Commun. Korean Math. Soc. **32** (2017), No. 4, pp. 933–957
https://doi.org/10.4134/CKMS.c170002
pISSN: 1225-1763 / eISSN: 2234-3024

GENERALIZED BI-QUASI-VARIATIONAL-LIKE INEQUALITIES ON NON-COMPACT SETS

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Abstract. In this paper, we prove some existence results of solutions for a new class of generalized bi-quasi-variational-like inequalities (GBQVLI) for (η-h)-quasi-pseudo-monotone type I and strongly (η-h)-quasi-pseudo-monotone type I operators defined on non-compact sets in locally convex Hausdorff topological vector spaces. To obtain our results on GBQVLI for (η-h)-quasi-pseudo-monotone type I and strongly (η-h)-quasi-pseudo-monotone type I operators, we use Chowdhury and Tan's generalized version of Ky Fan's minimax inequality as the main tool.

1. Introduction

Let E, F be topological spaces and let $g : E \to 2^F$ be a multi-valued mapping.

The mapping g is said to be *upper semi-continuous* on E if, for all $x_0 \in E$ and for each open set G in F with $g(x_0) \subset G$, there exists an open neighborhood $N(x_0)$ of x_0 such that $g(x) \subset G$ for all $x \in N(x_0)$. The mapping g is said to be *lower semi-continuous* on E if, for all $x_0 \in E$ and for each open set G in F with $g(x_0) \cap G \neq \emptyset$, there exists an open neighborhood $N(x_0)$ of x_0 such that $g(x) \cap G \neq \emptyset$ for all $x \in N(x_0)$. The mapping g is said to be *continuous* on E if g is both upper semi-continuous and lower semi-continuous on E.

Note that a multi-valued mapping g is upper semi-continuous (resp., lower semi-continuous) if the inverse image of a closed set (resp., an open set) is closed (resp., open), where, if $A \subset E$, then the set

$$g(A) = \cup_{x \in A} g(x) = \{y \in F : g^{-1}(y) \cap A \neq \emptyset\}$$

is called the *image* of A under g. If $B \subset F$, the set

$$g^{-1}(B) = \cup_{y \in B} g^{-1}(y) = \{x \in E : g(x) \cap B \neq \emptyset\}$$

Received January 1, 2017; Revised April 3, 2017; Accepted April 26, 2017.
2010 *Mathematics Subject Classification.* 49J30, 47H10, 47H17, 49M05.
Key words and phrases. generalized bi-quasi-variational-like inequalities, (η-h)-quasi-pseudomonotone type I operators, locally convex Hausdorff topological vector spaces.
The first author was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2013053358).

is called the *inverse image* of B under g.

Let E be a topological vector space over the field Φ, F be a vector space over Φ and $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional. For each $x_0 \in E$, for each nonempty subset A of E and $\varepsilon > 0$, let

$$W(x_0; \varepsilon) = \{y \in F : |\langle y, x_0 \rangle| < \varepsilon\}$$

and

$$U(A; \varepsilon) = \{y \in F : \sup_{x \in A} |\langle y, x \rangle| < \varepsilon\}.$$

Let $\sigma\langle F, E \rangle$ be the topology on F generated by the family

$$\{W(x_0; \varepsilon) : x \in E,\, \varepsilon > 0\}$$

as a subbase for the neighborhood system at 0 and let $\delta\langle F, E \rangle$ be the topology on F generated by the family

$$\{U(A; \varepsilon) : A \text{ is a nonempty compact subset of } E,\, \varepsilon > 0\}$$

as a base for the neighborhood system at

We note then that F, when equipped with the topology $\sigma\langle F, E \rangle$ or the topology $\delta\langle F, E \rangle$, becomes a locally convex topological vector space, but not necessarily a Hausdorff topological vector space. Furthermore, for a net $\{y_\alpha\}$ in F and $y \in F$, we have the following:

(1) $y_\alpha \to y$ in $\sigma\langle F, E \rangle$ if and only if $\langle y_\alpha, x \rangle \to \langle y, x \rangle$ for each $x \in E$;

(2) $y_\alpha \to y$ in $\delta\langle F, E \rangle$ if and only if $\langle y_\alpha, x \rangle \to \langle y, x \rangle$ uniformly for each $x \in A$, where A is a nonempty compact subset of E.

Definition 1.1. Let X be a nonempty subset of E. A mapping $T : X \to 2^F$ is said to be *monotone* with respect to the bilinear functional $\langle \cdot, \cdot \rangle$ if, for any $x, y \in X$, $\forall u \in T(x)$ and $\forall w \in T(y)$,

$$Re\langle w - u, y - x \rangle \geq 0.$$

Remark 1.1. (1) When $F = E^*$, the vector space of all continuous linear functionals on E, and $\langle \cdot, \cdot \rangle$ is the usual pairing between E^* and E, then the monotonicity notion coincides with the usual definition, i.e.,

$$Re\langle Ty - Tx, y - x \rangle \geq 0$$

for any $x, y \in X$, when $T : X \to E^*$ is single-valued, and

$$Re\langle w - u, y - x \rangle \geq 0$$

for any $x, y \in X$, $\forall u \in T(x)$ and $\forall w \in T(y)$, when $T : X \to 2^{E^*}$ is set-valued.

(2) A mapping $T : X \to 2^F$ is monotone if and only if its graph $G(T) = \{(x, y) : y \in T(x)\}$ is a monotone subset of $X \times F$, i.e., for all $(x_1, y_1), (x_2, y_2) \in G(T)$,

$$Re\langle y_2 - y_1, x_2 - x_1 \rangle \geq 0.$$

In 1989, Shih and Tan [30] introduced the following problem:
Let E and F be vector spaces over Φ, $\langle\cdot,\cdot\rangle : F\times E \to \Phi$ be a bilinear functional and X be a nonempty subset of E.

If $S : X \to 2^X$ and $M, T : X \to 2^F$, then the *generalized bi-quasi-variational inequality problem* (GBQVI) for the triple (S, M, T) is as follows:

Find $\hat{y} \in X$ such that

(1) $\hat{y} \in S(\hat{y})$;

(2) $\inf_{w\in T(\hat{y})} Re\langle f - w, \hat{y} - x\rangle \leq 0$ for all $x \in S(\hat{y})$ and $f \in M(\hat{y})$.

If T is a single-valued mapping, then a generalized bi-quasi-variational inequality problem will be called a *bi-quasi-variational inequality problem.*

We have the following special cases of the problem (GBQVI):

Suppose the E is a topological vector space, $F = E^*$, the vector space of all continuous linear functionals on E and $\langle\cdot,\cdot\rangle$ is the usual duality pairing between E^* and E.

(I) If $T = 0$, then a generalized bi-quasi-variational inequality problem for $(S, M, 0)$ becomes a generalized quasi-variational inequality problem:

Find $\hat{y} \in X$ such that

(1) $\hat{y} \in S(\hat{y})$;

(2) $Re\langle f, \hat{y} - x\rangle \leq 0$ for all $x \in S(\hat{y})$ and $f \in M(\hat{y})$.

This problem was studied by Chan and Pang [7] in the finite-dimensional case and, by Shih and Tan [31], in the infinite-dimensional case.

(II) If $T = 0$ and M is single-valued, then a generalized bi-quasi-variational inequality problem for $(S, M, 0)$ becomes a *quasi-variational inequality problem*:

Find $\hat{y} \in S(\hat{y})$ such that

$$Re\langle M(\hat{y}), \hat{y} - x\rangle \leq 0$$

for all $x \in S(\hat{y})$.

This problem was introduced by Bensoussan and Lions in 1973 in connection with impulse control (see Aubin [1], Baiocchi and Capelo [3], Bensoussan and Lions [4]).

(III) If $S(x) = X$, $M = 0$ and T is single-valued, then a generalized bi-quasi-variational inequality problem becomes a *variational inequality problem*:

Find $\hat{y} \in X$ such that

$$Re\langle T(\hat{y}), \hat{y} - x\rangle \geq 0$$

for all $x \in X$.

This problem was introduced by Stampacchia [32].

(IV) If $S(x) = X$ and $M = 0$, then a generalized bi-quasi-variational inequality problem becomes a *generalized variational inequality problem*:

Find $\hat{y} \in S(\hat{y})$ and $w \in T(\hat{y})$ such that

$$Re\langle w, \hat{y} - x\rangle \leq 0$$

for all $x \in S(x)$.

This problem was studied by Browder [6] and Yen [34].

Also, Shih and Tan proved the following theorems:

Theorem 1.1. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty compact convex subset of E and F be a topological vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that*

(a) $S : X \to 2^X$ *is an upper semi-continuous mapping such that each $S(x)$ is closed convex*;

(b) $M : X \to 2^F$ *is a monotone mapping with respect to $\langle \cdot, \cdot \rangle$*;

(c) $T : X \to 2^F$ *is an upper semi-continuous mapping such that each $T(x)$ is compact*;

(d) *the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} \sup_{f \in M(x)} \inf_{w \in T(y)} Re\langle f - w, y - x \rangle > 0\}$$

is open in X.

Then there exists a point $\hat{y} \in X$ such that

(1) $\hat{y} \in S(\hat{y})$;

(2) $\inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle \leq 0$ *for all $x \in S(\hat{y})$ and $f \in M(x)$.*

In addition, if M is lower semi-continuous along the line segments in X to the topology $\sigma\langle F, E \rangle$ on F, then

(3) $\inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle \leq 0$ *for all $x \in S(\hat{y})$ and $f \in M(\hat{y})$.*

Moreover, if $S(x) = X$ for all $x \in X$, then E is not required to be locally convex and, if $T \equiv 0$, then the continuity assumption on $\langle \cdot, \cdot \rangle$ can be weakened to the assumption that, for each $f \in F$, the mapping $x \mapsto \langle f, x \rangle$ is continuous on X.

Theorem 1.2. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty compact convex subset of E and F be a topological vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional which is continuous on X and let F equip with the topology $\delta\langle F, E \rangle$. Suppose that*

(a) $S : X \to 2^X$ *is an upper semi-continuous mapping such that each $S(x)$ is closed convex*;

(b) $M : X \to 2^F$ *is a monotone mapping with respect to $\langle \cdot, \cdot \rangle$ and lower semi-continuous*;

(c) $T : X \to 2^F$ *is an upper semi-continuous mapping such that each $T(x)$ is compact.*

Then there exists a point $\hat{y} \in X$ such that

(1) $\hat{y} \in S(\hat{y})$;

(2) $\inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle \leq 0$ *for all $x \in S(\hat{y})$ and $f \in M(\hat{y})$.*

Remark 1.2. Since the results of Shih and Tan, some authors have obtained many results on generalized (quasi-)variational inequalities, generalized (quasi-)variational-like inequalities and generalized bi-quasi-variational inequalities in topological vector spaces (see [9–24]).

In this paper, we obtain some existence results for solutions for a new class of generalized bi-quasi-variational-like inequalities (GBQVLI) for (η, h)-quasi-pseudo-monotone type I and strongly (η, h)-quasi-pseudomonotone type I operators defined on non-compact sets in locally convex Hausdorff topological vector spaces. In fact, the generalized bi-quasi-variational-like inequalities (GBQVLI) are the extensions of the generalized bi-quasi-variational inequalities (GBQVI) which was first introduced by Shih and Tan [31] in 1989.

2. Preliminaries

In 2010, Chowdhury and Tan [17] obtained the generalized bi-quasi-variational inequalities for quasi-pseudomonotone type I and strongly quasi-pseudomonotone type I operators on non-compact sets. As we have mentioned above, we are going to obtain some results for solutions for a new class of generalized bi-quasi-variational-like inequalities (GBQVLI) for (η, h)-quasi-pseudomonotone type I and strongly (η, h)-quasi-pseudomonotone type I operators on non-compact sets. For this, we now introduce the following definition of generalized bi-quasi-variational-like inequality (GBQVLI):

Let $S : X \to 2^X$ be a set-valued mapping, $M, T : X \to 2^F$ be two set-valued mappings and $\eta : X \times X \to E$ be a single-valued mapping. The *generalized bi-quasi-variational-like inequality problem* (GBQVLI) is as follows:

Find a point $\hat{y} \in X$ and a point $\hat{w} \in T(\hat{y})$ such that

(1) $\hat{y} \in S(\hat{y})$;

(2) $Re\langle f - \hat{w}, \eta(\hat{y}, x)\rangle \leq 0$ for all $x \in S(\hat{y})$ and $f \in M(\hat{y})$;

or

Find a point $\hat{y} \in X$, a point $\hat{w} \in T(\hat{y})$ and a point $\hat{f} \in M(\hat{y})$ such that

(1) $\hat{y} \in S(\hat{y})$;

(3) $Re\langle \hat{f} - \hat{w}, \eta(\hat{y}, x)\rangle \leq 0$ for all $x \in S(\hat{y})$.

If $\eta(\hat{y}, x) = \hat{y} - x$, then the generalized bi-quasi-variational-like inequality (GBQVLI) is equivalent to the generalized bi-quasi-variational inequality (GBQVI) introduced by Chowdhury and Tan in [14] and Shih and Tan in [31].

Now, we first introduce the following definition of (η, h)-quasi-pseudomonotone (resp., strongly (η, h)-quasi-pseudomonotone) type I operators which is a slight modification of the quasi-pseudomonotone (resp., strongly quasi-pseudomonotone) type I operators (see Definition 1.1 in [15] given by Chowdhury and Tan in 2010):

Definition 2.1. Let E be a topological vector space over Φ, X be a non-empty subset of E and F be a topological vector space over Φ which is equipped with the $\sigma\langle F, E\rangle$ topology. Let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional. Consider the following four mappings:

(1) $M : X \to 2^F$ is a multi-valued mapping;

(2) $T : X \to 2^F$ is a multi-valued mapping;

(3) $h : E \times E \to \mathbb{R}$ is a single-valued mapping;

(4) $\eta : X \times X \to E$ is a single-valued mapping.

Then the mapping T is said to be an (η, h)*-quasi-pseudomonotone type I* (resp., *strongly* (η, h)*-quasi-pseudomonotone type I*) *operator* if, for each $y \in X$ and net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y (resp., weakly to y) with

$$\limsup_{\alpha}[\inf_{f\in M(y)}\inf_{u\in T(y_\alpha)} Re\langle f-u, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)] \le 0,$$

we have

$$\begin{aligned}&\limsup_{\alpha}[\inf_{f\in M(x)}\inf_{u\in T(y_\alpha)} Re\langle f-u, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)]\\ \ge\ &\inf_{f\in M(x)}\inf_{w\in T(y)} Re\langle f-w, \eta(y, x)\rangle + h(y, x)\end{aligned}$$

for all $x \in X$.

Remark 2.1. The above operator T reduces to an h-quasi-pseudomonotone type I (resp., strongly h-quasi-pseudomonotone type I) operator due to Chowdhury and Tan in [17] if T is an (η, h)-quasi-pseudomonotone type I (resp., strongly (η, h)-quasi-pseudomonotone type I) operator with $\eta(x, y) = x-y$ for all $x, y \in X$ and, for some $h' : E \to \mathbb{R}$, $h(x, y) = h'(x) - h'(y)$ for all $x, y \in E$.

Also, T reduces to a quasi-pseudomonotone type I (resp., strongly quasi-pseudomonotone type I) operator due to Chowdhury and Tan in 15] if T is an h-quasi-pseudomonotone type I (resp., strongly h-quasi-pseudomonotone type I) operator with $h \equiv 0$.

Remark 2.2. (1) When $M \equiv 0$ and T is replaced by $-T$, an h-quasi-pseudomonotone type I operator is reduced to an h-pseudomonotone (or an h-demimonotone) operator defined in [10].

(2) The h-pseudomonotone (or h-demi-monotone) operators defined in [10] are slightly more general than the definition of h-pseudomonotone operators given in [13].

(3) Later, in the year 2000, Chowdhury renamed the above h-pseudomonotone (or h-demi-monotone) operators as *pseudomonotone type I operators* [8]. The pseudomonotone type I operators are set-valued generalization of the classical (single-valued) pseudomonotone operators with slight variations. The classical definition of a single-valued pseudomonotone operator was introduced by Brézis et al. in [5].

(4) The authors first introduced quasi-pseudomonotone type I operators in [15, Definition 1.1] as a generalization of pseudomonotone type I operators.

We state the following result given in [17]:

Proposition 2.1. *Let X be a non-empty subset of a topological vector space E. Let $T : X \to E^*$ and $M : X \to E^*$ be two single-valued maps. Suppose that the operator T is monotone, and both M and T are continuous maps from the relative weak topology on X to the weak* topology on E^*. Then T is both quasi-pseudomonotone type I and strongly quasi-pseudomonotone type I operator.*

For the proof, see in [17, pp. 424–425].

The following result justifies the validity of an (η-h)-quasi-pseudo-monotone type I and strongly (η-h)-quasi-pseudo-monotone type I operators:

Proposition 2.2. *Let X be a non-empty subset of a topological vector space E. Let $T : X \to E^*$ and $M : X \to E^*$ be two single-valued maps. Suppose that $h : X \times X \to \mathbb{R}$ is a real valued function such that for each $y \in X$, $h(\cdot, y)$ is continuous and $h(X \times X)$ is bounded. Let $\eta : X \times X \to E$ be a continuous mapping.*

Further suppose that the operators T and M are η-monotone (i.e., for each $x, y \in X$, we have $Re\langle T(y) - T(x), \eta(y, x)\rangle \geq 0$ (respectively, $Re\langle M(y) - M(x), \eta(y, x)\rangle \geq 0$)), and also both M and T are continuous mappings from the relative weak topology on X to the weak* topology on E^*. Then T is both (η-h)-quasi-pseudo-monotone type I and strongly (η-h)-quasi-pseudo-monotone type I operator.

Proof. Suppose that $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X and $y \in X$ with $y_\alpha \to y$ (respectively, $y_\alpha \to y$ weakly) and that

$$\limsup_\alpha Re\langle M(y) - T(y_\alpha), \eta(y_\alpha, y)\rangle + h(y_\alpha, y) \leq 0.$$

Let $x \in X$ be arbitrarily fixed. Then

$$\begin{aligned} &\limsup_\alpha [Re\langle M(x) - T(y_\alpha), \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] \\ \geq\ &\limsup_\alpha [Re\langle M(x) - T(y_\alpha), \eta(y_\alpha, x)\rangle] + \liminf_\alpha h(y_\alpha, x). \end{aligned} \tag{2.1}$$

Since M and T are η-monotone, we have

$$Re\langle (M(x) - T(y_\alpha)) - (M(x) - T(y)), \eta(y_\alpha, x)\rangle \geq 0.$$

Thus we have

$$Re\langle M(x) - T(y_\alpha), \eta(y_\alpha, x)\rangle \geq Re\langle M(x) - T(y), \eta(y_\alpha, x)\rangle.$$

Hence, we have,

$$\begin{aligned} &\limsup_\alpha [Re\langle M(x) - T(y_\alpha), \eta(y_\alpha, x)] \\ \geq\ &\limsup_\alpha [Re\langle M(x) - T(y), \eta(y_\alpha, x)\rangle]. \end{aligned} \tag{2.2}$$

Therefore, from equations (2.1) and (2.2) we have,

$$\begin{aligned} &\limsup_\alpha [Re\langle M(x) - T(y_\alpha), \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] \\ \geq\ &\limsup_\alpha [Re\langle M(x) - T(y), \eta(y_\alpha, x)\rangle] + \liminf_\alpha h(y_\alpha, x) \\ =\ &Re\langle M(x) - T(y), \eta(y, x)\rangle + h(y, x) \end{aligned}$$

for all $x \in X$.

Consequently, T is both (η-h)-quasi-pseudo-monotone type I and strongly (η-h)-quasi-pseudo-monotone type I operator. □

In this paper, we obtain some general theorems on solutions for a new class of generalized bi-quasi-variational-like inequalities for (η, h)-quasi-pseudomonotone type I and strongly (η, h)-quasi-pseudomonotone type I operators defined on non-compact spaces in topological vector spaces. To obtain these results, we mainly use the following generalized version of Ky Fan's minimax inequality [27] due to Chowdhury and Tan [10] which was stated and proved as Theorem 2.1 in [16] and is a slight modification of Theorem 1 in [10]:

Theorem 2.3. *Let E be a topological vector space, X be a nonempty convex subset of E, $\mathcal{F}(X)$ denote the family of all non-empty finite subsets of X and $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that*

(a) *for each $A \in \mathcal{F}(X)$ and fixed $x \in co(A)$, $y \mapsto f(x, y)$ is lower semi-continuous on $co(A)$;*

(b) *for each $A \in \mathcal{F}(X)$ and $y \in co(A)$, $\min_{x\in A} f(x, y) \leq 0$;*

(c) *for each $A \in \mathcal{F}(X)$ and $x, y \in co(A)$, every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y with $f(tx+(1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and $t \in [0, 1]$, we have $f(x, y) \leq 0$;*

(d) *there exist a nonempty closed and compact subset K of X and $x_0 \in K$ such that $f(x_0, y) > 0$ for all $y \in X \setminus K$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

Definition 2.2. A function $\phi : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be 0-*diagonally concave* (in short, 0-DCV) in the second argument [26] if, for any finite set $\{x_1, \ldots, x_n\} \subset X$ and $\lambda_i \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$, we have $\sum_{i=1}^n \lambda_i \phi(y, x_i) \leq 0$, where $y = \sum_{i=1}^n \lambda_i x_i$.

Let E be a topological vector space over Φ, F be a vector space over Φ and X be a non-empty subset of E. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional. Throughout this paper, Φ denotes either the real field $\mathbb{R}$ or the complex field $\mathcal{C}$.

Now, we state the following definition given in [25]:

Definition 2.3. Let X, E, F be the sets defined above and $T : X \to 2^F$, $\eta : X \times X \to E$, $g : X \to E$ be mappings.

(1) The mappings T and η are said to have 0-*diagonally concave relation* (in short, 0-DCVR) if the function $\phi : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\phi(x, y) = \inf_{w\in T(x)} Re\langle w, \eta(x, y)\rangle$$

is 0-DCV in y;

(2) The mappings T and g are said to have 0-*diagonally concave relation* if T and $\eta(x, y) = g(x) - g(y)$ have the 0-DCVR.

We first state the following result which is Lemma 1 of Shih and Tan in [25, pp. 334–335]:

Lemma 2.4. *Let X be a nonempty subset of a Hausdorff topological vector space E and $S : X \to 2^E$ be an upper semi-continuous mapping such that $S(x)$*

is a bounded subset of E for each $x \in X$. Then, for each continuous linear functional p on E, the functional $f_p : X \to \mathbb{R}$ defined by

$$f_p(y) = \sup_{x \in S(y)} Re\langle p, x\rangle$$

is upper semi-continuous, i.e., for each $\lambda \in \mathbb{R}$, the set

$$\{y \in X : f_p(y) = \sup_{x \in S(y)} Re\langle p, x\rangle < \lambda\}$$

is open in X.

The following result is Lemma 3 of Takahashi in [33, pp. 177] (see also Lemma 3 in [31, pp. 71–72]):

Lemma 2.5. *Let X and Y be topological spaces, $f : X \to \mathbb{R}$ be non-negative and continuous and $g : Y \to \mathbb{R}$ be lower semi-continuous. Then the functional $F : X \times Y \to \mathbb{R}$ defined by*

$$F(x, y) = f(x)g(y)$$

for all $(x, y) \in X \times Y$ is lower semi-continuous.

The following result, which was stated and proved as Lemma 2.2 in [16], follows from slight modification of Lemma 3 of Chowdhury and Tan given in [10]:

Lemma 2.6. *Let E be a Hausdorff topological vector space over Φ, $A \in \mathcal{F}(E)$ and $X = co(A)$ where $co(A)$ denotes the convex hull of A. Let F be a vector space over Φ and $\langle , \rangle : F \times E \to \phi$ be a bilinear functional such that $\langle , \rangle$ separates points in F. We equip F with the $\sigma\langle F, E\rangle$-topology. Suppose that, for each $w \in F$, $x \mapsto Re\langle w, x\rangle$ is continuous. Let $\eta : X \times X \to E$ be continuous. Let $T : X \to 2^F$ be upper semi-continuous from X into 2^F such that each $T(x)$ is $\sigma\langle F, E\rangle$-compact. Let $f : X \times X \to \mathbb{R}$ be defined by*

$$f(x, y) = \inf_{w \in T(y)} Re\langle w, \eta(y, x)\rangle$$

for all $x, y \in X$.

Suppose that $\langle , \rangle$ is continuous on the (compact) subset $[\cup_{y \in X} T(y)] \times \eta(X \times X)$ of $F \times E$. Then, for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semi-continuous on X.

For completeness we include the proof here given in [16]:

Proof. Let $\lambda \in \mathbb{R}$ be given and let $x \in X = co(A)$ be arbitrarily fixed. Let $A_\lambda = \{y \in X : f(x, y) \leq \lambda\}$. Suppose that $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in A_λ and $y_0 \in co(A) = X$ such that $y_\alpha \to y_0$. Then for each $\alpha \in \Gamma$,

$$\lambda \geq f(x, y_\alpha) = \inf_{w \in T(y_\alpha)} Re\langle w, \eta(y_\alpha, x)\rangle.$$

Since F is equipped with the $\sigma\langle F, E\rangle$-topology, for each $x \in E$, the function $w \mapsto Re\langle w, x\rangle$ is continuous. Also, $\eta(y_\alpha, x) \to \eta(y_0, x)$ because $\eta(\cdot, x)$ is continuous. By the $\sigma\langle F, E\rangle$-compactness of $T(y_\alpha)$, there exists $w_\alpha \in T(y_\alpha)$ such that

$$\lambda \geq \inf_{w \in T(y_\alpha)} Re\langle w, \eta(y_\alpha, x)\rangle = Re\langle w_\alpha, \eta(y_\alpha, x)\rangle.$$

Since T is upper semi-continuous from $X = co(A)$ to the $\sigma\langle F, E\rangle$-topology on F, X is compact, and each $T(z)$ is $\sigma\langle F, E\rangle$-compact, $\cup_{z\in X} T(z)$ is also $\sigma\langle F, E\rangle$-compact by Proposition 3.1.11 of Aubin and Ekeland [2]. Thus there is a subnet $\{w_{\alpha'}\}_{\alpha'\in\Gamma'}$ of $\{w_\alpha\}_{\alpha\in\Gamma}$ and $w_0 \in \cup_{z\in X} T(z)$ such that $w_{\alpha'} \to w_0$ in the $\sigma\langle F, E\rangle$-topology. Again, as T is upper semi-continuous with the $\sigma\langle F, E\rangle$-closed values, $w_0 \in T(y_0)$.

Suppose that $A = \{a_1, a_2, \ldots, a_n\}$ and let $t_1, t_2, \ldots, t_n \geq 0$ with $\sum_{i=1}^n t_i = 1$ such that $y_0 = \sum_{i=1}^n t_i a_i$. For each $\alpha' \in \Gamma$, let $t_1^{\alpha'}, t_2^{\alpha'}, \ldots, t_n^{\alpha'} \geq 0$ with $\sum_{i=1}^n t_i^{\alpha'} = 1$ such that $y_{\alpha'} = \sum_{i=1}^n t_i^{\alpha'} a_i$. Since E is Hausdorff and $y_{\alpha'} \to y_0$, we must have $t_i^{\alpha'} \to t_i$ for each $i = 1, 2, \ldots, n$. Thus

$$\begin{aligned} \lambda &\geq Re\langle w_{\alpha'}, \eta(y_{\alpha'}, x)\rangle = Re\langle w_{\alpha'}, \eta(\sum_{i=1}^n t_i^{\alpha'} a_i, x)\rangle \\ &\to Re\langle w_0, \eta(\sum_{i=1}^n t_i a_i, x)\rangle \qquad (2.1) \\ &= Re\langle w_0, \eta(y_0, x)\rangle \geq \inf_{w\in T(y_0)} Re\langle w, \eta(y_0, x)\rangle = f(x, y_0), \end{aligned}$$

where (2.1) is true since $\eta(\cdot, x)$ is continuous on X and $\langle\, ,\, \rangle$ is continuous on the compact subset $[\cup_{y\in X} T(y)] \times \eta(X \times X)$ of $F \times E$.

Hence $y_0 \in A_\lambda$. Thus A_λ is closed in $X = co(A)$ for each $\lambda \in \mathbb{R}$. Therefore $y \mapsto f(x, y)$ is lower semi-continuous on X. □

By a slight modification of Lemma 4.2 in [12], we obtain below a further modification of the result given in [24, Lemma 2.3]:

Lemma 2.7. *Let E be a topological vector over ϕ, X a nonempty convex subset of E and F a vector space over ϕ. Let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle\cdot,\cdot\rangle$ separates points in F. We equip F with the $\sigma\langle F, E\rangle$-topology such that for each $w \in F$, the function $x \mapsto Re\langle w, x\rangle$ is continuous. Let $\eta : X \times X \to E$ be such that for each fixed $y \in X$, $\eta(\cdot, y)$ is continuous and for each fixed $x \in X$, $\eta(x, \cdot)$ is affine. Let $h : X \times X \to \mathbb{R}$ be a mapping such that for each fixed $y \in X$, $h(\cdot, y)$ is lower semi-continuous and convex on $co(A)$ for each $A \in \mathcal{F}(X)$, and for each fixed $x \in X$, $h(x, \cdot)$ is concave, and $h(x, x) = 0$, $\eta(x, x) = 0$, and T and η have the 0-DCVR.*

Suppose that $S : X \to 2^X$ is a mapping, $M : X \to 2^F$ is a lower semi-continuous mapping along line segments in X to the $\sigma\langle F, E\rangle$-topology on F and $T : X \to 2^F$ is an upper hemi-continuous mapping along line segments in X.

Suppose further that there exists $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$, $S(\hat{y})$ is convex and

$$\inf_{f \in M(x)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y})$$

for all $x \in S(\hat{y})$. Then

$$\inf_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y})$$

for all $x \in S(\hat{y})$.

For completeness we give the detailed proof below:

Proof. Suppose that

$$\inf_{f \in M(x)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y}) \text{ for all } x \in S(\hat{y}).$$

Let $x \in S(\hat{y})$ be arbitrarily fixed. Let $z_t = tx + (1-t)\hat{y} = \hat{y} - t(\hat{y} - x)$ for all $t \in [0, 1]$. Then $z_t \in S(\hat{y})$ as $S(\hat{y})$ is convex.

Let $L = \{z_t : t \in [0, 1]\}$. Thus for every $t \in [0, 1]$

$$\inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, z_t)\rangle \leq h(z_t, \hat{y}).$$

Since for each $y \in S(\hat{y})$, $h(\cdot, y)$ is convex and for each $x \in S(\hat{y})$, $h(x, \cdot)$ is affine, we have

$$\inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, tx + (1-t)\hat{y})\rangle$$
$$\leq h(tx + (1-t)\hat{y}, \hat{y}) \leq t(h(x, \hat{y})) + (1-t)h(\hat{y}, \hat{y})$$

for all $t \in (0, 1]$; thus we have,

$$\inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} [Re\langle f - w, t\eta(\hat{y}, x) + (1-t)\eta(\hat{y}, \hat{y})\rangle] \leq t(h(x, \hat{y}));$$

therefore we have,

$$\inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} t[Re\langle f - w, \eta(\hat{y}, x)\rangle] \leq t(h(x, \hat{y})).$$

This implies that $\inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y})$ for all $t \in (0, 1]$. Since T is upper hemi-continuous on L, and M is lower semi-continuous on L, the function $f_{\eta(\hat{y}, x)} : L \to \mathbb{R} \cup \{+\infty\}$, defined by

$$f_{\eta(\hat{y}, x)}(z_t) = \inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \text{ for each } z_t \in L,$$

is lower semi-continuous on L. Thus the set

$$A = \{z_t \in L : f_{\eta(\hat{y}, x)}(z_t) \leq h(x, \hat{y})\}$$

is closed in L. Now $z_t \to \hat{y}$ in L as $t \to 0^+$. Since $z_t \in A$ for all $t \in (0, 1]$ we have $\hat{y} \in A$. Hence $f_{\eta(\hat{y}, x)}(\hat{y}) = \inf_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y})$. Since $x \in S(\hat{y})$ is arbitrary, we have

$$\inf_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y}) \text{ for all } x \in S(\hat{y}).$$

□

We need the following Kneser's minimax theorem in [28, pp. 2418–2420] (see also Aubin [1, pp. 40–41]):

Theorem 2.8. *Let X be a nonempty convex subset of a vector space and Y be a nonempty compact convex subset of a Hausdorff topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that, for each fixed $x \in X$, the mapping $y \mapsto f(x,y)$, i.e., $f(x,\cdot)$ is lower semi-continuous and convex on Y and, for each fixed $y \in Y$, the map $x \mapsto f(x,y)$, i.e., $f(\cdot,y)$ is concave on X. Then*

$$\min_{y\in Y}\sup_{x\in X} f(x,\, y) = \sup_{x\in X}\min_{y\in Y} f(x,\, y).$$

3. Generalized bi-quasi-variational-like inequalities

In this section, we obtain and prove some existence theorems for the solutions to the generalized bi-quasi-variational-like inequalities for (η,h)-quasi-pseudomonotone type I and strongly (η,h)-quasi-pseudomonotone type I operators T with non-compact domain in locally convex Hausdorff topological vector spaces. Our results extend and generalize the corresponding results in [31].

We first establish the following result:

Theorem 3.1. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty para-compact convex and bounded subset of E and F a Hausdorff topological vector space over Φ. Let $\langle\cdot,\cdot\rangle : F\times E\to\Phi$ be a bilinear functional which is continuous on compact subsets of $F\times X$. Suppose that*

(a) *$S : X \to 2^X$ is upper semi-continuous such that each $S(x)$ is compact and convex*;

(b) *$h : E\times E\to\mathbb{R}$ is convex and $h(X\times X)$ is bounded*;

(c) *$T : X\to 2^F$ is an $(\eta$-$h)$-quasi-pseudo-monotone type I (respectively, strongly $(\eta$-$h)$-quasi-pseudo-monotone type I) operator and is upper semi-continuous such that each $T(x)$ is compact (respectively, weakly compact) and convex and $T(X)$ is strongly bounded*;

(d) *$T : X\to 2^F$, and $\eta : X\times X\to E$ have the 0-DCVR and $\eta : X\times X\to E$ is convex and continuous*;

(e) *$M : X\to 2^F$ is a linear mapping in X (and is therefore single-valued for each $x\in X$)*;

(f) *for each fixed $y\in X$, $x\mapsto h(x,y)$, i.e., $h(\cdot,y)$ is lower semi-continuous on $co(A)$ for each $A\in\mathcal{F}(X)$ and, for each fixed $x\in X$, $h(x,\cdot)$ and $\eta(x,\cdot)$ are concave, and $\eta(x,\cdot)$ is affine and $h(x,x)=0$, $\eta(x,x)=0$*;

(g) *the set*

$$\Sigma = \{y\in X : \sup_{x\in S(y)}\big(\inf_{w\in T(y)} Re\langle M(x)-w, \eta(y,x)\rangle + h(y,x) > 0\}$$

is open in X.

Suppose further that there exist a nonempty closed and compact (*respectively, weakly closed and weakly compact*) *subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} Re\langle M(x_0) - w, \eta(y, x_0)\rangle + h(y, x_0) > 0$ for all $y \in X \setminus K$.*

Then there exists a point $\hat{y} \in X$ such that

(1) $\hat{y} \in S(\hat{y})$;

(2) *there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \le 0$ for all $x \in S(\hat{y})$.*

Moreover, if $S(x) = X$ for all $x \in X$, then E is not required to be locally convex and, if $T \equiv 0$, then the continuity assumption on $\langle \cdot, \cdot \rangle$ can be weakened to the assumption that, for each $f \in F$, the mapping $x \mapsto \langle f, x\rangle$ is continuous (*resp., weakly continuous*) *on X.*

Proof. We divide the proof into three steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{w \in T(\hat{y})} Re\langle M(x) - w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x)] \le 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that

$$\inf_{w \in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x) > 0,$$

that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$.

If $y \notin S(y)$, then, by a separation theorem for convex sets in locally convex Hausdorff topological vector spaces, there exist $p \in E^*$ and $\alpha \in \mathbb{R}$ such that $Re\langle p, x\rangle < \alpha < Re\langle p, y\rangle$ for all $x \in S(y)$. Therefore,

$$\sup_{x \in S(y)} Re\langle p, x\rangle \le \alpha < Re\langle p, y\rangle.$$

Hence we have, $Re\langle p, y\rangle - \sup_{x \in S(y)} Re\langle p, x\rangle > 0$. Let

$$\gamma(y) = \sup_{x \in S(y)} \inf_{w \in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x),$$

$$V_0 := \{y \in X | \gamma(y) > 0\} = \Sigma$$

and, for each $p \in E^*$, set

$$V_p := \{y \in X : Re\langle p, y\rangle - \sup_{x \in S(y)} Re\langle p, x\rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 2.1 and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X. Since X is para-compact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$ (see Theorem VIII, 4.2 of Dugundji in [23]), that is, for each $p \in E^*$, $\beta_p : X \to [0, 1]$ and $\beta_0 : X \to [0, 1]$ are continuous functions such that, for each $p \in E^*$, $\beta_p(y) = 0$ for all $y \in X \setminus V_p$, $\beta_0(y) = 0$ for all $y \in X \setminus V_0$, $\{\text{support } \beta_0, \text{support } \beta_p : p \in E^*\}$ is locally finite and $\beta_0(y) + \Sigma_{p\in E^*}\beta_p(y) = 1$ for each $y \in X$. Note that, for each $A \in \mathcal{F}(X)$, h is continuous on $co(A)$ (see [29, Corollary 10.1.1, p. 83]).

Define a function $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \beta_0(y)[\inf_{w\in T(y)} Re\langle M(x) - w, \eta(y,x)\rangle + h(y,x)] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle$$

for each $x, y \in X$. Then we have the following:

(1) Since E is Hausdorff, for each $A \in \mathcal{F}(X)$ and fixed $x \in co(A)$, the mapping

$$y \mapsto \inf_{w\in T(y)} Re\langle M(x) - w, \eta(y,x)\rangle + h(y,x)$$

is lower semi-continuous (resp., weakly lower semi-continuous) on $co(A)$ by Lemma 2.6 and the fact that h is continuous on $co(A)$ and therefore the map

$$y \mapsto \beta_0(y)[\inf_{w\in T(y)} Re\langle M(x) - w, \eta(y,x)\rangle + h(y,x)]$$

is lower semi-continuous (resp., weakly lower semi-continuous) on $co(A)$ by Lemma 2.5. Also, for each fixed $x \in X$,

$$y \mapsto \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle$$

is continuous on X. Hence, for each $A \in \mathcal{F}(X)$ and fixed $x \in co(A)$, the mapping $y \mapsto \phi(x,y)$ is lower semi-continuous (resp., weakly lower semi-continuous) on $co(A)$.

(2) For each $A \in \mathcal{F}(X)$ and $y \in co(A)$, $\min_{x\in A}\phi(x,y) \le 0$. Indeed, if this were false, then, for some $A = \{x_1, x_2, \ldots, x_n\} \in \mathcal{F}(X)$ and $y \in co(A)$ (say $y = \sum_{i=1}^n \lambda_i x_i$, where $\lambda_1, \lambda_2, \ldots, \lambda_n \ge 0$ with $\sum_{i=1}^n \lambda_i = 1$), we have $\min_{1\le i\le n}\phi(x_i, y) > 0$. Then, for each $i = 1, 2, \ldots, n$,

$$\beta_0(y)[\inf_{w\in T(y)} Re\langle M(x_i) - w, \eta(y,x_i)\rangle + h(y,x_i)] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x_i\rangle > 0$$

and so

$$\begin{aligned}
0 &= \phi(y,y) \\
&= \beta_0(y)[\inf_{w\in T(y)} Re\langle M(\sum_{i=1}^n \lambda_i x_i) - w, \eta(y, \sum_{i=1}^n \lambda_i x_i)\rangle + h(y, \sum_{i=1}^n \lambda_i x_i)] \\
&\quad + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - \sum_{i=1}^n \lambda_i x_i\rangle \\
&= \beta_0(y)[\inf_{w\in T(y)} Re\langle \sum_{i=1}^n \lambda_i M(x_i) - w, \eta(y, \sum_{i=1}^n \lambda_i x_i)\rangle + h(y, \sum_{i=1}^n \lambda_i x_i)] \\
&\quad + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - \sum_{i=1}^n \lambda_i x_i\rangle
\end{aligned}$$

$$\geq \sum_{i=1}^{n} \lambda_i(\beta_0(y)[\inf_{w\in T(y)} Re\langle M(x_i) - w, \eta(y, x_i)\rangle + h(y, x_i)] \\ + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x_i\rangle) > 0,$$

which is a contradiction.

(3) Suppose that $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X converging to y (resp., weakly to y) with $\phi(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and $t \in [0, 1]$.

Case (1): $\beta_0(y) = 0$.

Note that $\beta_0(y_\alpha) \geq 0$ for each $\alpha \in \Gamma$ and $\beta_0(y_\alpha) \to 0$. Since $T(X)$ is strongly bounded and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a bounded net, it follows that

$$\limsup_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle M(x) - w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] = 0. \tag{3.1}$$

Also, we have

$$\beta_0(y)[\min_{w\in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x)] = 0.$$

Thus it follows from (3.1) that

$$\begin{aligned} &\limsup_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle M(x) - w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] \\ &\quad + \sum_{p\in E^*}^{n} \beta_p(y) Re\langle p, y - x\rangle \\ &= \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle \\ &= \beta_0(y)[\min_{w\in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x)] \\ &\quad + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle. \end{aligned} \tag{3.2}$$

When $t = 1$, we have $\phi(x, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\begin{aligned} &\beta_0(y_\alpha)[\min_{w\in T(y_\alpha)} Re\langle M(x) - w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] \\ &\quad + \sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - x\rangle \leq 0 \end{aligned} \tag{3.3}$$

for all $\alpha \in \Gamma$. Therefore, by (3.3), we have

$$\begin{aligned} &\limsup_{\alpha}[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)} Re\langle M(x) - w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] \\ &\quad + \liminf_{\alpha}[\sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - x\rangle] \\ &\leq \limsup_{\alpha}[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)} Re\langle M(x) - w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x) \end{aligned} \tag{3.4}$$

$$+\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle] \le 0$$

and thus, by (3.4),

$$\limsup_\alpha[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)} Re\langle M(x)-w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] + \sum_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle \le 0.$$

Hence, by (3.2) and (3.4), we have $\phi(x, y) \le 0$.

Case (2): $\beta_0(y) > 0$.

Since $\beta_0(y_\alpha) \to \beta_0(y)$, there exists $\lambda \in \Gamma$ such that $\beta_0(y_\alpha) > 0$ for all $\alpha \ge \lambda$. When $t = 0$, we have $\phi(y, y_\alpha) \le 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha)[\inf_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)] + \sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle \le 0$$

for all $\alpha \in \Gamma$ and thus

$$\limsup_\alpha[\beta_0(y_\alpha)(\inf_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y) + \sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle] \le 0. \tag{3.5}$$

Hence it follows from (3.5) that

$$\begin{aligned}&\limsup_\alpha[\beta_0(y_\alpha)(\inf_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y\rangle + h(y_\alpha, y)] \\ &+ \liminf_\alpha[\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle] \\ \le\ &\limsup_\alpha[\beta_0(y_\alpha)(\inf_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y\rangle + h(y_\alpha, y) \\ &+ \sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle] \le 0.\end{aligned}$$

Since $\liminf_\alpha[\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle] = 0$, we have

$$\limsup_\alpha[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)] \le 0. \tag{3.6}$$

Since $\beta_0(y_\alpha) > 0$ for all $\alpha \ge \lambda$, it follows that

$$\begin{aligned}&\beta_0(y)\limsup_\alpha[\min_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)] \\ =\ &\limsup_\alpha[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)].\end{aligned} \tag{3.7}$$

Since $\beta_0(y) > 0$, by (3.6) and (3.7), we have

$$\limsup_\alpha[\min_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)] \le 0.$$

Since T is an $(\eta\text{-}h)$-quasi-pseudo-monotone type I (respectively, strongly $(\eta\text{-}h)$-quasi-pseudo-monotone type I) operator, we have

$$\begin{aligned}&\limsup_{\alpha}[\min_{w\in T(y_\alpha)} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)]\\ \geq\ &\min_{w\in T(y)} Re\langle M(x)-w,\eta(y,x)\rangle + h(y,x)\end{aligned}$$

for all $x\in X$. Since $\beta_0(y)>0$, we have

$$\begin{aligned}&\beta_0(y)[\limsup_{\alpha}(\min_{w\in T(y_\alpha} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)]\\ \geq\ &\beta_0(y)[\min_{w\in T(y)} Re\langle M(x)-w,\eta(y,x)\rangle + h(y,x)]\end{aligned}$$

and thus

$$\begin{aligned}&\beta_0(y)[\limsup_{\alpha}(\min_{w\in T(y_\alpha)} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)]\\ &+\sum_{p\in E^*}\beta_p(y)Re\langle p,y-x\rangle\\ \geq\ &\beta_0(y)[\min_{w\in T(y)} Re\langle M(x)-w,\eta(y,x)\rangle + h(y,x)]\\ &+\sum_{p\in E^*}\beta_p(y)Re\langle p,y-x\rangle.\end{aligned}\tag{3.8}$$

When $t=1$, we have $\phi(x,y_\alpha)\leq 0$ for all $\alpha\in\Gamma$, i.e.,

$$\begin{aligned}&\beta_0(y_\alpha)[\min_{w\in T(y_\alpha)} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)]\\ &+\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p,y_\alpha-x\rangle\leq 0\end{aligned}$$

for all $\alpha\in\Gamma$ and so, by (3.8),

$$\begin{aligned}0\geq\ &\limsup_{\alpha}[\beta_0(y_\alpha)\min_{w\in T(y_\alpha} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)\\ &+\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p,y_\alpha-x\rangle]\\ \geq\ &\limsup_{\alpha}[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)]\\ &+\liminf_{\alpha}[\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p,y_\alpha-x\rangle]\\ =\ &\beta_0(y)[\limsup_{\alpha}\{\min_{w\in T(y_\alpha)} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)\}]\\ &+\sum_{p\in E^*}\beta_p(y)Re\langle p,y-x\rangle\\ \geq\ &\beta_0(y)[\min_{w\in T(y)} Re\langle M(x)-w,\eta(y.x)\rangle + h(y,x)]\\ &+\sum_{p\in E^*}\beta_p(y)Re\langle p,y-x\rangle.\end{aligned}\tag{3.9}$$

Hence we have $\phi(x, y) \leq 0$.

(4) By hypothesis, there exist a nonempty compact and therefore closed (respectively, weakly closed and weakly compact) subset K of X and a point $x_0 \in X$ such that

$$x_0 \in K \cap S(y), \quad \inf_{w \in T(y)} [Re\langle M(x_0) - w, \eta(y, x_0)\rangle + h(y, x_0)] > 0$$

for all $y \in X \setminus K$. Thus it follows that, for all $y \in X \setminus K$,

$$\beta_0(y)[\inf_{w \in T(y)} Re\langle M(x_0) - w, \eta(y, x_0)\rangle + h(y, x_0)] > 0$$

whenever $\beta_0(y) > 0$, and $Re\langle p, y - x_0\rangle > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently, we have

$$\begin{aligned}\phi(x_0, y) = \beta_0(y)[\inf_{w \in T(y)} Re\langle M(x_0) - w, \eta(y, x_0)\rangle + h(y, x_0)] \\ + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x_0\rangle > 0\end{aligned}$$

for all $y \in X \setminus K$. (If T is a strongly (η-h)-quasi-pseudo-monotone type I operator, we equip E with the weak topology.) Thus ϕ satisfies all the hypotheses of Theorem 1.1. Hence, by Theorem 1.1, there exists a point $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$, i.e.,

$$\begin{aligned}\beta_0(\hat{y})[\inf_{w \in T(\hat{y})} Re\langle M(x) - w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x)] \\ + \sum_{p \in E^*} \beta_p(\hat{y}) Re\langle p, \hat{y} - x\rangle \leq 0\end{aligned} \tag{3.10}$$

for all $x \in X$.

On the other hand suppose for the above $\hat{y} \in X$, there exists $\hat{x} \in S(\hat{y})$ such that

$$\inf_{w \in T(\hat{y})} Re\langle M(\hat{x}) - w, \eta(\hat{y}, \hat{x})\rangle + h(\hat{y}, \hat{x}) > 0.$$

Then

$$\beta_0(\hat{y})[\inf_{w \in T(\hat{y})} Re\langle M(\hat{x}) - w, \eta(\hat{y}, \hat{x})\rangle + h(\hat{y}, \hat{x}) > 0$$

whenever $\beta_0(\hat{y}) > 0$.

Also if $\beta_p(\hat{y}) > 0$ for all $p \in E^*$, then $\hat{y} \in V_p$ and hence

$$Re\langle p, \hat{y}\rangle - \sup_{x \in S(\hat{y})} Re\langle p, x\rangle > 0.$$

Therefore, $Re\langle p, \hat{y}\rangle > \sup_{x \in S(\hat{y})} Re\langle p, x\rangle \geq Re\langle p, \hat{x}\rangle$. Hence, $Re\langle p, \hat{y} - \hat{x}\rangle > 0$. Then

$$\beta_p(\hat{y}) Re\langle p, \hat{y} - \hat{x}\rangle > 0$$

whenever $\beta_p(\hat{y}) > 0$ for all $p \in E^*$.

Since $\beta_p(\hat{y}) > 0$ for all $p \in E^*$, we have

$$\beta_0(\hat{y})[\inf_{w \in T(\hat{y})} Re\langle M(\hat{x}) - w, \eta(\hat{y}, \hat{x})\rangle + h(\hat{y}, \hat{x})]$$

$$+ \sum_{p \in E^*} \beta_p(\hat{y}) Re\langle p, \hat{y} - \hat{x}\rangle > 0$$

which contradicts (3.10). Therefore Step 1 is proved. Hence we have shown that there exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{w \in T(\hat{y})} Re\langle M(x) - w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x)] \leq 0.$$

Step 2. $\inf_{w \in T(\hat{y})} Re\langle M(\hat{y}) - w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$ for all $x \in S(\hat{y})$.

From Step 1, we have

$$\hat{y} \in S(\hat{y}), \qquad \inf_{w \in T(\hat{y})} Re\langle M(x) - w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$$

for all $x \in S(\hat{y})$. Since $S(\hat{y})$ is a convex subset of X and M is linear and so continuous along line segments in X, by Lemma 2.7, we have

$$\inf_{w \in T(\hat{y})} Re\langle M(\hat{y}) - w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$$

for all $x \in S(\hat{y})$.

Step 3. There exists a point $\hat{w} \in T(\hat{y})$ such that

$$Re\langle M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$$

for all $x \in S(\hat{y})$.

From Step 2 we have,

$$\sup_{x \in S(\hat{y})} [\inf_{w \in T(\hat{y})} Re\langle M(\hat{y}) - w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x)] \leq 0, \tag{3.11}$$

i.e.,

$$\sup_{x \in S(\hat{y})} [\inf_{(M(\hat{y}), w) \in M(\hat{y}) \times T(\hat{y})} Re\langle M(\hat{y}) - w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x)] \leq 0$$

where $M(\hat{y}) \times T(\hat{y})$ is a $\sigma\langle F, E\rangle$-compact convex subset of the Hausdorff topological vector space $F \times F$ and $S(\hat{y})$ is a convex subset of X.

Let us set $Q = M(\hat{y}) \times T(\hat{y})$ and define the mapping $g : S(\hat{y}) \times Q \to \mathbb{R}$ by $g(x, q) = g(x, (M(\hat{y}), w)) = Re\langle M(\hat{y}) - w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x)$ for each $x \in S(\hat{y})$ and each $q = (M(\hat{y}), w) \in Q = M(\hat{y}) \times T(\hat{y})$. Then, for each fixed $x \in S(\hat{y})$, the mapping $(M(\hat{y}), w) \mapsto g(x, (M(\hat{y}), w))$ is lower semi-continuous from the relative product topology on Q to $\mathbb{R}$ and also convex on Q. Clearly, for each fixed $q = (M(\hat{y}), w) \in Q$, the mapping $x \mapsto g(x, q) = g(x, (M(\hat{y}), w))$ is concave on $S(\hat{y})$.

So, we can apply Keneser's Minimax Theorem (Theorem 2.8) and obtain the following:

$$\min_{(M(\hat{y}), w) \in Q} \sup_{x \in S(\hat{y})} g(x, (M(\hat{y}), w)) = \sup_{x \in S(\hat{y})} \min_{(M(\hat{y}), w) \in Q} g(x, (M(\hat{y}), w)).$$

Hence, by (3.11), we obtain

$$\min_{(M(\hat{y}), w) \in Q} \sup_{x \in S(\hat{y})} Re\langle M(\hat{y}) - w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0.$$

Since $Q = M(\hat{y}) \times T(\hat{y})$ is compact, there exists $(M(\hat{y}), \hat{w}) \in M(\hat{y}) \times T(\hat{y})$ such that

$$\sup_{x \in S(\hat{y})} [Re\langle M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x)] \leq 0.$$

Therefore we have shown that

$$Re\langle M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$$

for all $x \in S(\hat{y})$. In other words, there exists a point $\hat{w} \in T(\hat{y})$ with

$$Re\langle M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$$

for all $x \in S(\hat{y})$.

We observe from the above proof that the requirement that E need to be locally convex is needed when and only when the separation theorem is applied to the case $y \notin S(y)$. Thus, if $S : X \to 2^X$ is the constant mapping $S(x) = X$ for all $x \in X$, then E is not required to be locally convex.

Finally, if $T \equiv 0$, in order to show that for each $x \in X$, $y \mapsto \phi(x, y)$ is lower semi-continuous (resp., weakly lower semi-continuous), Lemma 2.6 is no longer needed and the weaker continuity assumption on $\langle \cdot, \cdot \rangle$ that, for each $f \in F$, the mapping $x \mapsto \langle f, x\rangle$ is continuous (resp., weakly continuous) on X is sufficient. This completes the proof. □

Now, we establish our last result of this section:

Theorem 3.2. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty para-compact convex and bounded subset of E and F a Hausdorff topological vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that*

(a) *$S : X \to 2^X$ is a continuous mapping such that each $S(x)$ is compact and convex*;

(b) *$h : E \times E \to \mathbb{R}$ is convex and $h(X \times X)$ is bounded*;

(c) *$T : X \to 2^F$ is an $(\eta$-$h)$-quasi-pseudo-monotone type I* (*respectively, strongly $(\eta$-$h)$-quasi-pseudo-monotone type I*) *operator and is upper semi-continuous such that each $T(x)$ is compact and convex* (*respectively, weakly compact and convex, i.e., $\sigma\langle F, E\rangle$-compact and convex*) *and $T(X)$ is strongly bounded*;

(d) *$T : X \to 2^F$ and $\eta : X \times X \to E$ have the 0-DCVR and $\eta : X \times X \to E$ is convex and continuous*;

(e) *$M : X \to 2^F$ is a continuous linear mapping in X and for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)}[\inf_{w \in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x)] > 0\}$,*

$$\inf_{w \in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x) > 0$$

for some point $x \in S(y)$.

(f) *for each fixed $y \in X$, $x \mapsto h(x, y)$, i.e., $h(\cdot, y)$ is lower semi-continuous on $co(A)$ for each $A \in \mathcal{F}(X)$ and, for each fixed $x \in X$, $h(x, \cdot)$ and $\eta(x, \cdot)$ are concave, and $\eta(x, \cdot)$ is affine and $h(x, x) = 0$, $\eta(x, x) = 0$;*

(g) *for each open subset U of X and $x, y \in U$, $\eta(x,y) = x - y$ and there exists $h' : X \to \mathbb{R}$ such that $h(x,y) = h'(x) - h'(y)$;*

Suppose further that there exist a nonempty closed and compact (*respectively, weakly closed and weakly compact*) *subset K of X and a point $x_0 \in X$ such that*

$$x_0 \in K \cap S(y), \inf_{w \in T(y)} Re\langle M(x_0) - w, \eta(y, x_0)\rangle + h(y, x_0) > 0$$

for all $y \in X \setminus K$.

Then there exists a point $\hat{y} \in X$ such that

(1) *$\hat{y} \in S(\hat{y})$;*

(2) *there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$ for all $x \in S(\hat{y})$.*

Moreover, if $S(x) = X$ for all $x \in X$, then E is not required to be locally convex.

The proof is similar to the proof of Theorem 2 in [14]. For the completeness, we include the proof here.

Proof. The proof will follow from Theorem 3.1 if we can show that the set

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x)] > 0\}$$

is open in X. To show that Σ is open in X, we start as follows:

Let $y_0 \in \Sigma$ be an arbitrary point. We show that there exists an open neighbourhood N_0 of y_0 in X such that $N_0 \subset \Sigma$. Now, by the hypothesis (e), M is a continuous linear mapping on X and at some point x_0 in $S(y_0)$ we have

$$\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y_0, x_0)\rangle + h(y_0, x_0) > 0.$$

Let

$$\alpha := \inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y_0, x_0)\rangle + h(y_0, x_0).$$

Thus $\alpha > 0$. Again, let

$$W := \{w \in F : \sup_{z_1, z_2 \in X} |\langle w, z_1 - z_2\rangle| < \alpha/6\}.$$

Then W is an open neighbourhood of 0 in F and so $U_1 := T(y_0) + W$ is an open neighbourhood of $T(y_0)$ in F. Since T is upper semi-continuous at y_0, there exists an open neighbourhood N_1 of y_0 in X such that $T(y) \subset U_1$ for all $y \in N_1$.

Let $U_2 := M(x_0) + W$, then U_2 is an open neighbourhood of $M(x_0)$ in F. Since M is continuous at x_0, and therefore upper semi-continuous at x_0, there exists an open neighbourhood V_1 of x_0 in X such that $M(x) \in U_2$ for all $x \in V_1$.

Since the mapping $x \longmapsto \inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(x_0, x)\rangle + h(x_0, x)$ is continuous at x_0, there exists an open neighbourhood V_2 of x_0 in X such that

$$|\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(x_0, x)\rangle + h(x_0, x)| < \frac{\alpha}{6} \text{ for all } x \in V_2.$$

Let $V_0 := V_1 \cap V_2$. Then V_0 is an open neighborhood of x_0 in X. Since $x_0 \in V_0 \cap S(y_0) \neq \emptyset$ and S is lower semi-continuous at y_0, there exists an open neighborhood N_2 of y_0 in X such that $S(y) \cap V_0 \neq \emptyset$ for all $y \in N_2$.

Since the mapping $y \longmapsto \inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y, y_0)\rangle + h(y, y_0)$ is continuous at y_0, there exists an open neighborhood N_3 of y_0 in X such that

$$|\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y, y_0)\rangle + h(y, y_0)| < \frac{\alpha}{6} \text{ for all } y \in N_3.$$

Let $N_0 := N_1 \cap N_2 \cap N_3$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have the following:
(1) $T(y_1) \subset U_1 = T(y_0) + W$ as $y_1 \in N_1$;
(2) $S(y_1) \cap V_0 \neq \emptyset$ as $y_1 \in N_2$; so we can choose any $x_1 \in S(y_1) \cap V_0$;
(3) $|\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y_1, y_0)\rangle + h(y_1, y_0)| < \frac{\alpha}{6}$ as $y_1 \in N_3$;
(4) $M(x_1) \in U_2 = M(x_0) + W$ as $x_1 \in V_1$;
(5) $|\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(x_0, x_1)\rangle + h(x_0, x_1)| < \frac{\alpha}{6}$ as $x_1 \in V_2$.

Hence, using the assumption (g) of the theorem and by (1)-(5) above, we can obtain the following by omitting the details:

$$\begin{aligned}
&\inf_{w \in T(y_1)} Re\langle M(x_0) - w, \eta(y_1, x_1)\rangle + h(y_1, x_1) \\
\geq\ &\inf_{[w \in T(y_0) + W]} Re\langle M(x_0) - w, \eta(y_1, x_1)\rangle + h(y_1, x_1) \\
\geq\ &\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y_1, x_1)\rangle + h(y_1, x_1) \\
&+ \inf_{w \in W} Re\langle M(x_0) - w, \eta(y_1, x_1)\rangle \\
\geq\ &\inf_{w \in T(y_0)} Re\langle M(x_0) - w, y_1 - y_0\rangle + h'(y_1) - h'(y_0) \\
&+ \inf_{w \in T(y_0)} Re\langle M(x_0) - w, y_0 - x_0\rangle + h'(y_0) - h'(x_0) \\
&+ \inf_{w \in T(y_0)} Re\langle M(x_0) - w, x_0 - x_1\rangle + h'(x_0) - h'(x_1) \\
&+ Re\langle M(x_0), y_1 - x_1\rangle + \inf_{w \in W} Re\langle -w, y_1 - x_1\rangle \\
\geq\ &-\frac{\alpha}{6} + \alpha - \frac{\alpha}{6} - \frac{\alpha}{6} - \frac{\alpha}{6} \\
=\ &\frac{\alpha}{3} > 0.
\end{aligned}$$

Consequently, we have

$$\sup_{x \in S(y_1)} [\inf_{w \in T(y_1)} Re\langle M(x_0) - w, \eta(y_1, x)\rangle + h(y_1, x)] > 0$$

since $x_1 \in S(y_1)$. Hence $y_1 \in \Sigma$ for all $y_1 \in N_0$. Therefore, $y_0 \in N_0 \subset \Sigma$. But y_0 was arbitrary. Consequently, Σ is open in X.

Thus all the hypotheses of Theorem 3.1 are satisfied. Hence, the conclusion follows from Theorem 3.1. This completes the proof. □

Remark 3.1. (1) Theorems 3.1 and 3.2 in this paper are the extensions of Theorems 3.2 and 3.3 in [17], respectively, for generalized bi-quasi-variational-like inequalities (GBQVLI).

(2) The first paper on generalized bi-quasi-variational inequalities was written by Shih and Tan in 1989 in [31] and the results were obtained on compact sets where the set-valued mappings were either lower semi-continuous or upper semi-continuous. Our present paper is another extension of the original work in [31] using (η, h)-quasi-pseudomonotone type I and strongly (η, h)-quasi-pseudomonotone type I operators on non-compact spaces. The (η, h)-quasi-pseudomonotone type I and strongly (η, h)-quasi-pseudomonotone type I operators are generalizations of pseudomonotone type I operators introduced first in [10].

(3) In all our results on generalized bi-quasi-variational inequalities, if the operators $M \equiv 0$ and the operator T is replaced by $-T$, then we obtain results on generalized quasi-variational inequalities which generalize the corresponding results in the literature (see [30]).

(4) The results on generalized bi-quasi-variational inequalities given in [21] were obtained for set-valued quasi-semi-monotone and bi-quasi-semi-monotone operators and the corresponding results in [19] were obtained for set-valued upper-hemi-continuous operators introduced in [24]. Our results in this paper are also further extensions of the corresponding results in [21] and [9] using set-valued (η, h)-quasi-pseudomonotone type I and strongly (η, h)-quasi-pseudomonotone type I operators on non-compact spaces.

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