Commun. Korean Math. Soc. **32** (2017), No. 4, pp. 971–977
https://doi.org/10.4134/CKMS.c170006
pISSN: 1225-1763 / eISSN: 2234-3024

INCLUSION PROPERTIES REGARDING CLASSES OF MEROMORPHIC P-VALENT FUNCTIONS, INVOLVING THE OPERATOR $J^n_{p,\lambda}$

Petrică Dicu and Alina Totoi

Abstract. For $p \in \mathbb{N}^*$ let $\Sigma_{p,0}$ denote the class of meromorphic functions of the form $g(z) = \frac{1}{z^p} + a_0 + a_1 z + \cdots$, $z \in U$. In the present paper we introduce a new subclass of the class $\Sigma_{p,0}$, using the subordination and the operator $J^n_{p,\lambda}$. This class will be denoted by $B^n_{p,\lambda}(\alpha, h)$ and we study some inclusion properties of this subclass.

1. Introduction and preliminaries

Let $U = \{z \in \mathbb{C}/\ |z| < 1\}$ be the unit disc in the complex plane and $\dot{U} = U \setminus \{0\}$ the punctured disc.

We consider the sets of functions $H(U) = \{f : U \to \mathbb{C}/f$ is holomorphic in $U\}$ and $H_u(U) = \{f \in H(U)/f$ is univalent in $U\}$.

For $p \in \mathbb{N}$, $p \neq 0$, let Σ_p denote the class of meromorphic p-valent functions of the form

$$g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots,\ z \in U,\ a_{-p} \neq 0,$$

and $\Sigma_{p,0} = \{g \in \Sigma_p : a_{-p} = 1\}$.

For $n \in \mathbb{Z}$, $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$, let us consider, on the class Σ_p, the operator $J^n_{p,\lambda} : \Sigma_p \to \Sigma_p$, defined as

$$J^n_{p,\lambda} g(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\lambda - p}{k + \lambda}\right)^n a_k z^k, \text{ where } g(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k.$$

This operator was introduced for the first time by Alina Totoi in [7].

Obviously, we also have $J^n_{p,\lambda} : \Sigma_{p,0} \to \Sigma_{p,0}$.

We have the next properties for $J^n_{p,\lambda}$, when $\operatorname{Re}\lambda > p$:

(1) $J^0_{p,\lambda} g(z) = g(z),\ g \in \Sigma_p$;

(2) $J^1_{p,\lambda} g(z) = \frac{\lambda - p}{z^\lambda} \int_0^z t^{\lambda-1} g(t)dt = J_{p,\lambda} g(z),\ g \in \Sigma_p$;

Received January 5, 2017; Accepted April 26, 2017.
2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. meromorphic functions, multiplier transformations, subordination.

(3) $J^n_{p,\lambda}(J^m_{p,\lambda}g(z)) = J^{n+m}_{p,\lambda}g(z)$, $n, m \in \mathbb{Z}$, $g \in \Sigma_p$;
(4) $J^n_{p,\gamma}(J^m_{p,\lambda}g(z)) = J^m_{p,\lambda}(J^n_{p,\gamma}g(z))$, $n, m \in \mathbb{Z}$, $g \in \Sigma_p$, $\gamma > p$;
(5) $J^n_{p,\lambda}(g_1 + g_2)(z) = J^n_{p,\lambda}g_1(z) + J^n_{p,\lambda}g_2(z)$ for $g_1, g_2 \in \Sigma_p$, $n \in \mathbb{Z}$;
(6) $J^n_{p,\lambda}(cg)(z) = cJ^n_{p,\lambda}g(z)$, $c \in \mathbb{C}^*$, $n \in \mathbb{Z}$;
(7) $J^n_{p,\lambda}(zg'(z)) = z(J^n_{p,\lambda}g(z))' = (\lambda - p)J^{n-1}_{p,\lambda}g(z) - \lambda J^n_{p,\lambda}g(z)$, $n \in \mathbb{Z}$, $g \in \Sigma_p$.

Remark 1.1. (1) When $\lambda = 2$ and $p = 1$, we have

$$J^n_{1,2}g(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty}(k+2)^{-n}a_k z^k,$$

and this operator was studied by Cho and Kim [1] for $n \in \mathbb{Z}$ and by Uralegaddi and Somanatha [8] for $n < 0$.

(2) We also have the relation

$$z^2 J^n_{1,2}g(z) = D^n(z^2 g(z)),\ g \in \Sigma_{1,0}\ ,$$

where D^n is the well-known Sălăgean differential operator of order n [5], defined by $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$.

(3) $J^n_{p,\lambda}$ is an extension to the meromorphic functions of the operator K^n_p, defined on $A(p) = \{f \in H(U) : f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}\}$, introduced in [6]. Also, for $n \geq 0$ we find that K^n_p is the Komatu linear operator, defined in [3].

(4) It's easy to see that $J^n_{p,\lambda}$ with $n > 0$ is an integral operator while $J^{-n}_{p,\lambda}$, $n > 0$ is a differential operator with the property $J^{-n}_{p,\lambda}(J^n_{p,\lambda}g(z)) = g(z)$.

Similar operators are also used in [2].

Definition 1.1 ([4])**.** Let f and F be members of $H(U)$. The function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$.

Lemma 1.1 ([4])**.** *Let $f \in H(U)$ and $h \in H_u(U)$ convex in U, with $h(0) = f(0)$. If*

$$f(z) + \frac{1}{\mu}zf'(z) \prec h(z),$$

where $\operatorname{Re}\mu \geq 0$ *and* $\mu \neq 0$, *then* $f(z) \prec h(z)$.

2. Main results

Definition 2.1. For $p \in \mathbb{N}^*$, $n \in \mathbb{Z}$, λ, $\alpha \in \mathbb{C}$, with $\operatorname{Re}\lambda > p$, and $h \in H_u(U)$ convex in U with $h(0) = 1$, we define

$$B^n_{p,\lambda}(\alpha, h) = \left\{g \in \Sigma_{p,0} : z^p J^n_{p,\lambda}\left((1-\alpha)\,g(z) - \frac{\alpha}{p}zg'(z)\right) \prec h(z),\ z \in U\right\}.$$

Remark 2.1. Let $p \in \mathbb{N}^*$, $n \in \mathbb{Z}$, λ, $\alpha \in \mathbb{C}$, with $\operatorname{Re}\lambda > p$ and $h \in H_u(U)$ convex in U with $h(0) = 1$.

1. We have $B^n_{p,\lambda}(\alpha, h) \neq \emptyset$, since $g(z) = \frac{1}{z^p} \in B^n_{p,\lambda}(\alpha, h)$.
2. For every $g \in B^n_{p,\lambda}(\alpha, h)$, we have

$$z^p J^n_{p,\lambda}\left((1-\alpha)\, g(z) - \frac{\alpha}{p} z g'(z)\right)\big|_{z=0} = 1.$$

3. From the properties of the operator $J^n_{p,\lambda}$ we get

$$J^n_{p,\lambda}\left((1-\alpha)\, g(z) - \frac{\alpha}{p} z g'(z)\right) = (1-\alpha)\, J^n_{p,\lambda} g(z) - \frac{\alpha}{p} J^n_{p,\lambda}\left(z g'(z)\right).$$

4. Let $h_1, h_2 \in H_u(U)$ convex in U with $h_1(0) = h_2(0) = 1$, $h_1 \prec h_2$. It is obvious that we have

$$B^n_{p,\lambda}(\alpha, h_1) \subset B^n_{p,\lambda}(\alpha, h_2).$$

Theorem 2.1. *Let* $\alpha_2 < \alpha_1 \leq 0$. *Then*

$$B^n_{p,\lambda}(\alpha_2, h) \subset B^n_{p,\lambda}(\alpha_1, h).$$

Proof. Let $g \in B^n_{p,\lambda}(\alpha_2, h)$. We have

$$z^p J^n_{p,\lambda}\left((1-\alpha_2)\, g(z) - \frac{\alpha_2}{p} z g'(z)\right) \prec h(z),\ z \in U,$$

which is equivalent to

$$z^p\,(1-\alpha_2)\, J^n_{p,\lambda} g(z) - z^p \frac{\alpha_2}{p} J^n_{p,\lambda}\left(z g'(z)\right) \prec h(z).$$

Because $J^n_{p,\lambda}(z g'(z)) = z (J^n_{p,\lambda} g(z))'$, we obtain

$$z^p\,(1-\alpha_2)\, J^n_{p,\lambda} g(z) - z^{p+1} \frac{\alpha_2}{p} (J^n_{p,\lambda} g(z))' \prec h(z). \tag{1}$$

Suppose that

$$f(z) = z^p J^n_{p,\lambda} g(z). \tag{2}$$

It is easy to see that the function $f(z)$ is analytic in U with $f(0) = 1$. Differentiating both sides of (2) with respect to z, we get

$$f'(z) = p z^{p-1} J^n_{p,\lambda} g(z) + z^p \left(J^n_{p,\lambda} g(z)\right)'.$$

We have now

$$f(z) - \frac{\alpha_2}{p} z f'(z) = z^p\,(1-\alpha_2)\, J^n_{p,\lambda} g(z) - z^{p+1} \frac{\alpha_2}{p} (J^n_{p,\lambda} g(z))' \tag{3}$$

From (1) and (3) we obtain

$$f(z) - \frac{\alpha_2}{p} z f'(z) \prec h(z).$$

Since $\frac{\alpha_2}{p} < 0$, using Lemma 1.1 for the equality written above we get $f(z) \prec h(z)$, which means that

$$z^p J^n_{p,\lambda} g(z) \prec h(z). \tag{4}$$

We want to verify the fact that $g \in B^n_{p,\lambda}(\alpha_1, h)$, this meaning that

$$z^p J^n_{p,\lambda}\left((1-\alpha_1)\, g(z) - \frac{\alpha_1}{p} z g'(z)\right) \prec h(z),\ z \in U,$$

which is equivalent to

$$z^p\,(1-\alpha_1)\, J^n_{p,\lambda} g(z) - z^{p+1}\frac{\alpha_1}{p}(J^n_{p,\lambda} g(z))' \prec h(z). \tag{5}$$

It is not difficult to see that we have

$$\begin{aligned} &z^p\,(1-\alpha_1)\, J^n_{p,\lambda} g(z) - z^{p+1}\frac{\alpha_1}{p}(J^n_{p,\lambda} g(z))' \\ &= \frac{\alpha_1}{\alpha_2}\left(z^p\,(1-\alpha_2)\, J^n_{p,\lambda} g(z) - z^{p+1}\frac{\alpha_2}{p}(J^n_{p,\lambda} g(z))'\right) + \left(1-\frac{\alpha_1}{\alpha_2}\right) z^p J^n_{p,\lambda} g(z). \end{aligned} \tag{6}$$

Since $0 \le \frac{\alpha_1}{\alpha_2} < 1$ and $h \in H_u(U)$ convex, it follows from (1) and (4) that

$$\frac{\alpha_1}{\alpha_2}\left(z^p\,(1-\alpha_2)\, J^n_{p,\lambda} g(z) - z^{p+1}\frac{\alpha_2}{p}(J^n_{p,\lambda} g(z))'\right) + \left(1-\frac{\alpha_1}{\alpha_2}\right) z^p J^n_{p,\lambda} g(z) \prec h(z),$$

so

$$z^p\,(1-\alpha_2)\, J^n_{p,\lambda} g(z) - z^{p+1}\frac{\alpha_2}{p}(J^n_{p,\lambda} g(z))' \prec h(z).$$

Thus $g \in B^n_{p,\lambda}(\alpha_1, h)$ and the proof of Theorem 2.1 is completed. □

The following result gives a connection between the sets $B^n_{p,\lambda}(\alpha, h)$ and $B^{n-1}_{p,\lambda}(\alpha, h)$.

Theorem 2.2. *Let* $p \in \mathbb{N}^*$, $n \in \mathbb{Z}$, λ, $\alpha \in \mathbb{C}$, *with* $\operatorname{Re}\lambda > p$ *and* $h \in H_u(U)$ *convex in* U *with* $h(0) = 1$. *Then*

$$g \in B^n_{p,\lambda}(\alpha, h) \Leftrightarrow J_{p,\lambda}(g) \in B^{n-1}_{p,\lambda}(\alpha, h),$$

where $J_{p,\lambda}(g)(z) = \frac{\lambda-p}{z^\lambda}\int_0^z t^{\lambda-1} g(t)dt$.

Proof. Let be $g \in B^n_{p,\lambda}(\alpha, h)$ and consider $G = J_{p,\lambda}(g)$.

We have $G \in B^{n-1}_{p,\lambda}(\alpha, h)$ if and only if

$$z^p J^{n-1}_{p,\lambda}\left((1-\alpha)\, G(z) - \frac{\alpha}{p} z G'(z)\right) \prec h(z),\ z \in U,$$

which is equivalent to

$$z^p\,(1-\alpha)\, J^{n-1}_{p,\lambda} G(z) - z^p\frac{\alpha}{p} J^{n-1}_{p,\lambda}\left(zG'(z)\right) \prec h(z).$$

Because $J^{n-1}_{p,\lambda}(zG'(z)) = z(J^{n-1}_{p,\lambda} G(z))'$, we obtain

$$z^p\,(1-\alpha)\, J^{n-1}_{p,\lambda} G(z) - z^{p+1}\frac{\alpha}{p}(J^{n-1}_{p,\lambda} G(z))' \prec h(z). \tag{7}$$

Using the fact that $J^{n-1}_{p,\lambda}(J^1_{p,\lambda}(g)) = J^n_{p,\lambda}(g)$ and knowing that $J^1_{p,\lambda}(g) = J_{p,\lambda}(g)$, we obtain

$$J^{n-1}_{p,\lambda}(G) = J^{n-1}_{p,\lambda}\left(J_{p,\lambda}(g)\right) = J^{n-1}_{p,\lambda}\left(J^1_{p,\lambda}(g)\right) = J^n_{p,\lambda}(g).$$

From $J_{p,\lambda}^{n-1}(G) = J_{p,\lambda}^{n}(g)$ and (7) we deduce that $G \in B_{p,\lambda}^{n-1}(\alpha, h)$ if and only if

$$z^p\,(1-\alpha)\,J_{p,\lambda}^n g(z) - z^{p+1}\frac{\alpha}{p}(J_{p,\lambda}^n g(z))' \prec h(z). \tag{8}$$

It is easy to see that equality (8) is equivalent with

$$z^p J_{p,\lambda}^n\left((1-\alpha)\,g(z) - \frac{\alpha}{p}zg'(z)\right) \prec h(z),\ z \in U,$$

this meaning that

$$g \in B_{p,\lambda}^n(\alpha,h) \Leftrightarrow G = J_{p,\lambda}(g) \in B_{p,\lambda}^{n-1}(\alpha,h). \qquad \square$$

Theorem 2.3. *Let $p \in \mathbb{N}^*$, $n \in \mathbb{Z}$, $\lambda,\ \alpha, \gamma \in \mathbb{C}$, with $\operatorname{Re}\lambda > p$ and $\operatorname{Re}\gamma > p$. Let us consider $h \in H_u(U)$, convex in U, with $h(0) = 1$. Then*

$$g \in B_{p,\lambda}^n(\alpha,h) \Rightarrow G = J_{p,\gamma}(g) \in B_{p,\lambda}^n(\alpha,h).$$

Proof. Let be $g \in B_{p,\lambda}^n(\alpha,h)$ and $G = J_{p,\gamma}(g)$ with

$$J_{p,\gamma}(g)(z) = \frac{\gamma - p}{z^\gamma}\int_0^z t^{\gamma-1}g(t)dt.$$

We have $g \in B_{p,\lambda}^n(\alpha,h)$ if and only if

$$z^p J_{p,\lambda}^n\left((1-\alpha)\,g(z) - \frac{\alpha}{p}zg'(z)\right) \prec h(z),\ z \in U.$$

We denote by

$$u(z) = J_{p,\lambda}^n\left((1-\alpha)\,g(z) - \frac{\alpha}{p}zg'(z)\right),\ z \in \dot{U}, \tag{9}$$

and we get

$$g \in B_{p,\lambda}^n(\alpha,h) \Leftrightarrow z^p u(z) \prec h(z),\ z \in U. \tag{10}$$

We must prove that $G = J_{p,\gamma}(g) \in B_{p,\lambda}^n(\alpha,h)$.

We have $G = J_{p,\gamma}(g) \in B_{p,\lambda}^n(\alpha,h)$ if and only if

$$z^p J_{p,\lambda}^n\left((1-\alpha)\,J_{p,\gamma}g(z) - \frac{\alpha}{p}z(J_{p,\gamma}g)'(z)\right) \prec h(z),\ z \in U.$$

From the above subordination, using now the properties of the operator $J_{p,\gamma}$, we get

$$z^p J_{p,\lambda}^n\left(J_{p,\gamma}\left((1-\alpha)g(z) - \frac{\alpha}{p}zg'(z)\right)\right) \prec h(z),\ z \in U,$$

which is equivalent to

$$z^p J_{p,\gamma}\left(J_{p,\gamma}^n\left((1-\alpha)g(z) - \frac{\alpha}{p}zg'(z)\right)\right) \prec h(z),\ z \in U. \tag{11}$$

Using (9), the last subordination is equivalent to $z^p J_{p,\gamma}(u)(z) \prec h(z)$, this meaning that $G = J_{p,\gamma}(g) \in B_{p,\lambda}^n(\alpha,h)$ if and only if $z^p J_{p,\gamma}(u)(z) \prec h(z)$.

Let us denote $J_{p,\gamma}u$ by $\mathbf{U}$. It is easy to see that

$$\gamma\mathbf{U}(z) + z\mathbf{U}'(z) = (\gamma - p)u(z),\ z \in \dot{U}. \tag{12}$$

From (12) we obtain

$$z^p\gamma\mathbf{U}(z) + z^{p+1}\mathbf{U}'(z) = z^p(\gamma - p)u(z),\ z \in U,$$

which is equivalent to

$$z^p\mathbf{U}(z) + \frac{p}{\gamma - p}z^p\mathbf{U}(z) + \frac{1}{\gamma - p}z^{p+1}\mathbf{U}'(z) = z^p u(z),\ z \in U. \tag{13}$$

If we denote $\mathbf{V}(z) = z^p\mathbf{U}(z)$, we have

$$\mathbf{V}'(z) = \frac{p}{\gamma - p}z^p\mathbf{U}(z) + \frac{1}{\gamma - p}z^{p+1}\mathbf{U}'(z),$$

therefore, from (13), we obtain the equality

$$\mathbf{V}(z) + z\frac{1}{\gamma - p}\mathbf{V}'(z) = z^p u(z),\ z \in U.$$

From (10) we know that we have $z^p u(z) \prec h(z)$, $z \in U$, this meaning that we get the subordination

$$\mathbf{V}(z) + z\frac{1}{\gamma - p}\mathbf{V}'(z) \prec h(z),\ z \in U.$$

Since $\operatorname{Re}(\gamma - p) > 0$, from the above subordination, using Lemma1.1, we get

$$\mathbf{V}(z) = z^p\mathbf{U}(z) \prec h(z),\ z \in U.$$

Therefore, we get $z^p J_{p,\gamma}(u)(z) \prec h(z)$, $z \in U$, this meaning that

$$G = J_{p,\gamma}(g) \in B^n_{p,\lambda}(\alpha, h).$$ □

References

[1] N. E. Cho and J. A. Kim, *On certain classes of meromorphically starlike functions*, Internat. J. Math. & Math. Sci. **18** (1995), no. 3, 463–468.

[2] L. Cotârlă, *Properties of analytic functions defined by an integral operator*, Demonstratio Math., Polonia **2010** (2010), no. 4, 799–803.

[3] Y. Komatu, *Distorsion Theorems in Relation to Linear Integral Transforms*, Kluwer Academic Publishers, Dordrecht, Boston and London, 1996.

[4] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, Marcel Dekker Inc. New York, Basel, 2000.

[5] G. Şt. Sălăgean, *Subclasses of univalent functions*, Lectures Notes in Math., 1013, 362–372, Springer-Verlag, Heideberg, 1983.

[6] S. Shams, S. R. Kulkarni, and J. M. Jahangiri, *Subordination properties of p-valent functions defined by integral operators*, Int. J. Math. Math. Sci. **2006** (2006), Article ID 94572, 1–3.

[7] A. Totoi, *Meromorphic functions defined by a multiplier transformation*, Acta Univ. Apulensis Math. Inform. **25** (2011), 41–52.

[8] B. A. Uralegaddi and C. Somanatha, *Certain differential operators for meromorphic functions*, Houston J. Math. **17** (1991), no. 2, 279–284.

Petrică Dicu
Department of Mathematics
Faculty of Science
University "Lucian Blaga" of Sibiu
Str. Dr. Ion Raţiu, nr. 5-7, 550012-Sibiu, Romania
E-mail address: petrica.dicu@yahoo.com, petrica.dicu@ulbs.ro

Alina Totoi
Department of Mathematics
Faculty of Science
University "Lucian Blaga" of Sibiu
Str. Dr. Ion Raţiu, nr. 5-7, 550012-Sibiu, Romania
E-mail address: totoialina@yahoo.com