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ON THE GENERALIZED MODIFIED k-BESSEL FUNCTIONS OF THE FIRST KIND

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Abstract. The recent research investigates the generalization of Bessel function in different forms as its usefulness in various fields of applied sciences. In this paper, we introduce a new modified form of k-Bessel functions called the generalized modified k-Bessel functions and established some of its properties.

1. Introduction

The k-Bessel functions of the first kind are defined by the following series [14]:

$$J_{k,\upsilon}^{\gamma,\lambda}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n+\upsilon+1)} \frac{(-1)^n (z/2)^n}{(n!)^2}, \tag{1.1}$$

where $k \in \mathbb{R}$; $\alpha,\lambda,\gamma,\upsilon \in C$; $\Re(\lambda) > 0$ and $\Re(\upsilon) > 0$. Here $(\gamma)_{n,k}$ is the well-known k-Pochhammer symbol defined as follows (see [2])

$$(\gamma)_{n,k} = \begin{cases} \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & (k \in \mathbb{R}; \gamma \in \mathbb{C}\setminus\{0\}) \\ \gamma(\gamma+k)\cdots(\gamma+(n-1)k) & (n \in \mathbb{N}; \gamma \in \mathbb{C}), \end{cases} \tag{1.2}$$

where $\Gamma_k(z)$ denotes the k-gamma function defined by (see [2])

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}} t^{z-1} dt, \quad \Re(z) > 0, \quad k > 0. \tag{1.3}$$

Clearly, $\Gamma_k(z)$ reduces to the classical $\Gamma(z)$ function for $k = 1$. The k-Bessel functions and their properties are studied widely in many research articles (see, [5, 11, 14]). Recently Nisar and Saiful [10] defined a new generalized form of k-Bessel functions as follows:

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Let $k \in \mathbb{R}; \sigma, \gamma, \upsilon, c, b \in \mathbb{C}; \Re(\sigma) > 0, \Re(\upsilon) > 0$, the generalized k-Bessel functions of the first kind are given by the following series

$$J_{k,\upsilon}^{b,c,\gamma,\sigma}(z) = \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{n,k}}{\Gamma_k\left(\sigma n + \upsilon + \frac{b+1}{2}\right)} \frac{(z/2)^{\upsilon+2n}}{(n!)^2}, \tag{1.4}$$

where $J_{k,\upsilon}^{1,1,\gamma,\sigma}$ is the k-Bessel function and $J_{k,\upsilon}^{1,-1,\gamma,\sigma}$ denotes the modified k-Bessel function.

In this paper, we give a new modified form of (1.4) and define

$$I_{k,\upsilon}^{b,c,\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{n,k}}{\Gamma_k\left(\lambda n + \upsilon + \frac{b+1}{2} + k\right)} \frac{(z/2)^{\upsilon+2n}}{(n!)^2}, \tag{1.5}$$

where $k \in \mathbb{R}; \lambda, \gamma, \upsilon, c, b \in \mathbb{C}; \Re(\lambda) > 0, \Re(\upsilon) > 0$. Here our aim is to give some properties of generalized modified k-Bessel function defined in (1.5).

The generalized Wright hypergeometric functions ${}_p\Psi_q(z)$ are given by the series

$${}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\middle|z\right] = \frac{\prod_{j=1}^{q}\Gamma(\beta_j)}{\prod_{i=1}^{p}\Gamma(\alpha_i)}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{z^k}{k!}, \tag{1.6}$$

where $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}$ $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$. Asymptotic behavior of this function for large values of argument of $z \in \mathbb{C}$ were studied in [4] and under the condition

$$\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > -1 \tag{1.7}$$

was found in the work of [15, 16]. The properties of these generalized Wright functions were investigated in [6] (see also [7, 8]). In particular, it was proved [6] that ${}_p\Psi_q(z)$, $z \in \mathbb{C}$ is an entire function under the condition (1.7).

The generalized hypergeometric functions are represented as follows [13]:

$${}_pF_q\left[\begin{array}{c}(\alpha_p);\\(\beta_q);\end{array} z\right] = \sum_{n=0}^{\infty}\frac{\Pi_{j=1}^{p}(\alpha_j)_n}{\Pi_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \tag{1.8}$$

provided $p \leq q; p = q+1$ and $|z| < 1$ where $(\lambda)_n$ is well known Pochhammer symbol defined for $\lambda \in \mathbb{C}$ (see [13])

$$(\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \end{cases} \tag{1.9}$$

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \qquad \left(\lambda \in \mathbb{C}\backslash\mathbb{Z}_0^-\right), \tag{1.10}$$

where $\mathbb{Z}_0^-$ is the set of non positive integers.

If we put $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$ in (1.6), then (1.8) is a special case of the generalized Wright function:

(1.11)
$$ {}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{array}{c} (\alpha_1, 1), \ldots, (\alpha_p, 1); \\ (\beta_1, 1), \ldots, (\beta_q, 1); \end{array} z \right] = \frac{\prod_{j=1}^{q} \Gamma(\beta_j)}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \, {}_pF_q \left[\begin{array}{c} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{array} z \right]. $$

Also, we need the following relation of Γ_k with the classical gamma Euler function (see [14]):

(1.12)
$$ \Gamma_k(z+k) = z\Gamma_k(z), $$

(1.13)
$$ \Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right), $$

(1.14)
$$ (\gamma)_{n,k} = k^n \left(\frac{\gamma}{k}\right)_n. $$

2. Main results

In this section, we present some properties of the generalized modified k-Bessel functions given in (1.5).

Theorem 1. *For $\lambda, \mu, \upsilon, c \in \mathbb{C}$, $\gamma, k \in \mathbb{R}^+$ and $\Re(\upsilon) > 0$, the following formula holds*:
$$ \frac{d}{dz}\left(z^{\frac{\upsilon}{2}+\frac{b+1}{2}} I_{k,\upsilon}^{c,\gamma,1}\left(\sqrt{z}\right) \right) = z^{\frac{\upsilon+k+b}{2}-\frac{1}{2}} 2^{(\upsilon-k)+\frac{b+1}{2}} I_{k,\upsilon-k}^{c,\gamma,1}\left(\sqrt{z}\right). $$

Proof. Using the definition given in (1.5), we have
$$ \begin{aligned} z^{\frac{\upsilon}{2}+\frac{b+1}{2}} I_{k,\upsilon}^{c,\gamma,1}\left(\sqrt{z}\right) &= z^{\frac{\upsilon}{2}} \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k}}{\Gamma_k\left(n+\upsilon+\frac{b+1}{2}+k\right)} \frac{(\sqrt{z}/2)^{\upsilon+2n+\frac{b+1}{2}}}{(n!)^2} \\ &= \frac{1}{2^{\upsilon+\frac{b+1}{2}}} \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k}}{\Gamma_k\left(n+\upsilon+\frac{b+1}{2}+k\right)} \frac{z^{\upsilon+n+\frac{b+1}{2}}}{4^n (n!)^2}. \end{aligned} $$

Now,
$$ \begin{aligned} &\frac{d}{dz}\left[z^{\frac{\upsilon}{2}+\frac{b+1}{2}} I_{k,\upsilon}^{c,\gamma,1}\left(\sqrt{z}\right) \right] \\ &= \frac{1}{2^{\upsilon+\frac{b+1}{2}}} \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k}}{\Gamma_k\left(n+\upsilon+\frac{b+1}{2}+k\right)} \frac{\left(\upsilon+n+\frac{b+1}{2}\right) z^{\upsilon+n+\frac{b+1}{2}-1}}{4^n (n!)^2}. \end{aligned} $$

Using $\Gamma_k(z+k) = z\Gamma_k(z)$, we get
$$ \begin{aligned} &\frac{d}{dz}\left[z^{\frac{\upsilon}{2}+\frac{b+1}{2}} I_{k,\upsilon}^{c,\gamma,1}\left(\sqrt{z}\right) \right] \\ &= \frac{1}{2^{\upsilon+\frac{b+1}{2}}} \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k}}{\left(\upsilon+n+\frac{b+1}{2}\right)\Gamma_k\left(\lambda n+\upsilon+\frac{b+1}{2}\right)} \frac{\left(\upsilon+n+\frac{b+1}{2}\right) z^{\upsilon+n+\frac{b+1}{2}-1}}{4^n (n!)^2} \end{aligned} $$

$$= \frac{z^{\frac{k}{2}+\frac{v}{2}+\frac{b+1}{2}-1}}{2^{\frac{b+1}{2}-k}} \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k}}{\Gamma_k \left(n+v+\frac{b+1}{2}\right)} \frac{(\sqrt{z}/2)^{v+2n-k}}{(n!)^2}$$

$$= z^{\frac{v+k+b-1}{2}} 2^{\frac{b+1}{2}-k} I_{k,v-k}^{c,\gamma,1}\left(\sqrt{z}\right). \qquad \square$$

Corollary 2.1. *If we set $c = 1$ and $b = 1$, then we get*

$$\frac{d}{dz}\left(z^{\frac{v}{2}+1} I_{k,v}^{1,\gamma,1}\left(\sqrt{z}\right)\right) = z^{\frac{k}{2}-\frac{v}{2}} 2^{1-k} I_{k,v-k}^{1,\gamma,1}\left(\sqrt{z}\right)$$

which is the modified form of the result given in [1].

Theorem 2. *Let $I_{k,-v}^{c,\gamma,1}(z)$ be the generalized modified k-function of order $-v$ and let z be a real non-negative number. Then the following formula holds*:

$$\frac{d}{dz}\left(z^{\frac{v}{2}+\frac{b+1}{2}} I_{k,-v}^{c,\gamma,1}\left(\sqrt{z}\right)\right) = z^{\frac{b-v-1}{2}} 2^{-v-k} I_{k,-v-k}^{c,\gamma,1}\left(\sqrt{z}\right).$$

Proof. Using the definition given in (1.5), we have

$$z^{-\frac{v}{2}+\frac{b+1}{2}} I_{k,-v}^{c,\gamma,1}\left(\sqrt{z}\right) = z^{\frac{v}{2}+\frac{b+1}{2}} \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k}}{\Gamma_k \left(n-v+\frac{b+1}{2}+k\right)} \frac{(\sqrt{z}/2)^{2n-v}}{(n!)^2}$$

$$= z^{-\frac{v}{2}+\frac{b+1}{2}} \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k}}{2^v \Gamma_k \left(n-v+\frac{b+1}{2}+k\right)} \frac{z^{n-\frac{v}{2}}}{4^n (n!)^2}$$

$$= \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k}}{2^v \Gamma_k \left(n-v+\frac{b+1}{2}+k\right)} \frac{z^{n-v+\frac{b+1}{2}}}{4^n (n!)^2}.$$

Now,

$$\frac{d}{dz}\left[z^{\frac{v}{2}+\frac{b+1}{2}} I_{k,-v}^{c,\gamma,1}\left(\sqrt{z}\right)\right]$$

$$= \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k}}{2^v \Gamma_k \left(n-v+\frac{b+1}{2}+k\right)} \frac{\left(n-v+\frac{b+1}{2}\right) z^{n-v+\frac{b+1}{2}-1}}{4^n (n!)^2}.$$

Using $\Gamma_k(z+k) = z\Gamma_k(z)$, we get

$$= \frac{1}{2^v} z^{\frac{b+1}{2}-1} \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k}}{\Gamma_k \left(n-v+\frac{b+1}{2}\right)} \frac{z^{n-v}}{4^n (n!)^2}$$

$$= z^{\frac{b-1}{2}} 2^k z^{-\frac{v}{2}} \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{n,k}}{\Gamma_k \left(n-v+\frac{b+1}{2}\right)} \left(\frac{\sqrt{z}}{2}\right)^{-v-k+2n}$$

$$= z^{\frac{b-v-1}{2}} 2^k I_{k,-v-k}^{c,\gamma,1}\left(\sqrt{z}\right),$$

which is the required result. $\square$

Corollary 2.2. *If we set $c = 1$ and $b = 1$, then we get*

$$\frac{d}{dz}\left(z^{\frac{v}{2}+1} I_{k,-v}^{1,\gamma,1}\left(\sqrt{z}\right)\right) = z^{\frac{v}{2}} 2^k I_{k,-v-k}^{1,\gamma,1}\left(\sqrt{z}\right),$$

which is the modified form of the result given in [1].

In the next theorem we will use a fractional integral called k-fractional integral [12], which is the generalization of classical Reimann-Lioville fractional integral [9].

The k-fractional integral of order α is defined by

$$\mathcal{I}_k^{\alpha}(f)(x) = \frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)\,dt. \tag{2.1}$$

It is clear that when $k \to 1$, (2.1) reduces to the classical Reimann-Lioville fractional integral.

Theorem 3. *For $\lambda, \mu, \upsilon, c \in \mathbb{C}$, $\gamma, k \in \mathbb{R}^+$ and $\Re(\upsilon) > 0$, the following formula holds*:

$$\mathcal{I}_k^{\alpha}\left(z^{\frac{\upsilon}{2}+\frac{b+1}{2}} I_{k,\upsilon-k}^{c,\gamma,1}\left(2\sqrt{z}\right)\right) = \frac{z^{\upsilon+\frac{b+1}{2}+1}}{\Gamma\left(\frac{\gamma}{k}\right)k^{\upsilon+\frac{b}{2}+\alpha-\frac{1}{2}}}\ {}_1\Psi_2\left[\begin{matrix}\left(\frac{\gamma}{k},1\right); \\ (1,1),\left(\frac{\upsilon+\frac{b+1}{2}+\alpha}{k},\frac{1}{k}\right);\end{matrix}\ cz\right].$$

Proof. Using the definition given in (1.5), we have

$$\begin{aligned} z^{\frac{\upsilon}{2}+\frac{b+1}{2}} I_{k,\upsilon-k}^{c,\gamma,1}\left(2\sqrt{z}\right) &= z^{\frac{\upsilon+b+1}{2}} \sum_{n=0}^{\infty} \frac{c^n(\gamma)_{n,k}}{\Gamma_k\left(n+\upsilon+\frac{b+1}{2}\right)} \frac{\left(\frac{2\sqrt{z}}{2}\right)^{\upsilon+2n}}{(n!)^2} \\ &= z^{\frac{\upsilon+b+1}{2}} \sum_{n=0}^{\infty} \frac{c^n(\gamma)_{n,k}}{\Gamma_k\left(n+\upsilon+\frac{b+1}{2}\right)} \frac{z^{\frac{\upsilon}{2}+n}}{4^n(n!)^2} \\ &= \sum_{n=0}^{\infty} \frac{c^n(\gamma)_{n,k}}{\Gamma_k\left(n+\upsilon+\frac{b+1}{2}\right)} \frac{z^{\upsilon+n+\frac{b+1}{2}}}{4^n(n!)^2}. \end{aligned}$$

Now,

$$\begin{aligned} \mathcal{I}_k^{\alpha}\left[z^{\frac{\upsilon}{2}+\frac{b+1}{2}} I_{k,\upsilon-k}^{c,\gamma,1}\left(2\sqrt{z}\right)\right] &= \mathcal{I}_k^{\alpha}\left(\sum_{n=0}^{\infty} \frac{c^n(\gamma)_{n,k}}{\Gamma_k\left(n+\upsilon+\frac{b+1}{2}\right)} \frac{z^{\upsilon+n+\frac{b+1}{2}}}{4^n(n!)^2}\right) \\ &= \sum_{n=0}^{\infty} \frac{c^n(\gamma)_{n,k}}{4^n(n!)^2} I_k^{\alpha}\left(\frac{z^{\left(\frac{\upsilon+n+\frac{b+1}{2}}{k^2}\right)k+1-1}}{\Gamma_k\left(n+\upsilon+\frac{b+1}{2}\right)}\right). \end{aligned}$$

Using

$$\mathcal{I}_k^{\alpha}\left(\frac{x\frac{\beta}{k}-1}{\Gamma_k(\beta)}\right) = \frac{x^{\frac{\alpha}{k}+\frac{\beta}{k}+1}}{\Gamma(\alpha+\beta)}$$

we get

$$= \sum_{n=0}^{\infty} \frac{c^n\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)\,n!\Gamma(n+1)} \frac{z^{n+\upsilon+\frac{b+1}{2}+1}}{\Gamma_k\left(n+\upsilon+\frac{b+1}{2}+\alpha\right)}.$$

Now using the relation $\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$, we get

$$= \frac{z^{\upsilon+\frac{b+1}{2}+1}}{\Gamma\left(\frac{\gamma}{k}\right)k^{\upsilon+\frac{b}{2}+\alpha-\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{c^n\Gamma\left(\frac{\gamma}{k}+n\right)}{n!\Gamma(n+1)} \frac{z^n}{\Gamma\left(\frac{n}{k}+\frac{\upsilon}{k}+\frac{b+1}{2k}+\frac{\alpha}{k}\right)}.$$

In view of the definition of Wright hypergeometric functions, we obtain the desired result. □

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