Commun. Korean Math. Soc. **32** (2017), No. 4, pp. 915–932
https://doi.org/10.4134/CKMS.c160272
pISSN: 1225-1763 / eISSN: 2234-3024

A NEW TYPE OF THE ADDITIVE FUNCTIONAL EQUATIONS ON INTUITIONISTIC FUZZY NORMED SPACES

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Abstract. In this paper, we introduce a new type of additive functional equations and establish the generalized Ulam-Hyers stability for it in intuitionistic fuzzy normed space by using direct and fixed point methods.

1. Introduction

Fuzzy theory was initiated by Zadeh [31] in 1965. Nowadays, this theory is a powerful tool for modeling uncertainty and vagueness in miscellaneous problems arising in the field of science and engineering. The concept of intuitionistic fuzzy normed spaces initially was introduced by Saadati and Park in [27]. Later, Saadati et al. have obtained a modified case of intuitionistic fuzzy normed spaces by improving the separation condition and strengthening certain conditions in the definition of [28]. Intuitionistic fuzzy sets and intuitionistic fuzzy metric spaces are studied in [4] and [23], respectively.

The stability problems for functional equations is connected to a question of Ulam [29] regarding the stability of group homomorphisms and positively answered for an additive functional equation on Banach spaces by Hyers [15] and Aoki [2]. It was an advance generalized and admirable outcome obtained by a number of mathematicians; for instance, see [5,14,18,22,24,25,30]. On the other hand, Cădariu and Radu noticed that a fixed point alternative method is very important for the solution of the Ulam problem. In other words, they employed this fixed point method to the investigation of the Cauchy functional equation [13] and for the quadratic functional equation [12] (for more applications of this method, see [8–10]). The generalized Hyers-Ulam stability of different functional equations in intuitionistic fuzzy normed spaces has been studied by a number of authors (see [3, 6, 7, 11, 20, 21]). Over the last seven decades, the above problem has been tackled by numerous researchers and its

Received December 21, 2016; Revised May 17, 2017; Accepted July 28, 2017.
2010 *Mathematics Subject Classification.* Primary 39B52, 32B72, 32B82.
Key words and phrases. additive functional equation, generalized Ulam-Hyers stability, Banach space, intuitionistic fuzzy normed space, fixed point.

solutions via various forms of functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1, 16, 17, 26].

In this paper, we establish the generalized Ulam-Hyers stability of (r, s)-type additive functional equation

$$f(rx + sy) = \left(\frac{r+s}{2}\right) f(x+y) + \left(\frac{r-s}{2}\right) f(x-y), \tag{1.1}$$

where $r, s \in \mathbb{R}$ with $r \neq \pm s$, in intuitionistic fuzzy normed space using direct and fixed point methods.

2. Definitions and notations

In this section, we state the basic definitions and notations in the setting of intuitionistic fuzzy normed space.

Definition 2.1 ([31])**.** Let X be an arbitrary set. A fuzzy set M in X is a function with domain X and values in $[0, 1]$.

Definition 2.2. A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is said to be continuous t-norm if $*$ satisfies the following conditions:

(1) $*$ is commutative and associative;
(2) $*$ is continuous;
(3) $a * 1 = a$ for all $a \in [0, 1]$;
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3. A binary operation $\diamond : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is said to be continuous t-conorm if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative;
(ii) $\diamond$ is continuous;
(iii) $a \diamond 0 = a$ for all $a \in [0, 1]$;
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Using the notions of continuous t-norm and t-conorm, Saadati and Park [27] introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 2.4. The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an *intuitionistic fuzzy normed space* (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$

(IFN1) $\mu(x, t) + \nu(x, t) \leq 1$,
(IFN2) $\mu(x, t) > 0$,
(IFN3) $\mu(x, t) = 1$, if and only if $x = 0$,
(IFN4) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{\alpha}\right)$ for each $\alpha \neq 0$,
(IFN5) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$,
(IFN6) $\mu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous,
(IFN7) $\lim\limits_{t\to\infty} \mu(x, t) = 1$ and $\lim\limits_{t\to 0} \mu(x, t) = 0$,

(IFN8) $\nu(x,t) < 1$,
(IFN9) $\nu(x,t) = 0$, if and only if $x = 0$,
(IFN10) $\nu(\alpha x, t) = \nu\left(x, \frac{t}{\alpha}\right)$ for each $\alpha \neq 0$,
(IFN11) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$,
(IFN12) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous,
(IFN13) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case, (μ,ν) is called an *intuitionistic fuzzy norm* (on X).

Example 2.5. Let $(X, \|\cdot\|)$ be a normed space. Let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|} & \text{if} \quad t > 0; \\ 0 & \text{if} \quad t \leq 0; \end{cases} \qquad \text{and} \qquad \nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|} & \text{if} \quad t > 0; \\ 0 & \text{if} \quad t \leq 0. \end{cases}$$

Then, $(X, \mu, \nu, *, \diamond)$ is an IFNS.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are investigated in [27].

Definition 2.6. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $x = \{x_k\}$ is said to be *intuitionistic fuzzy convergent* to a point $L \in X$ if

$$\lim_{k\to\infty} \mu(x_k - L, t) = 1 \quad \text{and} \quad \lim_{k\to\infty} \nu(x_k - L, t) = 0$$

for all $t > 0$. In this case, we write $x_k \xrightarrow{IF} L$ as $k \to \infty$.

Definition 2.7. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, $x = \{x_k\}$ is said to be *intuitionistic fuzzy Cauchy sequence* if

$$\lim_{k\to\infty} \mu\left(x_{k+p} - x_k, t\right) = 1 \quad \text{and} \quad \lim_{k\to\infty} \nu\left(x_{k+p} - x_k, t\right) = 0$$

for all $t > 0$, and $p = 1, 2, \ldots$.

Definition 2.8. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, $(X, \mu, \nu, *, \diamond)$ is said to be *complete* if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent $(X, \mu, \nu, *, \diamond)$.

3. Stability results: direct method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.1) in INFS using direct and fixed point ways. Here and subsequently, assume that X is a linear space, (Z, μ', ν') is an intuitionistic fuzzy normed space and (Y, μ, ν) is an intuitionistic fuzzy Banach space. Now, we use the following notation for a given mapping $f : X \longrightarrow Y$ such that

$$Df_{(r,s)}(x,y) = f(rx+sy) - \left(\frac{r+s}{2}\right) f(x+y) - \left(\frac{r-s}{2}\right) f(x-y),$$

where $r, s \in \mathbb{R}$ with $r \neq \pm s$ for all $x, y \in X$.

Theorem 3.1. *Let $\tau \in \{1, -1\}$. Let $K_\mu, K_\nu : X \times X \longrightarrow Z$ be a mapping such that for some $0 < \left(\frac{p}{a}\right)^\tau < 1$, and*

$$\begin{cases} \mu'\left(K_\mu\left(a^{n\tau}x, a^{n\tau}y\right), t\right) \geq \mu'\left(p^{n\tau}K_\mu\left(x, y\right), t\right) \\ \nu'\left(K_\nu\left(a^{n\tau}x, a^{n\tau}y\right), t\right) \leq \nu'\left(p^{n\tau}K_\nu\left(x, y\right), t\right) \end{cases} \tag{3.1}$$

and

$$\begin{cases} \lim\limits_{n\to\infty} \mu'\left(K_\mu\left(a^{\tau n}x, a^{\tau n}y\right), a^{\tau n}t\right) = 1 \\ \lim\limits_{n\to\infty} \nu'\left(K_\nu\left(a^{\tau n}x, a^{\tau n}y\right), a^{\tau n}t\right) = 0 \end{cases} \tag{3.2}$$

for all $x, y \in X$ and all $t > 0$. Let $f : X \longrightarrow Y$ be a mapping satisfying the inequality

$$\begin{cases} \mu\left(Df_{(r,s)}(x, y), t\right) \geq \mu'\left(K_\mu\left(x, y\right), t\right) \\ \nu\left(Df_{(r,s)}(x, y), t\right) \leq \nu'\left(K_\nu\left(x, y\right), t\right) \end{cases} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Then, there exists a unique additive mapping $\mathcal{A} : X \longrightarrow Y$ satisfying (1.1) *and*

$$\begin{cases} \mu\left(f(x) - \mathcal{A}(x), t\right) \geq \mu'\left(K_\mu\left(x\right), 2|2-p|t\right) \\ \nu\left(f(x) - \mathcal{A}(x), t\right) \leq \nu'\left(K_\nu\left(x\right), 2|2-p|t\right), \end{cases} \tag{3.4}$$

where

$$\begin{cases} \mu'\left(K_\mu\left(x\right), t\right) \\ = \mu'\Big(K_\mu\Big(\frac{2rx}{r^2-s^2}, \frac{-2sx}{r^2-s^2}\Big), t\Big) * \mu'\Big(K_\mu\Big(\frac{x}{r+s}, \frac{x}{r+s}\Big), t\Big) * \mu'\Big(K_\mu\Big(\frac{x}{r-s}, \frac{-x}{r-s}\Big), t\Big) \\ \nu'\left(K_\nu\left(x\right), t\right) \\ = \nu'\Big(K_\nu\Big(\frac{2rx}{r^2-s^2}, \frac{-2sx}{r^2-s^2}\Big), t\Big) \diamond \nu'\Big(K_\nu\Big(\frac{x}{r+s}, \frac{x}{r+s}\Big), t\Big) \diamond \nu'\Big(K_\nu\Big(\frac{x}{r-s}, \frac{-x}{r-s}\Big), t\Big) \end{cases} \tag{3.5}$$

for all $x \in X$ and all $t > 0$.

Proof. **Case (i)** Let $\tau = 1$. Letting (x, y) by $\left(\frac{x+y}{2}, \frac{x+y}{2}\right)$ in (3.3), we get

$$\begin{cases} \mu\left(f\left(\frac{r+s}{2}(x+y)\right) - \frac{r+s}{2}f\left(x+y\right), t\right) \geq \mu'\left(K_\mu\left(\frac{x+y}{2}, \frac{x+y}{2}\right), t\right) \\ \nu\left(f\left(\frac{r+s}{2}(x+y)\right) - \frac{r+s}{2}f\left(x+y\right), t\right) \leq \nu'\left(K_\nu\left(\frac{x+y}{2}, \frac{x+y}{2}\right), t\right) \end{cases} \tag{3.6}$$

for all $x, y \in X$ and all $t > 0$. Putting (x, y) by $\left(\frac{x-y}{2}, \frac{y-x}{2}\right)$ in (3.3), we find

$$\begin{cases} \mu\left(f\left(\frac{r-s}{2}(x-y)\right) - \frac{r-s}{2}f\left(x-y\right), t\right) \geq \mu'\left(K_\mu\left(\frac{x-y}{2}, \frac{y-x}{2}\right), t\right) \\ \nu\left(f\left(\frac{r-s}{2}(x-y)\right) - \frac{r-s}{2}f\left(x-y\right), t\right) \leq \nu'\left(K_\nu\left(\frac{x+y}{2}, \frac{y-x}{2}\right), t\right) \end{cases} \tag{3.7}$$

for all $x, y \in X$ and all $t > 0$. It follows from (3.6), (3.7) and (IFN5), (IFN11) that

$$\begin{cases} \mu\left(f\left(\frac{r-s}{2}(x-y)\right) - \frac{r-s}{2}f\left(x-y\right), t\right) \\ \geq \mu'\left(K_\mu\left(x, y\right), t\right) * \mu'\left(K_\mu\left(\frac{x+y}{2}, \frac{x+y}{2}\right), t\right) * \mu'\left(K_\mu\left(\frac{x-y}{2}, \frac{y-x}{2}\right), t\right) \\ \nu\left(f\left(\frac{r-s}{2}(x-y)\right) - \frac{r-s}{2}f\left(x-y\right), t\right) \\ \leq \nu'\left(K_\nu\left(x, y\right), t\right) \diamond \nu'\left(K_\nu\left(\frac{x+y}{2}, \frac{x+y}{2}\right), t\right) \diamond \nu'\left(K_\nu\left(\frac{x-y}{2}, \frac{y-x}{2}\right), t\right) \end{cases} \tag{3.8}$$

for all $x, y \in X$ and all $t > 0$. Replacing x and y by $\frac{x}{r+s} + \frac{y}{r-s}$ and $\frac{x}{r+s} - \frac{y}{r-s}$, respectively in (3.8), we have

$$\text{(3.9)} \quad \begin{cases} \mu\left(f(x+y) - f\left(x\right) - f\left(y\right), t\right) \\ \geq \mu'\left(K_\mu\left(\frac{x}{r+s} + \frac{y}{r-s}, \frac{x}{r+s} - \frac{y}{r-s}\right), t\right) * \mu'\left(K_\mu\left(\frac{x}{r+s}, \frac{x}{r+s}\right), t\right) \\ *\mu'\left(K_\mu\left(\frac{y}{r-s}, \frac{-y}{r-s}\right), t\right) \\ \nu\left(f(x+y) - f\left(x\right) - f\left(y\right), t\right) \\ \leq \nu'\left(K_\nu\left(\frac{x}{r+s} + \frac{y}{r-s}, \frac{x}{r+s} - \frac{y}{r-s}\right), t\right) \diamond \nu'\left(K_\nu\left(\frac{x}{r+s}, \frac{x}{r+s}\right), t\right) \\ \diamond\nu'\left(K_\nu\left(\frac{y}{r-s}, \frac{-y}{r-s}\right), t\right) \end{cases}$$

for all $x, y \in X$ and all $t > 0$. Finally, by putting $y = x$ in (3.9), we obtain

$$\text{(3.10)} \quad \begin{cases} \mu\left(f(2x) - 2f\left(x\right), t\right) \geq \mu'\left(K_\mu\left(x\right), t\right) \\ \nu\left(f(2x) - 2f\left(x\right), t\right) \leq \nu'\left(K_\nu\left(x\right), t\right) \end{cases}$$

for all $x, y \in X$ and all $t > 0$, where

(3.11)

$$\begin{cases} \mu'\left(K_\mu\left(x\right), t\right) \\ = \mu'\left(K_\mu\left(\frac{2rx}{r^2-s^2}, \frac{-2sx}{r^2-s^2}\right), t\right) * \mu'\left(K_\mu\left(\frac{x}{r+s}, \frac{x}{r+s}\right), t\right) * \mu'\left(K_\mu\left(\frac{x}{r-s}, \frac{-x}{r-s}\right), t\right) \\ \nu'\left(K_\nu\left(x\right), t\right) \\ = \nu'\left(K_\nu\left(\frac{2rx}{r^2-s^2}, \frac{-2sx}{r^2-s^2}\right), t\right) \diamond \nu'\left(K_\nu\left(\frac{x}{r+s}, \frac{x}{r+s}\right), t\right) \diamond \nu'\left(K_\nu\left(\frac{x}{r-s}, \frac{-x}{r-s}\right), t\right). \end{cases}$$

Using (IFN4) and (IFN10) in (3.10), we arrive

$$\text{(3.12)} \quad \begin{cases} \mu\left(\frac{f(ax)}{2} - f(x), \frac{t}{2}\right) \geq \mu'\left(K_\mu(x), t\right) \\ \nu\left(\frac{f(ax)}{2} - f(x), \frac{t}{2}\right) \leq \nu'\left(K_\nu(x), t\right) \end{cases}$$

for all $x \in X$ and all $t > 0$. Substituting x by $2^n x$ in (3.12), we have

$$\text{(3.13)} \quad \begin{cases} \mu\left(\frac{f(2^{n+1}x)}{2} - f(2^n x), \frac{t}{2}\right) \geq \mu'\left(K_\mu(2^n x), t\right) \\ \nu\left(\frac{f(2^{n+1}x)}{2} - f(2^n x), \frac{t}{2}\right) \leq \nu'\left(K_\nu(2^n x), t\right) \end{cases}$$

for all $x \in X$ and all $t > 0$. It is easy to verify from (3.13) and using (3.1), (IFN4), (IFN4) that

$$\text{(3.14)} \quad \begin{cases} \mu\left(\frac{f(2^{n+1}x)}{2^{(n+1)}} - \frac{f(2^n x)}{2^n}, \frac{t}{2^{n+1}}\right) \geq \mu'\left(K_\mu(x), \frac{t}{p^n}\right) \\ \nu\left(\frac{f(2^{n+1}x)}{2^{(n+1)}} - \frac{f(2^n x)}{2^n}, \frac{t}{2^{n+1}}\right) \leq \nu'\left(K_\nu(x), \frac{t}{p^n}\right) \end{cases}$$

for all $x \in X$ and all $t > 0$. Interchanging t into $p^n r$ in (3.14), we have

$$\text{(3.15)} \quad \begin{cases} \mu\left(\frac{f(2^{n+1}x)}{2^{(n+1)}} - \frac{f(2^n x)}{2^n}, \frac{t \cdot p^n}{2^{n+1}}\right) \geq \mu'\left(K_\mu(x), t\right) \\ \nu\left(\frac{f(2^{n+1}x)}{2^{(n+1)}} - \frac{f(2^n x)}{2^n}, \frac{t \cdot p^n}{2^{n+1}}\right) \leq \nu'\left(K_\nu(x), t\right) \end{cases}$$

for all $x \in X$ and all $t > 0$. It is easy to see that

$$\frac{f(2^n x)}{2^n} - f(x) = \sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{2^{(i+1)}} - \frac{f(2^i x)}{2^i} \tag{3.16}$$

for all $x \in X$. From equations (3.14) and (3.15), we get

$$\begin{cases} \mu\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{i=0}^{n-1} \frac{p^i\ t}{2^{i+1}}\right) = \mu\left(\sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{2^{(i+1)}} - \frac{f(2^i x)}{2^i}, \sum_{i=0}^{n-1} \frac{p^i\ t}{2^{i+1}}\right) \\ \nu\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{i=0}^{n-1} \frac{p^i\ t}{2^{i+1}}\right) = \nu\left(\sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{2^{(i+1)}} - \frac{f(2^i x)}{2^i}, \sum_{i=0}^{n-1} \frac{p^i\ t}{2^{i+1}}\right) \end{cases} \tag{3.17}$$

for all $x \in X$ and all $t > 0$. From (3.16) and (3.17), we have

$$\begin{cases} \mu\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{i=0}^{n-1} \frac{p^i t}{2^{i+1}}\right) \geq \prod_{i=0}^{n-1} \mu\left(\frac{f(2^{i+1}x)}{2^{(i+1)}} - \frac{f(2^i x)}{2^i}, \frac{p^i\ t}{2^{i+1}}\right) \\ \nu\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{i=0}^{n-1} \frac{p^i t}{2^{i+1}}\right) \leq \coprod_{i=0}^{n-1} \nu\left(\frac{f(2^{i+1}x)}{2^{(i+1)}} - \frac{f(2^i x)}{2^i}, \frac{p^i\ t}{2^{i+1}}\right), \end{cases} \tag{3.18}$$

where $\prod_{i=0}^{n-1} c_j = c_1 * c_2 * \cdots * c_n$ and $\coprod_{i=0}^{n-1} d_j = d_1 \diamond d_2 \diamond \cdots \diamond d_n$ for all $x \in X$ and all $t > 0$. Hence

$$\begin{cases} \mu\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{i=0}^{n-1} \frac{p^i\ t}{2^{i+1}}\right) \geq \prod_{i=0}^{n-1} \mu'\left(K_\mu(x), t\right) = \mu'\left(K_\mu(x), t\right) \\ \nu\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{i=0}^{n-1} \frac{p^i\ t}{2^{i+1}}\right) \leq \coprod_{i=0}^{n-1} \nu'\left(K_\nu(x), t\right) = \nu'\left(K_\nu(x), t\right) \end{cases} \tag{3.19}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^m x$ in (3.19) and using (3.2), (IFN4), (IFN10), we obtain

$$\begin{cases} \mu\left(\frac{f(2^{n+m}x)}{2^{(n+m)}} - \frac{f(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{p^i\ t}{2 \cdot 2^{(i+m)}}\right) \geq \mu'\left(K_\mu(2^m x), t\right) = \mu'\left(K_\mu(x), \frac{t}{p^m}\right) \\ \nu\left(\frac{f(2^{n+m}x)}{2^{(n+m)}} - \frac{f(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{p^i\ t}{2 \cdot 2^{(i+m)}}\right) \leq \nu'\left(K_\nu(2^m x), t\right) = \nu'\left(K_\nu(x), \frac{t}{p^m}\right) \end{cases} \tag{3.20}$$

for all $x \in X$ and all $t > 0$ and all $m, n \geq 0$. Replacing t by $p^m t$ in (3.20), we get

$$\begin{cases} \mu\left(\frac{f(2^{n+m}x)}{2^{(n+m)}} - \frac{f(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{p^{i+m}\ t}{2 \cdot 2^{(i+m)}}\right) \geq \mu'\left(K_\mu(x), t\right) \\ \nu\left(\frac{f(2^{n+m}x)}{2^{(n+m)}} - \frac{f(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{p^{i+m}\ t}{2 \cdot 2^{(i+m)}}\right) \leq \nu'\left(K_\nu(x), t\right) \end{cases} \tag{3.21}$$

for all $x \in X$ and all $t > 0$ and all $m, n \geq 0$. The relation (3.20) implies that

$$\begin{cases} \mu\left(\frac{f(2^{n+m}x)}{2^{(n+m)}} - \frac{f(2^m x)}{2^m}, t\right) \geq \mu'\left(K_\mu(x), \frac{t}{\sum_{i=m}^{n-1} \frac{p^i}{2 \cdot 2^i}}\right) \\ \nu\left(\frac{f(2^{n+m}x)}{2^{(n+m)}} - \frac{f(2^m x)}{2^m}, t\right) \leq \nu'\left(K_\nu(x), \frac{t}{\sum_{i=m}^{n-1} \frac{p^i}{2 \cdot 2^i}}\right) \end{cases} \tag{3.22}$$

holds for all $x \in X$ and all $t > 0$ and all $m, n \geq 0$. Since $0 < p < 1$, we have $\sum_{i=0}^{n} \left(\frac{p}{1}\right)^i < \infty$. The Cauchy criterion for convergence in IFNS shows that the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is Cauchy in (Y,μ,ν). Since (Y,μ,ν) is a complete IFNS this sequence converges to some point $\mathcal{A}(x) \in Y$. That is,

$$\lim_{n\to\infty} \mu\left(\frac{f(2^n x)}{2^n} - \mathcal{A}(x), t\right) = 1, \lim_{n\to\infty} \nu\left(\frac{f(2^n x)}{2^n} - \mathcal{A}(x), t\right) = 0$$

for all $x \in X$ and all $t > 0$. So $\frac{f(2^n x)}{2^n} \xrightarrow{IF} \mathcal{A}(x)$ as $n \to \infty$. Letting $m = 0$ in (3.22), we arrive at

$$\begin{cases} \mu\Big(\frac{f(2^n x)}{2^n} - f(x), t\Big) \ge \mu'\left(K_\mu(x), \frac{t}{\sum_{i=0}^{n-1}\frac{p^i}{2\cdot 2^i}}\right) \\ \nu\Big(\frac{f(2^n x)}{2^n} - f(x), t\Big) \le \nu'\left(K_\nu(x), \frac{t}{\sum_{i=0}^{n-1}\frac{p^i}{2\cdot 2^i}}\right) \end{cases} \tag{3.23}$$

for all $x \in X$ and all $t > 0$. Letting n tend to infinity in (3.23), we have

$$\begin{cases} \mu\Big(\mathcal{A}(x) - f(x), t\Big) \ge \mu'\left(K_\mu(x), 2\ t|2-p|\right) \\ \nu\Big(\mathcal{A}(x) - f(x), t\Big) \le \nu'\left(K_\nu(x), 2\ t|2-p|\right) \end{cases} \tag{3.24}$$

for all $x \in X$ and all $t > 0$. To prove $\mathcal{A}$ satisfies (1.1), replacing (x,y) by $(2^n x, 2^n y)$ in (3.3) respectively, we obtain

$$\begin{cases} \mu\left(\frac{1}{2^n} Df_{(r,s)}(2^n x, 2^n y), t\right) \ge \mu'\left(K_\mu(2^n x, 2^n y), 2^n t\right) \\ \nu\left(\frac{1}{2^n} Df_{(r,s)}(2^n x, 2^n y), t\right) \le \nu'\left(K_\nu(2^n x, 2^n y), 2^n t\right) \end{cases} \tag{3.25}$$

for all $x, y \in X$ and all $t > 0$. Now,

$$\begin{aligned} &\mu\Big(\mathcal{A}(rx+sy) - \left(\frac{r+s}{2}\right)\mathcal{A}(x) - \left(\frac{r-s}{2}\right)\mathcal{A}(y)\Big) \\ \ge\ &\mu\Big(\mathcal{A}(rx+sy) - \frac{1}{2^n} f(rx+sy), \frac{t}{4}\Big) \\ &* \mu\Big(-\left(\frac{r+s}{2}\right)\mathcal{A}(x) + \frac{1}{2^n}\left(\frac{r+s}{2}\right) f(x), \frac{t}{4}\Big) \\ &* \mu\Big(-\left(\frac{r-s}{2}\right)\mathcal{A}(y) + \frac{1}{2^n}\left(\frac{r-s}{2}\right) f(y), \frac{t}{4}\Big) \\ &* \mu\Big(\frac{1}{2^n} f(rx+sy) - \frac{1}{2^n}\left(\frac{r+s}{2}\right) f(x) - \frac{1}{2^n}\left(\frac{r-s}{2}\right) f(y), \frac{t}{4}\Big) \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} &\nu\Big(\mathcal{A}(rx+sy) - \left(\frac{r+s}{2}\right)\mathcal{A}(x) - \left(\frac{r-s}{2}\right)\mathcal{A}(y)\Big) \\ \ge\ &\nu\Big(\mathcal{A}(rx+sy) - \frac{1}{2^n} f(rx+sy), \frac{t}{4}\Big) \\ &\diamond \nu\Big(-\left(\frac{r+s}{2}\right)\mathcal{A}(x) + \frac{1}{2^n}\left(\frac{r+s}{2}\right) f(x), \frac{t}{4}\Big) \\ &\diamond \nu\Big(-\left(\frac{r-s}{2}\right)\mathcal{A}(y) + \frac{1}{2^n}\left(\frac{r-s}{2}\right) f(y), \frac{t}{4}\Big) \end{aligned}$$

$$\diamond\, \nu\Big(\frac{1}{2^n} f(rx+sy) - \frac{1}{2^n}\left(\frac{r+s}{2}\right) f(x) - \frac{1}{2^n}\left(\frac{r-s}{2}\right) f(y), \frac{t}{4}\Big) \tag{3.27}$$

for all $x, y \in X$ and all $t > 0$. On the other hand,

$$\begin{cases} \lim\limits_{n\to\infty} \mu\Big(\frac{1}{2^n} Df_{(r,s)}(2^n x, 2^n y), \frac{t}{6}\Big) = 1 \\ \lim\limits_{n\to\infty} \nu\Big(\frac{1}{2^n} Df_{(r,s)}(2^n x, 2^n y), \frac{t}{6}\Big) = 0 \end{cases} \tag{3.28}$$

for all $x, y \in X$ and all $t > 0$. Letting $n \to \infty$ in (3.26), (3.27) and using (3.28), we observe that $\mathcal{A}$ fulfills (1.1). Therefore, $\mathcal{A}$ is an additive mapping. Let $\mathcal{A}'(x)$ be another mapping satisfying (1.1) and (3.4). Hence,

$$\begin{aligned} \mu(\mathcal{A}(x) - \mathcal{A}'(x), t) &\geq \mu\left(\mathcal{A}(2^n x) - f(2^n x), \tfrac{t\cdot 2^n}{2}\right) * \mu\left(f(2^n x) - \mathcal{A}'(2^n x), \tfrac{t\cdot 2^n}{2}\right) \\ &\geq \mu'\left(K_\mu(2^n x), \tfrac{2t\ 2^n|2-p|}{2}\right) \geq \mu'\left(K_\mu(x), \tfrac{2t\ 2^n|2-p|}{2\cdot p^n}\right), \\ \nu(\mathcal{A}(x) - \mathcal{A}'(x), t) &\leq \nu\left(\mathcal{A}(2^n x) - f(2^n x), \tfrac{t\cdot 2^n}{2}\right) \diamond \nu\left(f(2^n x) - \mathcal{A}'(2^n x), \tfrac{t\cdot 2^n}{2}\right) \\ &\leq \nu'\left(K_\nu(2^n x), \tfrac{2t\ 2^n|2-p|}{2}\right) \leq \nu'\left(K_\nu(x), \tfrac{2t\ 2^n|2-p|}{2\cdot p^n}\right) \end{aligned}$$

for all $x \in X$ and all $t > 0$. Since $\lim\limits_{n\to\infty} \frac{2t\ 2^n|2-p|}{2\ p^n} = \infty$, we obtain

$$\begin{cases} \lim\limits_{n\to\infty} \mu'\left(K_\mu(x), \frac{2t\ 2^n|2-p|}{2\cdot p^n}\right) = 1 \\ \lim\limits_{n\to\infty} \nu'\left(K_\nu(x), \frac{2t\ 2^n|2-p|}{2\cdot p^n}\right) = 0 \end{cases}$$

for all $x \in X$ and all $t > 0$. Thus

$$\begin{cases} \mu(\mathcal{A}(x) - \mathcal{A}'(x), t) = 1 \\ \nu(\mathcal{A}(x) - \mathcal{A}'(x), t) = 0 \end{cases}$$

for all $x \in X$ and all $t > 0$. Hence, $\mathcal{A}(x) = \mathcal{A}'(x)$. Therefore, $\mathcal{A}(x)$ is unique.

Case (ii) For $\tau = -1$. Putting x by $\frac{x}{2}$ in (3.10), we get

$$\begin{cases} \mu\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \mu'\left(K_\mu\left(\frac{x}{2}\right), t\right) \\ \nu\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \leq \nu'\left(K_\nu\left(\frac{x}{2}\right), t\right) \end{cases} \tag{3.29}$$

for all $x, y \in X$ and all $t > 0$. The rest of the proof is similar to that of **Case (i)**. This completes the proof. □

The following corollary is an immediate consequence of Theorem 3.1, regarding to the stability of (1.1).

Corollary 3.2. *Let X be a normed space with norm $\|\cdot\|$, and let $z_0 \in Z$. Suppose that the mapping $f : X \longrightarrow Y$ satisfies the double inequality*

(3.30)

$$\mu\left(Df_{(r,s)}(x,y),t\right) \geq \begin{cases} \mu'\left(\lambda z_0, t\right), \\ \mu'\left(\lambda\left(||x||^a + ||y||^b\right) z_0, t\right), \\ \mu'\left(\lambda ||x||^a ||y||^b z_0, t\right), \\ \mu'\left(\lambda\left\{||x||^a \; ||y||^b + \left(||x||^{a+b} + ||y||^{a+b}\right)\right\} z_0, t\right), \end{cases}$$
$$\nu\left(Df_{(r,s)}(x,y),t\right) \leq \begin{cases} \nu'\left(\lambda z_0, t\right), \\ \nu'\left(\lambda\left(||x||^a + ||y||^b\right) z_0, t\right), \\ \nu'\left(\lambda ||x||^a ||y||^b z_0, t\right), \\ \nu'\left(\lambda\left\{||x||^a \; ||y||^b + \left(||x||^{a+b} + ||y||^{a+b}\right)\right\} z_0, t\right) \end{cases}$$

for all $x, y \in X$ and all $t > 0$, where λ, a, b are constants with $\lambda > 0$. Then, there exists a unique additive mapping $\mathcal{A} : X \longrightarrow Y$ such that

(3.31)

$$\mu\left(f(x) - \mathcal{A}(x), t\right) \geq \begin{cases} \mu'\left(3\lambda z_0, 2|2-p|t\right), \\ \mu'\left(\left[\frac{\lambda||x||^a(2|r|^a+|r+s|^a+|r-s|^a)}{|r^2-s^2|^a}\right] z_0 \right. \\ \qquad \left. + \left[\frac{\lambda||x||^b(2|s|^b+|r+s|^b+|r-s|^b)}{|r^2-s^2|^b}\right] z_0, 2|2-p|t\right), \\ \mu'\left(\frac{\lambda||x||^{a+b}2^{(a+b)}r^a s^b+(r+s)^{a+b}+(r-s)^{a+b}}{(r^2-s^2)^{a+b}} z_0, 2|2-p|t\right), \\ \mu'\left(\frac{\lambda||x||^{a+b}}{|r^2-s^2|^{a+b}}\left(2|r|^{a+b} + 2|s|^{a+b} + 2|r+s|^{a+b}\right.\right. \\ \left.\left. +2|r-s|^{a+b} + 2^{(a+b)}r^a s^b + (r+s)^{a+b}\right) z_0, 2|2-p|t\right), \end{cases}$$
$$\nu\left(f(x) - \mathcal{A}(x), t\right) \leq \begin{cases} \nu'\left(3\lambda z_0, 2|2-p|t\right), \\ \nu'\left(\left[\frac{\lambda||x||^a(2|r|^a+|r+s|^a+|r-s|^a)}{|r^2-s^2|^a}\right] z_0 \right. \\ \qquad \left. + \left[\frac{\lambda||x||^b(2|s|^b+|r+s|^b+|r-s|^b)}{|r^2-s^2|^b}\right] z_0, 2|2-p|t\right), \\ \nu'\left(\frac{\lambda||x||^{a+b}2^{(a+b)}r^a s^b+(r+s)^{a+b}+(r-s)^{a+b}}{(r^2-s^2)^{a+b}} z_0, 2|2-p|t\right), \\ \nu'\left(\frac{\lambda||x||^{a+b}}{|r^2-s^2|^{a+b}}\left(2|r|^{a+b} + 2|s|^{a+b} + 2|r+s|^{a+b}\right.\right. \\ \left.\left. +2|r-s|^{a+b} + 2^{(a+b)}r^a s^b + (r+s)^{a+b}\right) z_0, 2|2-p|t\right) \end{cases}$$

for all $x \in X$ and all $t > 0$.

We close this section by an example related to Theorem 3.1.

Example 3.3. Let X be a normed space. Also μ, ν and μ', ν' be intuitionistic fuzzy norms on X and $\mathbb{R}$, respectively defined by

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|} & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X, \end{cases}$$

$$\mu'(x,t) = \begin{cases} \frac{t}{t+|x|} & t > 0, \quad x \in \mathbb{R}, \\ 0, & t \leq 0, \quad x \in \mathbb{R}, \end{cases}$$

and

$$\nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|} & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X, \end{cases}$$

$$\nu'(x,t) = \begin{cases} \frac{|x|}{t+|x|} & t > 0, \quad x \in \mathbb{R}, \\ 0, & t \leq 0, \quad x \in \mathbb{R}. \end{cases}$$

Let $\alpha_\mu = \alpha_\nu : \mathbb{R} \longrightarrow \mathbb{R}$ be functions such that $\alpha_\mu(2t) < p\alpha_\mu(t)$ for all $t > 0$ and $1 \leq p < 2$. Define

$$\begin{aligned} \beta_\mu(x,y) &= \beta_\nu(x,y) \\ &= \alpha_\mu(\|rx+sy\|) - \left|\frac{r+s}{2}\right| \alpha_\mu(\|x-y\|) - \left|\frac{r-s}{2}\right| \alpha_\mu(\|x-y\|) \end{aligned}$$

for all $x, y \in X$, where $r, s \in \mathbb{R}$ with $r \neq \pm s$. Let $x_0 \in X$ be a unit vector and define $f : X \longrightarrow X$ through $f(x) = \alpha_\mu(|x|)\, x_0$. Now for any $x, y \in X$ and $r > 0$, we have

$$\begin{aligned} \mu\left(Df_{(r,s)}(x,y), t\right) &= \frac{t}{t + |\beta_\mu(x,y)|\, \|x_0\|} \\ &\geq \frac{t}{t + |\beta_\mu(x,y)|} = \mu'(\beta_\mu(x,y), t) \end{aligned}$$

and

$$\begin{aligned} \mu'(\alpha_\mu(2x,2y), t) &= \frac{t}{t + |\beta_\mu(2x,2y)|} \\ &\geq \frac{t}{t + p|\beta_\mu(x,y)|} = \mu'(p\ \beta_\mu(x,y), t). \end{aligned}$$

Also, for any $x, y \in X$ and $t > 0$, we have

$$\begin{aligned} \nu\left(Df_{(r,s)}(x,y), t\right) &= \frac{|\beta_\nu(x,y)|\, \|x_0\|}{t + |\beta_\nu(x,y)|\, \|x_0\|} \\ &\leq \frac{|\beta_\nu(x,y)|}{t + |\beta_\nu(x,y)|} = \nu'(\beta_\nu(x,y), t) \end{aligned}$$

and

$$\begin{aligned} \nu'(\beta_\nu(2x,2y), t) &= \frac{|\beta_\nu(2x,2y)|}{t + |\beta_\nu(2x,2y)|} \\ &\leq \frac{p|\beta_\nu(x,y)|}{t + p|\beta_\nu(x,y)|} = \nu'(p\ \beta_\nu(x,y), t). \end{aligned}$$

Hence, the inequalities (3.1) and (3.3) are satisfied. Using Theorem 3.1, there exists a unique additive mapping $A : X \longrightarrow Y$ such that

$$\begin{cases} \mu(f(x) - A(x), t) \geq \mu'(\beta_\mu(x), 2\,|2-p|\, t) \\ \nu(f(x) - A(x), t) \leq \nu'(\beta_\nu(x), 2\,|2-p|\, t), \end{cases}$$

where

$$\begin{aligned}\mu'\left(\beta_\mu\left(x\right),t\right) = \mu'&\left(\beta_\mu\left(\frac{2rx}{r^2-s^2},\frac{-2sx}{r^2-s^2}\right),t\right) * \mu'\left(\beta_\mu\left(\frac{x}{r+s},\frac{x}{r+s}\right),t\right)\\ &* \mu'\left(\beta_\mu\left(\frac{x}{r-s},\frac{-x}{r-s}\right),t\right)\end{aligned}$$

for all $x \in X$ and $t > 0$.

4. Stability results: fixed point method

In this section, we apply a fixed point theorem for achieving stability of the additive functional equation (1.1). Here, we present the upcoming result due to Margolis and Diaz [19] for fixed point theory.

Theorem 4.1. *Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \longrightarrow \Omega$ with constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \ \forall \ n \geq 0,$$

or there exists a natural number n_0 such that

(FP1) *$d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;*
(FP2) *The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T;*
(FP3) *y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;*
(FP4) *$d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.*

Applying the above theorem, we now obtain the generalized Ulam-Hyers stability of the functional equation (1.1).

Theorem 4.2. *Let $f : X \longrightarrow Y$ be a mapping for which there exists a mapping $K : X \times X \longrightarrow Z$ with the double condition*

$$\left\{\begin{array}{l}\lim\limits_{n\to\infty} \mu'\left(K\left(\chi_i^n x, \chi_i^n y\right), \chi_i^n t\right) = 1\\ \lim\limits_{n\to\infty} \nu'\left(K\left(\chi_i^n x, \chi_i^n y\right), \chi_i^n t\right) = 0\end{array}\right. \tag{4.1}$$

for all $x, y \in X$ and all $t > 0$ where

$$\chi_i = \left\{\begin{array}{lll} 2 & \textit{if} & i = 0\\ \frac{1}{2} & \textit{if} & i = 1\end{array}\right. \tag{4.2}$$

and the double functional inequalities hold

$$\left\{\begin{array}{l}\mu\left(Df_{(r,s)}(x,y),t\right) \geq \mu'\left(K\left(x,y\right),t\right)\\ \nu\left(Df_{(r,s)}(x,y),t\right) \leq \nu'\left(K\left(x,y\right),t\right)\end{array}\right. \tag{4.3}$$

for all $x, y \in X$ and all $t > 0$. If there exists $L = L(i)$ such that the mapping

$$\rho(x) = \frac{1}{2}K\left(\frac{x}{2},\frac{x}{2}\right), \tag{4.4}$$

has the property

$$\left\{ \begin{array}{l} \mu'\left(L\frac{\rho(\chi_i x)}{\chi_i}, t\right) = \mu'\left(\rho(x), t\right) \\ \nu'\left(L\frac{\rho(\chi_i x)}{\chi_i}, t\right) = \nu'\left(\rho(x), t\right) \end{array} \right. \tag{4.5}$$

for all $x \in X$ and all $t > 0$, then there exists a unique additive mapping $\mathcal{A} : X \longrightarrow Y$ satisfying the functional equation (1.1) *and*

$$\left\{ \begin{array}{l} \mu\left(f(x) - \mathcal{A}(x), t\right) \geq \mu'\left(\rho(x), \frac{L^{1-i}}{1-L}t\right) \\ \nu\left(f(x) - \mathcal{A}(x), t\right) \leq \nu'\left(\rho(x), \frac{L^{1-i}}{1-L}t\right) \end{array} \right. \tag{4.6}$$

for all $x \in X$ and all $t > 0$.

Proof. Consider the set $\Lambda = \{h \,|\, h : X \longrightarrow Y,\ h(0) = 0\}$ and introduce the generalized metric on Λ,

(4.7)
$$d(h, f) = \inf\left\{M \in (0,\infty) : \left\{ \begin{array}{l} \mu(h(x) - f(x), t) \geq \mu'(\rho(x), Mt), x \in X \\ \nu(h(x) - f(x), t) \leq \nu'(\rho(x), Mt), x \in X \end{array} \right\}\right\}.$$

It is easy to see that Λ is complete with respect to the above metric. Define $J : \Lambda \longrightarrow \Lambda$ by $Jh(x) = \frac{1}{\chi_i}h(\chi_i x)$ for all $x \in X$. If $h, f \in \Lambda$ so that $d(h, f) \leq M$, then

$$\left\{ \begin{array}{l} \quad\ \mu(h(x) - f(x), t) \geq \mu'(\rho(x), Mt) \qquad (x \in X) \\ \Longrightarrow \mu(\frac{1}{\chi_i}h(\chi_i x) - \frac{1}{\chi_i}f(\chi_i x), t) \geq \mu'(\rho(\chi_i x), \chi_i Mt) \qquad (x \in X) \\ \Longrightarrow \mu(\frac{1}{\chi_i}h(\chi_i x) - \frac{1}{\chi_i}f(\chi_i x), t) \geq \mu'(\rho(x), MLt) \qquad (x \in X) \\ \Longrightarrow \mu(Jh(x) - Jf(x), t) \geq \mu'(\rho(x), MLt) \qquad (x \in X) \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \quad\ \nu(h(x) - f(x), t) \leq \nu'(\rho(x), Mt) \qquad (x \in X) \\ \Longrightarrow \nu(\frac{1}{\chi_i}h(\chi_i x) - \frac{1}{\chi_i}f(\chi_i x), t) \leq \nu'(\rho(\chi_i x), \chi_i Mt) \qquad (x \in X) \\ \Longrightarrow \nu(\frac{1}{\chi_i}h(\chi_i x) - \frac{1}{\chi_i}f(\chi_i x), t) \leq \nu'(\rho(x), MLt) \qquad (x \in X) \\ \Longrightarrow \nu(Jh(x) - Jf(x), t) \leq \nu'(\rho(x), MLt) \qquad (x \in X). \end{array} \right.$$

This implies $d(Jh, Jf) \leq Ld(h, f)$, i.e., J is a strictly contractive mapping on Λ with Lipschitz constant L. It follows from (4.7) and (3.10) that

$$\left\{ \begin{array}{l} \mu\Big(f(2x) - 2f(x), t\Big) \geq \mu'\left(K(x, x), t\right) \\ \nu\Big(f(2x) - 2f(x), t\Big) \leq \nu'\left(K(x, x), t\right) \end{array} \right. \tag{4.8}$$

for all $x \in X$ and all $t > 0$. Now, from (4.8) and (4.5) for the case $i = 0$, we reach

$$
\tag{4.9}
\begin{cases}
\qquad \mu\Big(f(2x) - 2f(x), t\Big) \geq \mu'\left(K(x,x), t\right) \\
\Longrightarrow \mu\Big(\frac{f(2x)}{2} - f(x), t\Big) \geq \mu'\left(K(x,x), 2t\right) \\
\Longrightarrow \mu\Big(Jf(x) - f(x), t\Big) \geq \mu'\left(\rho(x), Lt\right) \\
\Longrightarrow \mu\Big(Jf(x) - f(x), t\Big) \geq \mu'\left(\rho(x), Lt\right) \\
\Longrightarrow \mu\Big(Jf(x) - f(x), t\Big) \geq \mu'\left(\rho(x), Lt\right) \\
\qquad \nu\Big(f(2x) - 2f(x), t\Big) \leq \nu'\left(K(x,x), t\right) \\
\Longrightarrow \nu\Big(\frac{f(2x)}{2} - f(x), t\Big) \leq \nu'\left(K(x,x), 2t\right) \\
\Longrightarrow \nu\Big(Jf(x) - f(x), t\Big) \leq \nu'\left(\rho(x), Lt\right) \\
\Longrightarrow \nu\Big(Jf(x) - f(x), t\Big) \leq \nu'\left(\rho(x), Lt\right) \\
\Longrightarrow \nu\Big(Jf(x) - f(x), t\Big) \leq \nu'\left(\rho(x), Lt\right)
\end{cases}
$$

for all $x \in X$ and all $t > 0$, i.e.,

$$
\tag{4.10}
d(Jf, f) \leq L = L^{1-0} = L^{1-i}.
$$

Again by interchanging x into $\frac{x}{2}$ in (4.9) and (4.5) for the case $i = 1$, we get

$$
\tag{4.11}
\begin{cases}
\qquad \mu\Big(f(2x) - 2f(x), t\Big) \geq \mu'\left(K(\tfrac{x}{2}, \tfrac{x}{2}), at\right) \\
\Longrightarrow \mu\Big(f(x) - Jf(x), t\Big) \geq \mu'\left(\rho(x), t\right) \\
\Longrightarrow \mu\Big(f(x) - Jf(x), t\Big) \geq \mu'\left(\rho(x), t\right) \\
\Longrightarrow \mu\Big(f(x) - Jf(x), t\Big) \geq \mu'\left(\rho(x), t\right) \\
\qquad \nu\Big(f(2x) - 2f(x), t\Big) \leq \nu'\left(K(\tfrac{x}{2}, \tfrac{x}{2}), at\right) \\
\Longrightarrow \nu\Big(f(x) - Jf(x), t\Big) \leq \nu'\left(\rho(x), t\right) \\
\Longrightarrow \nu\Big(f(x) - Jf(x), t\Big) \leq \nu'\left(\rho(x), t\right) \\
\Longrightarrow \nu\Big(f(x) - Jf(x), t\Big) \leq \nu'\left(\rho(x), t\right)
\end{cases}
$$

for all $x \in X$ and all $t > 0$, i.e.,

$$
\tag{4.12}
d(f, Jf) \leq 1 = L^{1-1} = L^{1-i}.
$$

Thus, from (4.10) and (4.12), we see that the property (FP1) holds. By (FP2), it follows that there exists a fixed point $\mathcal{A}$ of J in Λ such that

$$
\lim_{n\to\infty} \mu\left(\frac{f(\chi_i^n x)}{\chi_i^n} - \mathcal{A}(x), t\right) = 1, \lim_{n\to\infty} \nu\left(\frac{f(\chi_i^n x)}{\chi_i^n} - \mathcal{A}(x), t\right) = 0
$$

for all $x \in X$ and all $t > 0$. Replacing (x, y) by $(\chi_i^n x, \chi_i^n y)$ and dividing by χ_i^t in (4.3) and using the definition of $\mathcal{A}(x)$, and then letting $t \to \infty$, we see that

$\mathcal{A}$ satisfies (1.1) for all $x, y \in X$ and all $z \in X$. By (FP3), $\mathcal{A}$ is the unique fixed point of J in the set $\Delta = \{\mathcal{A} \in \Lambda : d(f, A) < \infty\}, \mathcal{A}$ is the unique mapping such that

$$\begin{aligned}\mu(f(x)-\mathcal{A}(x),t) &\geq \mu'(\rho(x),L^{1-i}t),\ x\in X\\ \nu(f(x)-\mathcal{A}(x),t) &\leq \nu'(\rho(x),L^{1-i}t),\ x\in X\end{aligned}$$

for all $x \in X$ and all $t > 0$. Finally by (FP4), we obtain

$$\begin{aligned}\mu\left(f(x)-\mathcal{A}(x),t\right) &\geq \mu'\left(\rho(x),\tfrac{L^{1-i}}{1-L}t\right)\\ \nu\left(f(x)-\mathcal{A}(x),t\right) &\leq \nu'\left(\rho(x),\tfrac{L^{1-i}}{1-L}t\right)\end{aligned}$$

for all $x \in X$ and all $t > 0$. So, the proof is complete. □

The next corollary is a direct consequence of Theorem 4.2 which shows that (1.1) can be stable.

Corollary 4.3. *Let X be a normed space with norm $\|\cdot\|$, and let $z_0 \in Z$. Suppose that a mapping $f : X \longrightarrow Y$ satisfies the double inequalities*

(4.13)

$$\mu\left(Df_{(r,s)}(x,y),t\right) \geq \begin{cases}\mu'\left(\lambda z_0,t\right),\\ \mu'\left(\lambda\left(||x||^a+||y||^a\right)z_0,t\right),\\ \mu'\left(\lambda||x||^a||y||^a z_0,t\right),\\ \mu'\left(\lambda\left\{||x||^a\ ||y||^a+\left(||x||^{2a}+||y||^{2a}\right)\right\}z_0,t\right),\end{cases}$$
$$\nu\left(Df_{(r,s)}(x,y),t\right) \leq \begin{cases}\nu'\left(\lambda z_0,t\right),\\ \nu'\left(\lambda\left(||x||^a+||y||^a\right)z_0,t\right),\\ \nu'\left(\lambda||x||^a||y||^a z_0,t\right),\\ \nu'\left(\lambda\left\{||x||^a\ ||y||^a+\left(||x||^{2a}+||y||^{2a}\right)\right\}z_0,t\right)\end{cases}$$

for all $x, y \in X$ and all $t > 0$, where λ, a are constants with $\lambda > 0$. Then, there exists a unique additive mapping $\mathcal{A} : X \longrightarrow Y$ such that the double inequalities

(4.14)

$$\mu\left(f(x)-\mathcal{A}(x),t\right)$$
$$\geq \begin{cases}\mu'\left(3\lambda z_0,2|2-p|t\right),\\ \mu'\left(\frac{\lambda||x||^a}{|r^2-s^2|^a}(2|r|^a+2|s|^a+2|r+s|^a+2|r-s|^a)z_0,2|2-p|t\right),\\ \mu'\left(\frac{\lambda||x||^{2a}}{(r^2-s^2)^{2a}}(2^{2a}|r|^a|s|^a+|r+s|^{2a}+|r-s|^{2a})z_0,2|2-p|t\right),\\ \mu'\left(\frac{\lambda||x||^{2a}}{(r^2-s^2)^{2a}}(2^{2a}[|r|^a|s|^a+|r|^{2a}+|s|^{2a}]+3[|r+s|^{2a}+|r-s|^{2a}])z_0,2|2-p|t\right),\end{cases}$$
$$\nu\left(f(x)-\mathcal{A}(x),t\right)$$
$$\leq \begin{cases}\nu'\left(3\lambda z_0,2|2-p|t\right),\\ \nu'\left(\frac{\lambda||x||^a}{|r^2-s^2|^a}(2|r|^a+2|s|^a+2|r+s|^a+2|r-s|^a)z_0,2|2-p|t\right),\\ \nu'\left(\frac{\lambda||x||^{2a}}{(r^2-s^2)^{2a}}(2^{2a}|r|^a|s|^a+|r+s|^{2a}+|r-s|^{2a})z_0,2|2-p|t\right),\\ \nu'\left(\frac{\lambda||x||^{2a}}{(r^2-s^2)^{2a}}(2^{2a}[|r|^a|s|^a+|r|^{2a}+|s|^{2a}]+3[|r+s|^{2a}+|r-s|^{2a}])z_0,2|2-p|t\right)\end{cases}$$

hold for all $x \in X$ and all $t > 0$.

Proof. Set

$$\mu'\left(K(\chi_i^n x, \chi_i^n y), \chi_i^k t\right) = \begin{cases} \mu'\left(\lambda z_0, \chi_i^k t\right), \\ \mu'\left(\lambda\left(||x||^a + ||y||^a\right) z_0, \chi_i^{k-a} t\right), \\ \mu'\left(\lambda ||x||^a ||y||^a z_0, \chi_i^{k-2a} t\right), \\ \mu'\left(\lambda\left\{||x||^a||y||^a + \left(||x||^{2a} + ||y||^{2a}\right)\right\} z_0, \chi_i^{k-2a} t\right) \end{cases}$$

$$= \begin{cases} \to 1 \text{ as } k \to \infty \\ \to 1 \text{ as } k \to \infty \\ \to 1 \text{ as } k \to \infty \\ \to 1 \text{ as } k \to \infty \end{cases}$$

$$\nu'\left(K(\chi_i^n x, \chi_i^n y), \chi_i^k t\right) = \begin{cases} \nu'\left(\lambda z_0, \chi_i^k t\right), \\ \nu'\left(\lambda\left(||x||^a + ||y||^a\right) z_0, \chi_i^{k-a} t\right), \\ \nu'\left(\lambda ||x||^a ||y||^a z_0, \chi_i^{k-2a} t\right), \\ \nu'\left(\lambda\left\{||x||^a||y||^a + \left(||x||^{2a} + ||y||^{2a}\right)\right\} z_0, \chi_i^{k-2a} t\right) \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } k \to \infty \\ \to 0 \text{ as } k \to \infty \\ \to 0 \text{ as } k \to \infty \\ \to 0 \text{ as } k \to \infty \end{cases}$$

for all $x \in X$ and all $t > 0$. Thus, the relation (4.1) holds. It follows from (4.4), (4.5) and (4.13) that

$$\mu'\left(\frac{1}{2}K\left(\frac{x}{2}, \frac{x}{2}\right), t\right)$$
$$\geq \begin{cases} \mu'\left(\lambda z_0, t\right) \\ \mu'\left(\frac{(2^a(|r|^a+|s|^a)+2[|r-s|^a+|r+s|^a])\lambda||x||^a}{|r^2-s^2|^a} z_0, 3t\right) \\ \mu'\left(\frac{(2^{2a}\ |r|^a|s|^a+|r+s|^{2a}+|r-s|^{2a})\lambda||x||^{2a}}{|r^2-s^2|^{2a}} z_0, t\right) \\ \mu'\left(\frac{(2^{2a}[|r|^a|s|^a+|r|^{2a}+|s|^{2a}]+3[|r+s|^{2a}+|r-s|^{2a}])\lambda||x||^{2a}}{|r^2-s^2|^{2a}} z_0, t\right) \end{cases}$$

and

$$\nu'\left(\frac{1}{2}K\left(\frac{x}{2}, \frac{x}{2}\right), t\right)$$
$$\geq \begin{cases} \nu'\left(\lambda z_0, t\right) \\ \nu'\left(\frac{(2^a(|r|^a+|s|^a)+2[|r-s|^a+|r+s|^a])\lambda||x||^a}{|r^2-s^2|^a} z_0, t\right) \\ \nu'\left(\frac{(2^{2a}\ |r|^a|s|^a+|r+s|^{2a}+|r-s|^{2a})\lambda||x||^{2a}}{|r^2-s^2|^{2a}} z_0, t\right) \\ \nu'\left(\frac{(2^{2a}[|r|^a|s|^a+|r|^{2a}+|s|^{2a}]+3[|r+s|^{2a}+|r-s|^{2a}])\lambda||x||^{2a}}{|r^2-s^2|^{2a}} z_0, t\right) \end{cases}$$

for all $x, y \in X$ and all $t > 0$. Also from (4.5), we have

$$\mu'\left(\frac{\rho(\chi_i x)}{\chi_i}, t\right) = \begin{cases} \nu'\left(\lambda z_0, {\chi_i}^k t\right) \\ \mu'\left(\frac{(2^a(|r|^a+|s|^a)+2[|r-s|^a+|r+s|^a])\lambda||x||^a}{|r^2-s^2|^a} z_0, {\chi_i}^{k-a} t\right) \\ \mu'\left(\frac{(2^{2a}\ |r|^a|s|^a+|r+s|^{2a}+|r-s|^{2a})\lambda||x||^{2a}}{|r^2-s^2|^{2a}} z_0, {\chi_i}^{k-2a} t\right) \\ \mu'\left(\frac{(2^{2a}[|r|^a|s|^a+|r|^{2a}+|s|^{2a}]+3[|r+s|^{2a}+|r-s|^{2a}])\lambda||x||^{2a}}{|r^2-s^2|^{2a}} z_0, {\chi_i}^{k-2a} t\right) \end{cases}$$

$$\nu'\left(\frac{\rho(\chi_i x)}{\chi_i}, t\right) = \begin{cases} \nu'\left(\lambda z_0, {\chi_i}^k t\right) \\ \nu'\left(\frac{(2^a(|r|^a+|s|^a)+2[|r-s|^a+|r+s|^a])\lambda||x||^a}{|r^2-s^2|^a} z_0, 3{\chi_i}^{k-a} t\right) \\ \nu'\left(\frac{(2^{2a}\ |r|^a|s|^a+|r+s|^{2a}+|r-s|^{2a})\lambda||x||^{2a}}{|r^2-s^2|^{2a}} z_0, {\chi_i}^{k-2a} t\right) \\ \nu'\left(\frac{(2^{2a}[|r|^a|s|^a+|r|^{2a}+|s|^{2a}]+3[|r+s|^{2a}+|r-s|^{2a}])\lambda||x||^{2a}}{|r^2-s^2|^{2a}} z_0, {\chi_i}^{k-2a} t\right) \end{cases}$$

for all $x \in X$ and all $t > 0$. Hence, the inequality (4.6) is true for

	L	$a(i=0)$	L	$a(i=1)$
(I)	2	0	2^{-1}	0
(II)	2^{k-a}	$a < 1$	2^{a-k}	$a > 1$
(III)	2^{k-2a}	$2a < 1$	2^{2a-k}	$2a > 1$
(IV)	2^{k-2a}	$2a < 1$	2^{2a-k}	$2a > 1.$

Now, for the condition **(I)** and $i = 0$, we have

$$\begin{cases} \mu\left(f(x) - \mathcal{A}(x), t\right) \geq \mu'\left(\rho(x), \frac{2^{1-0}}{1-2} t\right) = \mu'\left(\rho(x), -2t\right) \\ \nu\left(f(x) - \mathcal{A}(x), t\right) \leq \nu'\left(\rho(x), \frac{2^{1-0}}{1-2} t\right) = \nu'\left(\rho(x), -2t\right) \end{cases}$$

for all $x \in X$ and all $t > 0$. Also, for the condition **(I)** and $i = 1$, we get

$$\begin{cases} \mu\left(f(x) - \mathcal{A}(x), t\right) \geq \mu'\left(\rho(x), \frac{((2)^{-1})^{1-1}}{1-(2)^{-1}} t\right) = \mu'\left(\rho(x), 2t\right) \\ \nu\left(f(x) - \mathcal{A}(x), t\right) \leq \nu'\left(\rho(x), \frac{((2)^{-1})^{1-1}}{1-(2)^{-1}} t\right) = \nu'\left(\rho(x), 2t\right) \end{cases}$$

for all $x \in X$ and all $t > 0$. Again, for the condition **(II)** and $i = 0$, we obtain

$$\begin{cases} \mu\left(f(x) - \mathcal{A}(x), t\right) \geq \mu'\left(\rho(x), \frac{(2^{k-a})^{1-0}}{1-(2^{k-a})} t\right) = \mu'\left(\rho(x), \frac{2^k}{2^a-2^k} t\right) \\ \nu\left(f(x) - \mathcal{A}(x), t\right) \leq \nu'\left(\rho(x), \frac{(2^{k-a})^{1-0}}{1-(2^{k-a})} t\right) = \nu'\left(\rho(x), \frac{2^k}{2^a-2^k} t\right) \end{cases}$$

for all $x \in X$ and all $t > 0$. Also, for the condition **(II)** and $i = 1$, we arrive

$$\begin{cases} \mu\left(f(x) - \mathcal{A}(x), t\right) \geq \mu'\left(\rho(x), \frac{(2^{a-k})^{1-1}}{1-(2^{a-k})} t\right) = \mu'\left(\rho(x), \frac{2^k}{2^k-2^a} t\right) \\ \nu\left(f(x) - \mathcal{A}(x), t\right) \leq \nu'\left(\rho(x), \frac{(2^{a-k})^{1-1}}{1-(2^{a-k})} t\right) = \nu'\left(\rho(x), \frac{2^k}{2^k-2^a} t\right) \end{cases}$$

for all $x \in X$ and all $t > 0$. The rest of the proof is similar to that of previous cases. This finishes the proof. $\square$

Acknowledgement. The authors express their sincere thank to the reviewers for the careful and detailed reading of the manuscript and very helpful suggestions that improved the manuscript substantially. The second author would like to thanks the Young Researchers and Elite Club of Islamic Azad University of Islamshahr Branch for its financial support.

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