive self mappings on K. Then the sequence $\{x_n\}$ defined by* (6) *converges strongly to $p \in F$ if and only if* $\lim_{n\to\infty} d(x_n, F) = 0$.

Proof. It follows from (11), the sequence $\{x_n\}$ is Fejér monotone with respect to F and $\lim_{n\to\infty} d(x_n, F)$ exists for all $p \in F$. Hence, the result follows from Lemma 3.3. ☐

We now establish the strong convergence theorems of the iterative scheme (6) based on Lemma 3.4.

Theorem 3.5. *Let $X, K, \{T_i : i \in I\}$ and $\{x_n\}$ satisfy the hypotheses of Theorem 3.1. Then the sequence $\{x_n\}$ converges strongly to some $p \in F$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. If $\{x_n\}$ converges to $p \in F$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F) = 0$. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. It follows from (7) that $\lim_{n\to\infty} d(x_n, F)$ exists. Thus by hypothesis, we get $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Lemma 3.4 implies that $\{x_n\}$ converges strongly to a point p in F. ☐

Remark 3.6. In Theorem 3.5, the condition $\liminf_{n\to\infty} d(x_n, F) = 0$ may be replaced with $\limsup_{n\to\infty} d(x_n, F) = 0$.

Theorem 3.7. *Under the assumptions of Theorem 3.1, if $\{T_i : i \in I\}$ satisfies Condition* (A)*, then the sequence $\{x_n\}$ defined by* (6) *converges strongly to a point in F.*

Proof. It follows from (7) that $\lim_{n\to\infty} d(x_n, F)$ exists for all $p \in F$. Also, by (12), we have $\lim_{n\to\infty} d(x_n, T_l x_n) = 0$ for each $l \in I$. So Condition (A) guarantees that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a non-decreasing function with $f(0) = 0$ and $f(r) > 0, \forall r > 0$, we have $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Lemma 3.4 implies that $\{x_n\}$ converges strongly to a point p in F. □

Note that the Condition (A) is weaker than both of the compactness of K and the semi-compactness of nonexpansive mappings $\{T_i : i \in I\}$ (see [30]) therefore we already have the following result.

Theorem 3.8. *Under the assumptions of Theorem 3.1, if either K is compact or one of the mappings in $\{T_i : i \in I\}$ is semi-compact, then the sequence $\{x_n\}$ defined by* (6) *converges strongly to a point in F.*

Remark 3.9. (1) The strong convergence results using the iterative scheme (5) can now be obtained as corollaries from Theorems 3.5, 3.7, 3.8.

(2) Our results generalize the corresponding results Khan and Abbas [18] in two ways: (i) from a nonexpansive mapping to a finite family of nonexpansive mapping, (ii) from CAT(0) space to general setup of hyperbolic space.

4. Some data dependence and stability results

First, we prove the strong convergence theorem of the faster iterative scheme (5) for a mapping satisfying (4) in a hyperbolic space.

Theorem 4.1. *Let K be a nonempty closed convex subset of a hyperbolic space X, and let T be a mapping satisfying* (4) *with $F(T) \neq \emptyset$. Define an iterative sequence $\{x_n\}$ by*

$$\begin{cases} y_n = W(x_n, Tx_n, \beta_n), \\ z_n = W(x_n, y_n, \gamma_n), \\ x_{n+1} = W(Tz_n, Ty_n, \alpha_n), \quad \forall n \geq 0, \end{cases} \tag{21}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$. Then the sequence $\{x_n\}$ converges strongly to the unique fixed point p of T.

Proof. From (W1), (4) and (21), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(Tz_n, Ty_n, \alpha_n), p) \\ &\leq (1 - \alpha_n) d(Tz_n, p) + \alpha_n d(Ty_n, p) \\ &\leq (1 - \alpha_n) \{ad(z_n, p) + Ld(p, Tp)\} + \alpha_n \{ad(y_n, p) + Ld(p, Tp)\} \\ &= (1 - \alpha_n) ad(z_n, p) + \alpha_n ad(y_n, p). \end{aligned} \tag{22}$$

Similarly, we obtain

$$\begin{aligned} d(y_n, p) &= d(W(x_n, Tx_n, \beta_n), p) \\ &\leq (1 - \beta_n) d(x_n, p) + \beta_n d(Tx_n, p) \\ &\leq (1 - \beta_n) d(x_n, p) + \beta_n \{ad(x_n, p) + Ld(p, Tp)\} \end{aligned}$$

$$= (1 - \beta_n(1-a))d(x_n, p)$$
$$\le d(x_n, p) \tag{23}$$

and so

$$d(z_n, p) = d(W(x_n, y_n, \gamma_n), p)$$
$$\le (1-\gamma_n)d(x_n, p) + \gamma_n d(y_n, p)$$
$$\le d(x_n, p). \tag{24}$$

Then from (22), (23) and (24), we get that

$$\begin{aligned} d(x_{n+1}, p) &\le (1-\alpha_n)ad(x_n, p) + \alpha_n ad(x_n, p) \\ &\le ad(x_n, p) \\ &\vdots \\ &\le a^{n+1}d(x_0, p). \end{aligned}$$

If $a \in (0,1)$, we obtain

$$\lim_{n\to\infty} d(x_{n+1}, p) = 0.$$

Thus we have $x_n \to p \in F(T)$. If $a = 0$, the result is clear. This completes the proof. □

Example 4.2. Let $\mathbb{R}$ be the real line with the usual metric $|\cdot|$ and let $K = [0,1]$. Define a mapping $T : K \to K$ by $Tx = \frac{x}{2}$. This mapping satisfies the condition (4) with $a = \frac{1}{2}$ and $L = 0$. It is clear that T has unique fixed point at 0. Let $\alpha_n = \beta_n = \gamma_n = 0$ for $n = 1, 2, 3$ and $\alpha_n = \beta_n = \gamma_n = \frac{2}{\sqrt{n}}$ for all $n \ge 4$. It is easy to see that the conditions of Theorem 4.1 are satisfied.

Example 4.3. Let $\mathbb{R}$ be the real line with the usual metric $|\cdot|$ and let $K = [0,1]$. Define a mapping $T : K \to K$ by

$$Tx = \begin{cases} \frac{1}{6}, & x \in (0.5, 1] \\ 0, & x \in [0, 0.5]. \end{cases}$$

It is easy to see that T satisfies (4). Clearly, $F(T) = \{0\}$. Let $\alpha_n = \beta_n = \gamma_n = 0$ for $n = 1, 2, \ldots, 15$ and $\alpha_n = \beta_n = \gamma_n = \frac{4}{\sqrt{n}}$ for all $n \ge 16$. So, the conditions of Theorem 4.1 are satisfied.

Data dependence of fixed points has become an important subject for research. The data dependence of various iterative schemes has been studied by many authors; see [13, 29, 33].

Definition ([5, p. 166])**.** Let $T, \widetilde{T} : X \to X$ be two operators. We say that $\widetilde{T}$ is an approximate operator for T if, for all $x \in X$ and for a fixed $\varepsilon > 0$, we have $d(Tx, \widetilde{T}x) \le \varepsilon$.

By using this definition, we prove the data dependence result for the iterative scheme defined by (21) in hyperbolic space.

Theorem 4.4. *Let* X, K *and* T *be the same as in Theorem 4.1. Suppose that* $\widetilde{T}$ *is approximate operator of* T*, that is,* $d(Tx, \widetilde{T}x) \leq \varepsilon$*. Let* $\{x_n\}$ *and* $\{u_n\}$ *be two iterative sequences defined by* (21) *and*

$$\begin{cases} v_n = W(u_n, \widetilde{T}u_n, \beta_n), \\ w_n = W(u_n, v_n, \gamma_n), \\ u_{n+1} = W(\widetilde{T}w_n, \widetilde{T}v_n, \alpha_n), \quad n \in \mathbb{N}, \end{cases} \tag{25}$$

respectively, where $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are real sequences in* $[0,1]$ *satisfying* $\alpha_n \geq \frac{1}{2}, \forall n \in \mathbb{N}$*. If* $p = Tp$ *and* $q = \widetilde{T}q$*, then we have*

$$d(p,q) \leq \frac{4\varepsilon}{1-a}.$$

Proof. From (W4), (4), (21) and (25), we have the following estimates:

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &= d(W(Tz_n, Ty_n, \alpha_n), W(\widetilde{T}w_n, \widetilde{T}v_n, \alpha_n)) \\ &\leq (1-\alpha_n) d(Tz_n, \widetilde{T}w_n) + \alpha_n d(Ty_n, \widetilde{T}v_n) \\ &\leq (1-\alpha_n)\{d(Tz_n, Tw_n) + d(Tw_n, \widetilde{T}w_n)\} \\ &\quad + \alpha_n\{d(Ty_n, Tv_n) + d(Tv_n, \widetilde{T}v_n)\} \\ &\leq (1-\alpha_n)\{ad(z_n, w_n) + Ld(z_n, Tz_n) + \varepsilon\} \\ &\quad + \alpha_n\{ad(y_n, v_n) + Ld(y_n, Ty_n) + \varepsilon\} \end{aligned} \tag{26}$$

with

$$\begin{aligned} d(y_n, v_n) &= d(W(x_n, Tx_n, \beta_n), W(u_n, \widetilde{T}u_n, \beta_n)) \\ &\leq (1-\beta_n) d(x_n, u_n) + \beta_n d(Tx_n, \widetilde{T}u_n) \\ &\leq (1-\beta_n) d(x_n, u_n) + \beta_n\{d(Tx_n, Tu_n) + d(Tu_n, \widetilde{T}u_n)\} \\ &\leq (1-\beta_n) d(x_n, u_n) + \beta_n\{ad(x_n, u_n) + Ld(x_n, Tx_n) + \varepsilon\} \\ &= (1-\beta_n(1-a)) d(x_n, u_n) + \beta_n L d(x_n, Tx_n) + \beta_n \varepsilon \end{aligned} \tag{27}$$

and

$$\begin{aligned} d(z_n, w_n) &= d(W(x_n, y_n, \gamma_n), W(u_n, v_n, \gamma_n)) \\ &\leq (1-\gamma_n) d(x_n, u_n) + \gamma_n d(y_n, v_n) \\ &\leq (1-\gamma_n) d(x_n, u_n) + \gamma_n(1-\beta_n(1-a)) d(x_n, u_n) \\ &\quad + \gamma_n\beta_n L d(x_n, Tx_n) + \gamma_n\beta_n\varepsilon \\ &= (1-\gamma_n\beta_n(1-a)) d(x_n, u_n) + \gamma_n\beta_n L d(x_n, Tx_n) + \gamma_n\beta_n\varepsilon. \end{aligned} \tag{28}$$

Combining (26), (27) and (28), we get

$$\begin{aligned} & d(x_{n+1}, u_{n+1}) \\ \leq\ & \{(1-\alpha_n)a(1-\gamma_n\beta_n(1-a)) + \alpha_n a(1-\beta_n(1-a))\} d(x_n, u_n) \\ & + \{(1-\alpha_n)\gamma_n\beta_n a + \alpha_n\beta_n a\} L d(x_n, Tx_n) + \alpha_n L d(y_n, Ty_n) \end{aligned}$$

$$+ (1-\alpha_n)Ld(z_n, Tz_n) + (1-\alpha_n)\gamma_n\beta_n a\varepsilon$$

$$+ (1-\alpha_n)\varepsilon + \alpha_n\beta_n a\varepsilon + \alpha_n\varepsilon. \tag{29}$$

Since $a \in [0,1)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$, we have

$$\begin{gathered} 1-\gamma_n\beta_n(1-a) \leq 1,\ (1-\alpha_n)a \leq 1-\alpha_n,\ 1-\beta_n(1-a) \leq 1, \\ \gamma_n\beta_n a \leq 1,\ \alpha_n\beta_n a \leq \alpha_n. \end{gathered} \tag{30}$$

It follows from the condition $\alpha_n \geq \frac{1}{2}$ for all $n \in \mathbb{N}$ that

$$1-\alpha_n \leq \alpha_n, \quad \forall n \in \mathbb{N}. \tag{31}$$

By substituting (30) and (31) into (29), we obtain

$$\begin{aligned} d(x_{n+1}, u_{n+1}) \leq\ & (1-\alpha_n(1-a))d(x_n, u_n) \\ & + 2\alpha_n Ld(x_n, Tx_n) + \alpha_n Ld(y_n, Ty_n) + \alpha_n Ld(z_n, Tz_n) \\ & + 4\alpha_n\varepsilon, \end{aligned}$$

or, equivalently,

$$\begin{aligned} & d(x_{n+1}, u_{n+1}) \\ \leq\ & (1-\alpha_n(1-a))d(x_n, u_n) \\ & + \alpha_n(1-a)\frac{2Ld(x_n, Tx_n) + Ld(y_n, Ty_n) + Ld(z_n, Tz_n) + 4\varepsilon}{1-a}. \end{aligned} \tag{32}$$

Now define

$$\begin{aligned} a_n &= d(x_n, u_n), \\ r_n &= \alpha_n(1-a), \\ t_n &= \frac{2Ld(x_n, Tx_n) + Ld(y_n, Ty_n) + Ld(z_n, Tz_n) + 4\varepsilon}{1-a}. \end{aligned}$$

Thus, the inequality (32) becomes

$$a_{n+1} \leq (1-r_n)a_n + r_n t_n. \tag{33}$$

From Theorem 4.1, it follows that $\lim_{n\to\infty} d(x_n, p) = 0$ and $\lim_{n\to\infty} d(u_n, q) = 0$. Since the mapping T satisfies (4) and $p = Tp$, we get

$$\begin{aligned} 0 &\leq d(x_n, Tx_n) \\ &\leq d(x_n, p) + d(Tp, Tx_n) \\ &\leq d(x_n, p) + ad(p, x_n) + Ld(p, Tp) \\ &= (1+a)d(x_n, p) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{34}$$

It is easy to see from (34) that this result is also valid for $d(y_n, Ty_n)$ and $d(z_n, Tz_n)$. Therefore, using Lemma 2.4, the inequality (33) yields

$$d(p,q) \leq \frac{4\varepsilon}{1-a}.$$

□

The stability of iterative schemes has extensively been studied by various authors [3,14,15,20,26,28,35] due to its increasing importance in computational mathematics, especially due to revolution in computer programming.

Definition ([8])**.** Two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are said to be equivalent sequences if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition ([35])**.** Let (X, d) be a metric space and let T be a self mapping on X. Let $\{x_n\}_{n=0}^{\infty} \subset X$ be an iterative sequence generated by the general algorithm of the form

$$\begin{cases} x_0 \in X, \\ x_{n+1} = f(T, x_n), \quad \forall n \geq 0, \end{cases} \tag{35}$$

where x_0 is an initial approximation and f is a function. Suppose that $\{x_n\}$ converges strongly to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ be an equivalent sequence of $\{x_n\}_{n=0}^{\infty} \subset X$. Then, the iterative scheme (35) is said to be *weak w^2-stable with respect to T* if and only if $\lim_{n\to\infty} d\left(y_{n+1}, f(T, y_n)\right) = 0$ implies that $\lim_{n\to\infty} y_n = p$.

We now study an open problem of Sintunavarat and Pitea [32] in hyperbolic space.

Theorem 4.5. *Let X, K and T be the same as in Theorem 4.1. Then, for $x_0 \in K$, the sequence $\{x_n\}$ defined by* (21) *is weak w^2-stable with respect to T.*

Proof. Suppose that $\{p_n\}_{n=0}^{\infty} \subset K$ is an equivalent sequence of $\{x_n\}$,

$$\epsilon_n = d(p_{n+1}, W(Tr_n, Tq_n, \alpha_n)),$$

where $q_n = W(p_n, Tp_n, \beta_n), r_n = W(p_n, q_n, \gamma_n)$, and let $\lim_{n\to\infty} \epsilon_n = 0$. Then, using (4) we have

$$\begin{aligned} d(p_{n+1}, p) &\leq d(p_{n+1}, x_{n+1}) + d(x_{n+1}, p) \\ &\leq d(p_{n+1}, W(Tr_n, Tq_n, \alpha_n)) \\ &\quad + d(W(Tr_n, Tq_n, \alpha_n), W(Tz_n, Ty_n, \alpha_n)) + d(x_{n+1}, p) \\ &\leq \epsilon_n + (1-\alpha_n) d(Tr_n, Tz_n) + \alpha_n d(Tq_n, Ty_n) + d(x_{n+1}, p) \\ &\leq \epsilon_n + (1-\alpha_n)\left\{ad(r_n, z_n) + Ld(z_n, Tz_n)\right\} \\ &\quad + \alpha_n \left\{ad(q_n, y_n) + Ld(y_n, Ty_n)\right\} + d(x_{n+1}, p) \\ &= \epsilon_n + (1-\alpha_n) ad(r_n, z_n) + \alpha_n ad(q_n, y_n) \\ &\quad + (1-\alpha_n) Ld(z_n, Tz_n) + \alpha_n Ld(y_n, Ty_n) + d(x_{n+1}, p). \end{aligned} \tag{36}$$

Again from (4), we have the following estimates:

$$\begin{aligned} d(q_n, y_n) &\leq (1-\beta_n) d(p_n, x_n) + \beta_n d(Tp_n, Tx_n) \\ &\leq (1-\beta_n) d(p_n, x_n) + \beta_n \left\{ad(p_n, x_n) + Ld(x_n, Tx_n)\right\} \\ &= (1-\beta_n(1-a)) d(p_n, x_n) + \beta_n Ld(x_n, Tx_n) \end{aligned} \tag{37}$$

and

$$\begin{aligned} d(r_n, z_n) &\le (1-\gamma_n)d(p_n, x_n) + \gamma_n d(q_n, y_n) \\ &\le (1-\gamma_n)d(p_n, x_n) + \gamma_n(1-\beta_n(1-a))d(p_n, x_n) \\ &\quad + \gamma_n\beta_n L d(x_n, Tx_n) \\ &= (1-\gamma_n\beta_n(1-a))d(p_n, x_n) + \gamma_n\beta_n L d(x_n, Tx_n). \end{aligned} \tag{38}$$

Using (36), (37) and (38), we arrive that

$$\begin{aligned} & d(p_{n+1}, p) \\ &\le \epsilon_n + \{(1-\alpha_n)a(1-\gamma_n\beta_n(1-a)) + \alpha_n a(1-\beta_n(1-a))\} d(p_n, x_n) \\ &\quad + \{(1-\alpha_n)\gamma_n\beta_n a + \alpha_n\beta_n a\} L d(x_n, Tx_n) + \alpha_n L d(y_n, Ty_n) \\ &\quad + (1-\alpha_n)L d(z_n, Tz_n) + d(x_{n+1}, p). \end{aligned} \tag{39}$$

Now, from Theorem 4.1, we have $\lim_{n\to\infty} d(x_n, p) = 0$. Since $\{x_n\}$ and $\{p_n\}$ are equivalent sequences, therefore we have $\lim_{n\to\infty} d(x_n, p_n) = 0$. Also as in the proof of Theorem 4.4, we can get

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(y_n, Ty_n) = \lim_{n\to\infty} d(z_n, Tz_n) = 0.$$

By taking the limit on both sides of (39), we obtain $\lim_{n\to\infty} p_n = p$. This shows that $\{x_n\}$ is weak w^2-stable with respect to T. □

Remark 4.6. The similar results of Theorems 4.1, 4.4, 4.5 can also be proved for a finite family of mappings satisfying (4).

References

[1] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal. **8** (2007), no. 1, 61–79.

[2] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer-Verlag, New York, 2011.

[3] V. Berinde, *On the stability of some fixed point procedures*, Bul. Ştiinţ. Univ. Baia Mare Ser. B Fasc. Mat.-Inform. **18** (2002), no. 1, 7–14.

[4] ______, *A convergence theorem for some mean value fixed point iterations procedures*, Demonstratio Math. **38** (2005), no. 1, 177–184.

[5] ______, *Iterative Approximation of Fixed Points*, Springer, 2007.

[6] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, 1999.

[7] F. E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.

[8] T. Cardinali and P. Rubbioni, *A generalization of the Caristi fixed point theorem in metric spaces*, Fixed Point Theory **11** (2010), no. 1, 3–10.

[9] S. K. Chatterjea, *Fixed point theorems*, C. R. Acad. Bulgare Sci. **25** (1972), 727–730.

[10] H. Fukhar-ud-din and M. A. Khamsi, *Approximating common fixed points in hyperbolic spaces*, Fixed Point Theory Appl. **2014** (2014), Article ID 113, 15 pages.

[11] K. Goebel and W. A. Kirk, *Iteration processes for nonexpansive mappings*, In: Topological methods in nonlinear functional analysis (Toronto, Ont., 1982), 115–123, Contemp. Math., 21, Amer. Math. Soc., Providence, RI, 1983.

[12] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York, 1984.

[13] F. Gürsoy, V. Karakaya, and B. E. Rhoades, *Data dependence results of new multi-step and S-iterative schemes for contractive-like operators*, Fixed Point Theory Appl. **2013** (2013), Article ID 76, 12 pages.

[14] A. M. Harder and T. L. Hicks, *Stability results for fixed point iteration procedures*, Math. Japon. **33** (1988), no. 5, 693–706.

[15] C. O. Imoru and M. O. Olantinwo, *On the stability of Picard and Mann iteration processes*, Carpathian J. Math. **19** (2003), no. 2, 155–160.

[16] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[17] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **10** (1968), 71–76.

[18] S. H. Khan and M. Abbas, *Strong and Δ-convergence of some iterative schemes in $CAT(0)$ spaces*, Comput. Math. Appl. **61** (2011), no. 1, 109–116.

[19] A. R. Khan, H. Fukhar-ud-din, and M. A. A. Khan, *An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces*, Fixed Point Theory Appl. **2012** (2012), Article ID 54, 12 pages.

[20] A. R. Khan, F. Gürsoy, and V. Kumar, *Stability and data dependence results for the Jungck-Khan iterative scheme*, Turkish J. Math. **40** (2016), no. 3, 631–640.

[21] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis*, Trans. Amer. Math. Soc. **357** (2005), no. 1, 89–128.

[22] T. Kuczumow, *An almost convergence and its applications*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A **32** (1978), 79–88.

[23] L. Leustean, *Nonexpansive iterations in uniformly convex W-hyperbolic spaces*, In: Nonlinear analysis and optimization I. Nonlinear analysis, 193–210, Contemp. Math., 513, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, 2010.

[24] T. C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc. **60** (1976), 179–182.

[25] W. R. Mann, *Mean value methods in iterations*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[26] M. O. Osilike, *Some stability results for fixed point iteration procedures*, J. Nigerian Math. Soc. **14/15** (1995/96), 17–29.

[27] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal. **15** (1990), no. 6, 537–558.

[28] B. E. Rhoades, *Fixed point theorems and stability results for fixed point iteration procedures*, Indian J. Pure Appl. Math. **21** (1990), no. 1, 1–9.

[29] A. Şahin and M. Başarır, *Convergence and data dependence results of an iteration process in a hyperbolic space*, Filomat **30** (2016), no. 3, 569–582.

[30] H. F. Senter and W. G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. **44** (1974), no. 2, 375–380.

[31] T. Shimizu and W. Takahashi, *Fixed points of multivalued mappings in certain convex metric spaces*, Topol. Methods Nonlinear Anal. **8** (1996), no. 1, 197–203.

[32] W. Sintunavarat and A. Pitea, *On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis*, J. Nonlinear Sci. Appl. **9** (2016), no. 5, 2553–2562.

[33] Ş. M. Şoltuz and T. Grosan, *Data dependence for Ishikawa iteration when dealing with contractive-like operators*, Fixed Point Theory Appl. **2008** (2008), Article ID 242916, 7 pages.

[34] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings*, Kodai Math. Semin. Rep. **22** (1970), 142–149.

[35] I. Timiş, *On the weak stability of Picard iteration for some contractive type mappings*, An. Univ. Craiova Ser. Mat. Inform. **37** (2010) no. 2, 106–114.

[36] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math. (Basel) **23** (1972), 292–298.

Metin Başarir
Department of Mathematics
Sakarya University
Sakarya 54050, Turkey
E-mail address: basarir@sakarya.edu.tr

Aynur Şahin
Department of Mathematics
Sakarya University
Sakarya 54050, Turkey
E-mail address: ayuce@sakarya.edu.tr