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THE GEOMETRY OF $\mathcal{L}_s(^3l^2_\infty)$

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Abstract. We classify the extreme, exposed and smooth symmetric 3-linear forms of the unit ball of $\mathcal{L}_s(^3l^2_\infty)$, respectively.

1. Introduction

We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. $x \in B_E$ is called an *exposed point* of B_E if there is a $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $extB_E, expB_E$ and smB_E the sets of extreme, exposed and smooth points of B_E, respectively. Let $n \in \mathbb{N}, n \geq 2$. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a unique continuous symmetric n-linear form L on the product $E \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{L}_s(^nE)$ the Banach space of all continuous symmetric n-linear forms on E endowed with the norm $\|L\| = \sup_{\|x_j\|=1, 1\leq j\leq n} |L(x_1, \ldots, x_n)|$. $\mathcal{P}(^nE)$ denotes the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [8].

In 1998, Choi *et al.* [3, 4] characterized the extreme points of the unit ball of $\mathcal{P}(^2l^2_1)$ and $\mathcal{P}(^2l^2_2)$. In 2007, the author [13] classified the exposed 2-homogeneous polynomials on $\mathcal{P}(^2l^2_p)$ $(1 \leq p \leq \infty)$. Recently, the author [15, 17, 21] classify the extreme, exposed, smooth points of the unit ball of $\mathcal{P}(^2d_*(1,w)^2)$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm of weight w.

In 2009, the author [14] classified the extreme, exposed, smooth points of the unit ball of $\mathcal{L}_s(^2l^2_\infty)$. Recently, the author [16, 18–20] classify the extreme, exposed, smooth points of the unit balls of $\mathcal{L}_s(^2d_*(1,w)^2)$ and $\mathcal{L}(^2d_*(1,w)^2)$.

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We refer to ([1–7], [9–28] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces. In this paper, we classify the extreme, exposed and smooth symmetric 3-linear forms of the unit ball of $\mathcal{L}_s({}^3 l_\infty^2)$, respectively.

2. The extreme points of the unit ball of $\mathcal{L}_s({}^3 l_\infty^2)$

Let $T((x_1, x_2), (y_1, y_2), (z_1, z_2)) = ax_1y_1z_1 + bx_2y_2z_2 + c(x_2y_1z_1 + y_2x_1z_1 + z_2x_1y_1) + d(x_1y_2z_2 + y_1x_2z_2 + z_1x_2y_2) \in \mathcal{L}_s({}^3 l_\infty^2)$ for some $a, b, c, d \in \mathbb{R}$. For simplicity, we will denote T by (a, b, c, d).

Theorem 2.1. *Let $T = (a, b, c, d) \in \mathcal{L}_s({}^3 l_\infty^2)$. Then*

$$\|T\| = \max\{|a + 3d| + |b + 3c|, |a - d| + |b - c|\}.$$

Proof. Since $\{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$ is the set of all extreme points of the unit ball of l_∞^2 and T is a symmetric 3-linear form,

$$\begin{aligned}\|T\| = &\max\{|T((1, 1), (1, 1), (1, 1))|, |T((1, -1), (1, 1), (1, 1))|,\\ &|T((1, -1), (1, -1), (1, 1))|, |T((1, -1), (1, -1), (1, -1))|\}\\ = &\max\{|a + 3d| + |b + 3c|, |a - d| + |b - c|\}.\end{aligned}$$ □

Note that if $\|T\| = 1$, then $|a| \le 1, |b| \le 1, |c| \le \frac{1}{3}, |d| \le \frac{1}{3}$.

Theorem 2.2. *Let $T = (a, b, c, d) \in \mathcal{L}_s({}^3 l_\infty^2)$. Then $T \in extB_{\mathcal{L}_s({}^3 l_\infty^2)}$ if and only if $(b, a, d, c) \in extB_{\mathcal{L}_s({}^3 l_\infty^2)}$.*

Proof. Let

$$S((x_1, x_2),\ (y_1, y_2), (z_1, z_2)) := T((x_2, x_1),\ (y_2, y_1), (z_2, z_1)) \in \mathcal{L}_s({}^3 l_\infty^2).$$

Then $S = (b, a, d, c)$. Note that $T \in extB_{\mathcal{L}_s({}^3 l_\infty^2)}$ if and only if $(b, a, d, c) \in extB_{\mathcal{L}_s({}^3 l_\infty^2)}$. □

Theorem 2.3.

$$\begin{aligned}extB_{\mathcal{L}_s({}^3 l_\infty^2)} = \{&\pm(1, 0, 0, 0), \pm(0, 1, 0, 0), \pm(\tfrac{1}{2}, 0, 0, -\frac{1}{2}), \pm(0, \tfrac{1}{2}, -\tfrac{1}{2}, 0),\\ &\pm(\tfrac{1}{4}, -\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}), \pm(-\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}), \pm(\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{1}{4}), \pm(\tfrac{1}{4}, \tfrac{3}{4}, -\tfrac{1}{4}, \tfrac{1}{4})\}.\end{aligned}$$

Proof. Let $T = (a, b, c, d) \in \mathcal{L}_s({}^3 l_\infty^2)$.

Claim: $T \in extB_{\mathcal{L}_s({}^3 l_\infty^2)}$ if and only if

$$(*)\qquad \begin{aligned}1 &= |T((1, 1), (1, 1), (1, 1))| = |T((1, -1), (1, 1), (1, 1))|\\ &= |T((1, -1), (1, -1), (1, 1))| = |T((1, -1), (1, -1), (1, -1))|.\end{aligned}$$

($\Leftarrow$): Let $T_1 = (a+\epsilon, b+\delta, c+\gamma, d+\rho), T_2 = (a-\epsilon, b-\delta, c-\gamma, d-\rho) \in \mathcal{L}_s({}^3 l_\infty^2)$ be such that $\|T_1\| = \|T_2\| = 1$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since, for $j = 1, 2$,

$$\begin{aligned}1 &\ge |T_j((1, 1), (1, 1), (1, 1))|,\\ 1 &\ge |T_j((1, -1), (1, 1), (1, 1))|,\\ 1 &\ge |T_j((1, -1), (1, -1), (1, 1))|,\end{aligned}$$

$$1 \geq |T_j((1,-1),(1,-1),(1,-1))|,$$

we have

$$\begin{aligned} 0 &= \epsilon + \delta + 3\gamma + 3\rho, \\ 0 &= \epsilon - \delta - 3\gamma + 3\rho, \\ 0 &= \epsilon + \delta - \gamma - \rho, \\ 0 &= \epsilon - \delta + \gamma - \rho, \end{aligned}$$

which show that $0 = \epsilon = \delta = \gamma = \rho$. Hence, T is extreme.

($\Rightarrow$): Otherwise. Then four cases may happen as follows:

(Case 1) $|T((1,1),(1,1),(1,1))| < 1$ or

(Case 2) $|T((1,-1),(1,1),(1,1))| < 1$ or

(Case 3) $|T((1,-1),(1,-1),(1,1))| < 1$ or

(Case 4) $|T((1,-1),(1,-1),(1,-1))| < 1$.

Case 1: $|T((1,1),(1,1),(1,1))| < 1$

Without loss of generality, we may assume that

$$\begin{aligned} 1 &= |T((1,-1),(1,1),(1,1))| \\ &= |T((1,-1),(1,-1),(1,1))| \\ &= |T((1,-1),(1,-1),(1,-1))|. \end{aligned}$$

Let $n \in \mathbb{N}$ be such that $a + b + 3c + 3d + \frac{8}{n} < 1$. Let $T_1 = (a + \frac{1}{n}, b + \frac{1}{n}, c + \frac{1}{n}, d + \frac{1}{n}), T_2 = (a - \frac{1}{n}, b - \frac{1}{n}, c - \frac{1}{n}, d - \frac{1}{n}) \in \mathcal{L}_s({}^3l_\infty^2)$. By Theorem 2.1, $\|T_1\| = \|T_2\| = 1$, which shows that T is not extreme. It is a contradiction. Similar to the Case 1, if (Case 2) or (Case 3) or (Case 4) is true, then we may show that T is not extreme. It is a contradiction. Therefore, we have shown the Claim. By $(*)$, we complete the proof. □

3. The exposed points of the unit ball of $\mathcal{L}_s({}^3l_\infty^2)$

Theorem 3.1. *Let $f \in \mathcal{L}_s({}^3l_\infty^2)^*$ and $\alpha = f(x_1y_1z_1)$, $\beta = f(x_2y_2z_2)$, $\gamma = f(x_2y_1z_1 + y_2x_1z_1 + z_2x_1y_1)$, $\delta = f(x_1y_2z_2 + y_1x_2z_2 + z_1x_2y_2)$. Then*

$$\begin{aligned} \|f\| = \ & \max\{|\alpha|, |\beta|, \tfrac{1}{2}|\alpha - \delta|, \tfrac{1}{2}|\beta - \gamma|, \tfrac{1}{4}(|\alpha + \delta| + |3\beta - \gamma|), \\ & \tfrac{1}{4}(|3\alpha - \delta| + |\beta + \gamma|)\}. \end{aligned}$$

Proof. It follows from Theorem 2.3 and the fact that

$$\|f\| = \max_{T \in extB_{\mathcal{L}({}^2l_\infty^2)}} |f(T)|.$$

□

Theorem 3.2 ([19, Theorem 2.3])**.** *Let E be a real Banach space such that $extB_E$ is finite. Suppose that $x \in extB_E$ satisfies that there exists an $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in extB_E \backslash \{\pm x\}$. Then $x \in expB_E$.*

Theorem 3.3. *Let $T = (a, b, c, d) \in \mathcal{L}_s({}^3l_\infty^2)$. Then $T \in expB_{\mathcal{L}_s({}^3l_\infty^2)}$ if and only if $(b, a, d, c) \in expB_{\mathcal{L}_s({}^3l_\infty^2)}$.*

Proof. Let

$$S((x_1,x_2),\ (y_1,y_2),(z_1,z_2)) := T((x_2,x_1),\ (y_2,y_1),(z_2,z_1)) \in \mathcal{L}_s({}^3l_\infty^2).$$

Then $S = (b,a,d,c)$. Note that $T \in expB_{\mathcal{L}_s({}^3l_\infty^2)}$ if and only if $(b,a,d,c) \in expB_{\mathcal{L}_s({}^3l_\infty^2)}$. $\square$

Now we are in position to describe all the exposed points of the unit ball of $\mathcal{L}_s({}^3l_\infty^2)$.

Theorem 3.4. $expB_{\mathcal{L}_s({}^3l_\infty^2)} = extB_{\mathcal{L}_s({}^3l_\infty^2)}$.

Proof. It is enough to show that $extB_{\mathcal{L}_s({}^3l_\infty^2)} \subset expB_{\mathcal{L}_s({}^3l_\infty^2)}$.

Claim: $T = (1,0,0,0)$ is exposed.

Let $f \in \mathcal{L}_s({}^3l_\infty^2)^*$ be such that $\alpha = 1, 0 = \beta = \delta = \gamma$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^3l_\infty^2)} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed. By Theorem 3.3, $(0,1,0,0)$ is exposed.

Claim: $T = (0, \frac{1}{2}, -\frac{1}{2}, 0)$ is exposed.

Let $f \in \mathcal{L}_s({}^3l_\infty^2)^*$ be such that $\alpha = 0 = \beta = \delta, \gamma = -2$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^3l_\infty^2)} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed. By Theorem 3.3, $(\frac{1}{2}, 0, 0, -\frac{1}{2})$ is exposed.

Claim: $T = (\frac{1}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{1}{4})$ is exposed.

Let $f \in \mathcal{L}_s({}^3l_\infty^2)^*$ be such that $\alpha = \frac{1}{2} = -\beta, \gamma = 1 = \delta$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^3l_\infty^2)} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed. By Theorem 3.3, $(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is exposed.

Claim: $T = (\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, \frac{1}{4})$ is exposed.

Let $f \in \mathcal{L}_s({}^3l_\infty^2)^*$ be such that $\alpha = \frac{1}{2} = \beta, -\gamma = 1 = \delta$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}_s({}^3l_\infty^2)} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed. By Theorem 3.3, $(\frac{3}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4})$ is exposed. $\square$

4. The smooth points of the unit ball of $\mathcal{L}_s({}^3l_\infty^2)$

Theorem 4.1. *Let $T = (a,b,c,d) \in \mathcal{L}_s({}^3l_\infty^2)$. Then $T \in smB_{\mathcal{L}_s({}^3l_\infty^2)}$ if and only if*

$$\begin{aligned}
&(|T((1,1),(1,1),(1,1))| = 1, |T((1,-1),(1,1),(1,1))| < 1,\\
&|T((1,-1),(1,-1),(1,1))| < 1, |T((1,-1),(1,-1),(1,-1))| < 1) \textit{ or}\\
&(|T((1,1),(1,1),(1,1))| < 1, |T((1,-1),(1,1),(1,1))| = 1,\\
&|T((1,-1),(1,-1),(1,1))| < 1, |T((1,-1),(1,-1),(1,-1))| < 1) \textit{ or}\\
&(|T((1,1),(1,1),(1,1))| < 1, |T((1,-1),(1,1),(1,1))| < 1,\\
&|T((1,-1),(1,-1),(1,1))| = 1, |T((1,-1),(1,-1),(1,-1))| < 1) \textit{ or}\\
&(|T((1,1),(1,1),(1,1))| < 1, |T((1,-1),(1,1),(1,1))| < 1, |T((1,-1),\\
&(1,-1),(1,1))| < 1, |T((1,-1),(1,-1),(1,-1))| = 1).
\end{aligned}$$

Proof.

Case 1 : $|T((1,1),(1,1),(1,1))| = 1,\ |T((1,-1),(1,1),(1,1))| < 1,$
$|T((1,-1),(1,-1),(1,1))| < 1,\ |T((1,-1),(1,-1),(1,-1))| < 1.$

Let $l := T((1,1),(1,1),(1,1)) = a+b+3c+3d$ for some $l \in \{1,-1\}$. Let $f \in \mathcal{L}_s({}^3l_\infty^2)^*$ be such that $f(T) = 1 = \|f\|$ with $\alpha = f(x_1y_1z_1), \beta = f(x_2y_2z_2), \gamma = f(x_2y_1z_1 + y_2x_1z_1 + z_2x_1y_1), \delta = f(x_1y_2z_2 + y_1x_2z_2 + z_1x_2y_2)$. We will show that $\alpha = l = \beta, \gamma = 3l = \delta$. Let $n \in \mathbb{N}$ be such that

$$|a+3d-b-3c| + \frac{6}{n} < 1, |a-d+b-c| + \frac{4}{3n} < 1, |a-d-b+c| + \frac{2}{n} < 1.$$

By Theorem 2.1,

$$1 = \|(a \pm \frac{1}{n}, b \mp \frac{1}{n}, c, d)\| = \|(a, b, c \pm \frac{1}{n}, d \mp \frac{1}{n})\| = \|(a \pm \frac{1}{n}, b, c \mp \frac{1}{3n}, d)\|.$$

Since

$$1 \geq |f((a \pm \frac{1}{n}, b \mp \frac{1}{n}, c, d))| = 1 \pm \frac{1}{n}(\alpha - \beta),$$

so, $\alpha = \beta$. Since

$$1 \geq |f((a, b, c \pm \frac{1}{n}, d \mp \frac{1}{n}))| = 1 \pm \frac{1}{n}(\gamma - \delta),$$

so, $\gamma = \delta$. Since

$$1 \geq |f((a \pm \frac{1}{n}, b, c \mp \frac{1}{3n}, d))| = 1 \pm \frac{1}{n}(\alpha - \frac{1}{3}\gamma),$$

so, $\alpha = \frac{1}{3}\gamma$. Therefore,

$$1 = f(T) = a\alpha + b\beta + c\gamma + d\delta = (a+b+3c+3d)\alpha = l\alpha,$$

hence, $\alpha = l = \beta, \gamma = 3l = \delta$. Therefore, $T \in smB_{\mathcal{L}_s({}^3l_\infty^2)}$.

Case 2 : $|T((1,1),(1,1),(1,1))| < 1,\ |T((1,-1),(1,1),(1,1))| = 1,$
$|T((1,-1),(1,-1),(1,1))| < 1,\ |T((1,-1),(1,-1),(1,-1))| < 1.$

Let $f \in \mathcal{L}_s({}^3l_\infty^2)^*$ be such that $f(T) = 1 = \|f\|$ with $\alpha = f(x_1y_1z_1), \beta = f(x_2y_2z_2), \gamma = f(x_2y_1z_1 + y_2x_1z_1 + z_2x_1y_1), \delta = f(x_1y_2z_2 + y_1x_2z_2 + z_1x_2y_2)$.

By the similar argument in the Case 1, we show that $\alpha = l = -\beta = \gamma = -\delta$. Therefore, $T \in smB_{\mathcal{L}_s({}^3l_\infty^2)}$.

Case 3 : $|T((1,1),(1,1),(1,1))| < 1,\ |T((1,-1),(1,1),(1,1))| < 1,$
$|T((1,-1),(1,-1),(1,1))| = 1,\ |T((1,-1),(1,-1),(1,-1))| < 1.$

Let $f \in \mathcal{L}_s({}^3l_\infty^2)^*$ be such that $f(T) = 1 = \|f\|$ with $\alpha = f(x_1y_1z_1), \beta = f(x_2y_2z_2), \gamma = f(x_2y_1z_1 + y_2x_1z_1 + z_2x_1y_1), \delta = f(x_1y_2z_2 + y_1x_2z_2 + z_1x_2y_2)$.

By the similar argument in the Case 1, we show that $\alpha = l = \beta = -\gamma = -\delta$. Therefore, $T \in smB_{\mathcal{L}_s({}^3l_\infty^2)}$.

Case 4 : $|T((1,1),(1,1),(1,1))| < 1,\ |T((1,-1),(1,1),(1,1))| < 1,$
$|T((1,-1),(1,-1),(1,1))| < 1,\ |T((1,-1),(1,-1),(1,-1))| = 1.$

Let $f \in \mathcal{L}_s({}^3l_\infty^2)^*$ be such that $f(T) = 1 = \|f\|$ with $\alpha = f(x_1y_1z_1), \beta = f(x_2y_2z_2), \gamma = f(x_2y_1z_1 + y_2x_1z_1 + z_2x_1y_1), \delta = f(x_1y_2z_2 + y_1x_2z_2 + z_1x_2y_2)$.

By the similar argument in the Case 1, we show that $\alpha = l = -\beta, \gamma = -3l = -\delta$. Therefore, $T \in smB_{\mathcal{L}_s({}^3l_\infty^2)}$. □

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