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# FIXED POINTS OF BSC-SEQUENCES

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Abstract. We call a sequence $(T_n)_n$ of bounded operators on a Banach space X, BSC-Sequence if it is a Cauchy sequence in the strong operator topology and is uniformly bounded below. We determine conditions under which such sequences has a fixed point.

## 1. Introduction

In this paper by an operator we mean a linear transformation. First we extend some notations from [8]. Let $X$ be a complex Banach space and denote by $B(X)$ the space of all bounded operators on X. For $\mathcal{T} = (T_n)_n$ a subset of $B(X)$, we define

$$orb(\mathcal{T}, x) = \{T_n x : n \in \mathbb{N}\}$$

for every $x$ in $X$. The $J$-set of $\mathcal{T}$ under $x$ is denoted by $J(\mathcal{T}, x)$ that is defined by

$$J(\mathcal{T}, x) = \{y : \text{there exist a sequence}\, (x_n)_n \in X \,\text{and a strictly increasing} \\ \text{sequence}\, (k_n)_n \,\text{in}\, \mathbb{N} \,\text{such that}\; x_n \to x \;\text{and}\; T_{k_n} x_n \to y\}.$$

Also, $J^{mix}$-set of $\mathcal{T}$ under x is defined by

$$J^{mix}(\mathcal{T}, x) = \{y : \text{there exists a sequence}\, (x_n)_n \in X \,\text{such that}\; x_n \to x \\ \text{and}\; T_n x_n \to y\}.$$

Note that if $T \in B(X)$ and we put $T_n = T^n$, then by the notations of [8], $orb(\mathcal{T}, x) = orb(T, x)$, $J(\mathcal{T}, x) = J(T, x)$ and $J^{mix}(\mathcal{T}, x) = J^{mix}(T, x)$.

A sequence $\mathcal{T} = (T_n)_n$ is called $J$-class ($J^{mix}$-class) sequence if $J(\mathcal{T}, x) = X$ ($J^{mix}(\mathcal{T}, x) = X$), and in this case, $x$ is called a $J$-vector ($J^{mix}$-vector) of $\mathcal{T}$. In a similar way, we can define $A_{\mathcal{T}}$ and $A_{\mathcal{T}}^{mix}$. The closedness of $A_{\mathcal{T}}$ and $A_{\mathcal{T}}^{mix}$ can be proved by the same method used in [8, Lemma 2.12].

**Definition 1.1.** A sequence $\mathcal{T} = (T_n)_n$ of operators on a normed space $X$ is called bounded below if each $T_n$ is bounded below. We call $\mathcal{T}$ uniformly bounded below if there exists $c > 0$ such that $\|T_n x\| \geq c\|x\|$ for all $n$, and

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for all $x$. Also, $\mathcal{T}$ is called strongly bounded below if for every $n$ there exists $c_n \geq 1$ such that $\|T_n x\| \geq c_n \|x\|$ for all $x$ in $X$.

**Definition 1.2.** Let $\mathcal{T} = (T_n)_n$ be a sequence of operators on a normed space $X$. A vector $x \in X$ is called a fixed point for $\mathcal{T}$ if $T_n x = x$ for all $n$.

In the next section, by using linear dynamics we give a necessary and sufficient condition for a certain sequence of operators has a fixed point.

## 2. Main results

Fixed point theory has been the focus of attention for several decades and many properties relating these topics have been studied (e.g. [2, 4–6, 13, 14, 18, 20–22, 24]). R. Agarwal, M. Meehan and D. O'Regan in their book have provided a clear exposition of the flourishing field of fixed point theory ([1]). In this book authors start from the basics of Banach's contraction theorem, most of the main results and techniques have been developed: fixed point results are established for several classes of maps and the three main approaches to establishing continuation principles have been presented. Also, L. Barbet and K. Nachi considered $k$-contractions $T_n$ which are only defined on a subset $X_n$ of a metric space. They introduced a new notion of convergence and obtained a convergence result for the fixed points ([3]). S. Mishra, S. L. Singh and S. Stofile proved stability results for a pair of sequences of mappings and their common fixed points in a Hausdorff uniform space using certain new notions of convergence ([17]). M. Ertürk and V. Karakaya in [9] studied existence and uniqueness of fixed points of operator $F : X^n \to X$ where $n$ is an arbitrary positive integer and $X$ is partially ordered complete metric space. A. Soliman in [26] introduced the concept of new notions related to $n$-tupled fixed point and proved some related results for an asymptotically regular one-parameter semigroup, $\Im = \{F(t) : t \in G,$ where $G$ is an unbounded subset of $[0, \infty)\}$ of Lipschitzian self-mappings on $\prod_{i=1}^{n} X$ in the case when $X$ is a complete bounded metric space with uniform normal structure. In [25] authors compared relation between $n$-tuple fixed point results and fixed point theorems in abstract metric spaces and metric-like spaces. Actually, they showed that the results of $n$-tuple fixed point can be obtained from fixed point theorems and conversely.

Regarding the theory of linear dynamics, in 1982, C. Kitai introduced linear dynamics which mainly concerned with the behaviour of iterates of linear transformations ([15]). Some considerable works have been done by G. Godefroy, J. H. Shapiro, K. Grosse, and G. Erdmann to develop this theory ([11, 12, 23]). We recall that the concept of $J$-sets has been given by localizing dynamic behaviour (hypercyclicity) of an operator. G. Costakis and A. Manoussos have done some valuable works about $J$-sets of operators ([7, 8]). The purpose of their work is to treat a new notion related to linear dynamics, which can be viewed as a "localization" of the notion of hypercyclicity. They investigated the class of operators satisfying the property of topologically transitivity and provided various examples. It is worthwhile to mention that many results from the theory of hypercyclic operators have their analogues in this setting. For example they established results related to the Bourdon-Feldman theorem and they characterized the $J$-class weighted shifts. In [16], A. Manoussos generalized concept of coarse hypercyclicity, introduced by N. S. Feldman in [10], to that of coarse topological transitivity on open cones. He localized these concepts by introducing two new classes of operators called coarsely $J$-class and coarsely $D$-class operators. Namely, he showed that if a backward unilateral weighted shift on $l^2(\mathbb{N})$ is coarsely $J$-class (or $D$-class) on an open cone, then it is hypercyclic. Then he gave an example of a bilateral weighted shift on $l^\infty(\mathbb{Z})$ which is coarsely $J$-class, hence it is coarsely $D$-class, and not $J$-class. Finally, he showed that there exists a non-separable Banach space which supports no coarsely $D$-class operators on open cones. In [19], A. B. Nasseri in his Ph.D. dissertation investigated $J$-class operators on non-separable space $l^\infty$.

Here, we state and prove some properties of $J$-sets and then we give theorems concerning the fixed points of sequences.

**Lemma 2.1.** *Let $\mathcal{T} = (T_n)$ be in B(X). We have*

(a) *If $\mathcal{T}$ is a Cauchy sequence and it has a subsequence converges to a bounded operator $U$ in the strong operator topology, then for every $x$, both sets $J(\mathcal{T}, x)$ and $J^{mix}(\mathcal{T}, x)$ are equal to the singleton set $\{Ux\}$.*

(b) *If $(T_n)_n$ converges to an operator $U$ in the strong operator topology, then*

$$2J(\mathcal{T}, x_1 + x_2) - U(x_1 + x_2) \subseteq J(\mathcal{T}, x_1) + J(\mathcal{T}, x_2)$$

*for all $x_1, x_2 \in X$.*

*Proof.* (a) Suppose that $x \in X$ and $y \in J(\mathcal{T}, x)$, then there exist $(x_n)_n$ in X and a strictly increasing sequence of positive integers $(k_n)_n$ such that $x_n \to x$ and $T_{k_n} x_n \to y$. Now since

$$\|y - Ux\| \leq \|y - T_{k_n} x_n\| + \|T_{k_n} - U\| \|x_n\| + \|U\| \|x_n - x\|$$

for sufficiently large $n$, we get $y = Ux$. By putting $k_n = n$, we conclude the same result for $J^{mix}(T, x)$.

(b) Suppose that $y \in J(\mathcal{T}, x_1 + x_2)$, then there exist a strictly increasing sequence of positive integers $(k_n)_n$ and a sequence $(w_n)_n$ in X such that $w_n \to x_1 + x_2$ and $T_{k_n} w_n \to y$. Now we have

$$w_n - x_1 \to x_2,\ T_{k_n}(w_n - x_1) \to y - Ux_1$$

and

$$w_n - x_2 \to x_1,\ T_{k_n}(w_n - x_2) \to y - Ux_2.$$

Hence we get

$$y - Ux_1 \in J(\mathcal{T}, x_2),\ y - Ux_2 \in J(\mathcal{T}, x_1).$$

And so

$$2y - U(x_1 + x_2) \in J(\mathcal{T}, x_1) + J(\mathcal{T}, x_2),$$

which gives the result. □

Recall that for $\mathcal{T} \subseteq B(X)$, by $\mathcal{T}'$ we mean the collection of all bounded operators on X which commutes with each member of $\mathcal{T}$.

**Lemma 2.2.** *Suppose that* $\mathcal{T} = (T_n)_n \in B(X)$ *and* $U \in \mathcal{T}'$. *For all positive integers* $m$ *and* $x \in X$, *we have*
(a) $U^m(J(\mathcal{T}, x)) \subseteq J(\mathcal{T}, U^m x)$.
(b) $U^m(J^{mix}(\mathcal{T}, x)) \subseteq J^{mix}(\mathcal{T}, U^m x)$, *and equality holds if* $U$ *has bounded inverse. Moreover, if* $U$ *is surjective, then* $A_{\mathcal{T}}$ *is a* $U$*-invariant subset and* $A_{\mathcal{T}}^{mix}$ *is a* $U$*-invariant subspace of* $X$.

*Proof.* (a) Without loss of generality suppose that $J(\mathcal{T}, x) \neq \emptyset$. Let $y \in J(\mathcal{T}, x)$, then there exist a strictly increasing sequence of positive integers $(k_n)_n$ and a sequence $(x_n)_n$ in $X$ such that $x_n \to x$ and $T_{k_n} x_n \to y$. For any positive integer $m$, we have

$$U^m x_n \to U^m x, \;\; T_{k_n}(U^m x_n) \to U^m y$$

as $n \to \infty$. Hence $U^m y \in J(\mathcal{T}, U^m x)$.
(b) Put, $k_n = n$ in the proof of part (a).
Now we suppose that $U$ has bounded inverse. If $z \in J(\mathcal{T}, U^m x)$, then there exist a strictly increasing sequence of positive integers $(k_n)_n$ and a sequence $(u_n)_n$ in $X$ such that $u_n \to U^m x$ and $T_{k_n} u_n \to z$. So we get

$$U^{-m} u_n \to x, \;\; T_{k_n}(U^{-m} u_n) \to U^{-m} z.$$

This yields that $U^{-m} z \in J(\mathcal{T}, x)$ and hence $z \in U^m(J(\mathcal{T}, x))$. By putting, $k_n = n$, we can prove for $J^{mix}$-sets. Now suppose that $T$ is surjective, if $x \in A_{\mathcal{T}}$, then $J(\mathcal{T}, x) = X$ and so we have

$$X = U(X) = U(J(\mathcal{T}, x)) \subseteq J(\mathcal{T}, Ux) \subseteq X.$$

Hence $U(A_{\mathcal{T}}) \subseteq A_{\mathcal{T}}$. Similarly, we can prove that $A_{\mathcal{T}}^{mix}$ is a $U$-invariant subspace. □

**Corollary 2.3.** *If* $\mathcal{T} = (T_n)_n \subseteq B(X)$ *and* $\lambda \neq 0$, *then* $J(\mathcal{T}, \lambda x) = \lambda J(\mathcal{T}, x)$, *and* $J^{mix}(\mathcal{T}, \lambda x) = \lambda J^{mix}(\mathcal{T}, x)$.

*Proof.* Put $Ux = \lambda x$, and $m = 1$ in Lemma 2.2. □

By Corollary 2.3, if $x$ is a $J$-vector ($J^{mix}$-vector) for $\mathcal{T}$, then for any nonzero scaler $\lambda$, $\lambda x$ is a $J$-vector ($J^{mix}$-vector) for $T$. So if $A_{\mathcal{T}}$ ($A_{\mathcal{T}}^{mix}$) is nonempty, then it should be infinite set.

**Definition 2.4.** We mean by a BSC-Sequence, a sequence $\mathcal{T}$ of operators on a normed space $X$ that holds in the following conditions:
(a) $\mathcal{T}$ is a uniformly bounded below sequence.
(b) $\mathcal{T}$ is a Cauchy sequence in the strong operator topology.
If furthermore $\mathcal{T}$ is a Cauchy sequence in the norm topology, then we say that $\mathcal{T}$ is a BC-Sequence.

Obviously BC-Sequences are BSC-Sequence. Also, it's clear that a constant sequence of operators is a BSC-Sequence. Here we give other examples.

**Example 2.5.** Let $n \in \mathbb{N}$ and $S_n$ be defined on $l^p\,(1 \le p < \infty)$ by $S_n = (1+\frac{1}{n})B$, where $B$ is forward shift operator. Then

$$\|S_n x\| = (1+\frac{1}{n})\|Bx\| = (1+\frac{1}{n})\|x\| \ge \|x\|$$

and so $(S_n)_n$ is uniformly bounded below. On the other hand, $(S_n)_n$ is a Cauchy sequence, since for $m, n \in \mathbb{N}$. We have

$$\|S_n x - S_m x\| = |\frac{1}{n} - \frac{1}{m}|\|Bx\| = |\frac{1}{n} - \frac{1}{m}|\|x\|.$$

Thus $\|S_n - S_m\| = |\dfrac{1}{n} - \dfrac{1}{m}|$. Hence $(S_n)_n$ is a BC-Sequence.

**Example 2.6.** Suppose that $n \in \mathbb{N}$ is arbitrary and $T_n$ is defined on $c_0$ by

$$T_n(x_1, x_2, \ldots) = (x_1, \ldots, x_{n-1}, 2x_n, x_{n+1}, \ldots),$$

then $T_n \in B(c_0)$. In fact $\|x\| \le \|T_n x\| \le 2\|x\|$ for all $x \in c_0$. To show that $(T_n)_n$ is a Cauchy sequence in the strong operator topology, suppose that $n < m$ and $x = (x_1, \ldots) \in c_0$, we have

$$\|T_n x - T_m x\| = \|(0, \ldots, x_n, 0, \ldots, 0, -x_m, 0, \ldots)\| = \sup\{x_n, -x_m\},$$

which tends to zero as $n$ tends to infinity. Thus $(T_n)_n$ is a BSC-Sequence.

**Theorem 2.7.** *Suppose that $\mathcal{T} = (T_n)_n \subseteq B(X)$ holds in the following conditions.*
(a) $(T_n)_n$ *is a bounded sequence.*
(b) $J^{mix}(\mathcal{T}, x) \neq \emptyset$ *for all* $x$.
*Then $\mathcal{T}$ is a Cauchy sequence in the strong operator topology.*

*Proof.* For every $x$ in $X$, we choose $y$ in $J^{mix}(\mathcal{T}, x)$, so there exists $(x_n)_n$ in $X$ which tends to $x$ and $T_n x_n \to y$. Now we have

$$\|T_n x - y\| \le \|T_n x - T_n x_n\| + \|T_n x_n - y\| \le M\|x - x_n\| + \|T_n x_n - y\|,$$

where $M$ is a bound of $\mathcal{T}$. Therefore, for every $x$ in $X$ we can find $y$ in $X$ such that $\|T_n x - y\| \to 0$. By Banach-Steinhaus Theorem there exists $U \in B(X)$ such that $\|T_n x - Ux\| \to 0$ for all $x$. Therefore $(T_n)_n$ converges in the strong operator topology and so it is a Cauchy sequence in the strong operator topology. □

Now immediately we have:

**Corollary 2.8.** *If $\mathcal{T} = (T_n)_n \subseteq B(X)$ is uniformly bounded below such that*
(a) $\mathcal{T}$ *is bounded.*
(b) $J^{mix}(\mathcal{T}, x) \neq \emptyset$ *for all* $x$.
*Then $\mathcal{T}$ is a BSC-Sequence.*

In the following we state and prove some properties of certain operator sequences.

**Theorem 2.9.** *Let $\mathcal{T} = (T_n)_n \subseteq B(X)$ and $x, y \in X$ be such that $x \neq y$. If $\mathcal{T}$ is uniformly bounded below or $\mathcal{T}$ is strongly bounded below, then we have*

$$J^{mix}(\mathcal{T}, x) \ \cap J^{mix}(\mathcal{T}, y) = \emptyset.$$

*Proof.* Suppose that $z \in J^{mix}(\mathcal{T}, x) \cap J^{mix}(\mathcal{T}, y)$, then there exist sequences $(x_n)_n$ and $(y_n)_n$ in $X$ such that $x_n \to x$, $y_n \to y$ and

$$T_n x_n \to z, \ \ T_n y_n \to z.$$

Let $\epsilon > 0$, be given. Then there exists a natural number $N$ such that

$$\|x_N - x\| < \epsilon, \ \ \|y_N - y\| < \epsilon,$$

and

$$\|T_N x_N - z\| < \epsilon/2, \ \ \|T_N y_N - z\| < \epsilon/2.$$

So

$$\|T_N x_N - T_N y_N\| < \epsilon.$$

If $\mathcal{T}$ is uniformly bounded below, then there exists $c > 0$ such that

$$\|x_N - y_N\| \leq c\|T_N x_N - T_N y_N\| < c\epsilon.$$

Since

$$\|x - y\| \leq \|x - x_N\| + \|x_N - y_N\| + \|y_N - y\|,$$

we conclude that $x = y$.

If $T_n$'s are strongly bounded below, then there exists $c_N \in (0, 1]$ such that

$$\|x_N - y_N\| \leq c_N\|T_N x_N - T_N y_N\| < \epsilon.$$

and again we obtained $x = y$, that is a contradiction. □

**Corollary 2.10.** *If $\mathcal{T} = (T_n)_n \subseteq B(X)$ is a $J^{mix}$-class sequence, then it is not uniformly bounded below. Also, there exists $n \in \mathbb{N}$ such that $T_n$ is not strongly bounded below.*

*Proof.* If $\mathcal{T}$ is uniformly bounded below (or strongly bounded below), then we can take $x, y \in A^{mix}_{\mathcal{T}}$ such that $x \neq y$. Since

$$J^{mix}(\mathcal{T}, x) = X, \ J^{mix}(\mathcal{T}, y) = X,$$

so $J^{mix}(\mathcal{T}, x)$ and $J^{mix}(\mathcal{T}, y)$ have nonempty intersection which is a contradiction by Theorem 2.9. □

**Theorem 2.11.** *Let $\mathcal{T} = (T_n)_n \subseteq B(X)$ and $x, y \in X$ such that $x \neq y$. If $\mathcal{T}$ is a bounded BSC-Sequence (or BC-Sequence), then $J(\mathcal{T}, x) \cap J(\mathcal{T}, y) = \emptyset$.*

*Proof.* Suppose that $\mathcal{T}$ is a bounded BSC-Sequence and $z \in J(\mathcal{T}, x) \cap J(\mathcal{T}, y)$. Then there exist strictly increasing sequences of positive integers $(i_n)_n$ and $(k_n)_n$ and sequences $(x_n)_n$ and $(y_n)_n$ in $X$ such that $x_n \to x$, $y_n \to y$ and

$$T_{i_n} x_n \to z, \ \ T_{k_n} y_n \to z.$$

Let $M$ be a bound of $\mathcal{T}$. For every $\epsilon > 0$ there exists a natural number $N_1$ such that

$$\|x_{N_1} - x\| < \epsilon/9M, \;\; \|y_{N_1} - y\| < \epsilon/9M$$

and

$$\|T_{i_{N_1}} x_{N_1} - z\| < \epsilon/3, \;\; \|T_{k_{N_1}} y_{N_1} - z\| < \epsilon/3.$$

Also, there exists $N_2 \in \mathbb{N}$ such that for $i, k \geq N_2$ we have

$$\|T_i y - T_k y\| < \epsilon/9.$$

Since the sequences $(i_n)_n$ and $(k_n)_n$ are strictly increasing, there exists a natural number $N_3$ such that $i_{N_3}, k_{N_3} \geq N_2$ and so

$$\|T_{i_{N_3}} y - T_{k_{N_3}} y\| < \epsilon/9.$$

Put $N = \max\{N_1, N_3\}$. Therefore we have

$$\begin{aligned}\|T_{i_N} y_N - T_{k_N} y_N\| &\leq \|T_{i_N} y_N - T_{i_N} y\| + \|T_{i_N} y - T_{k_N} y\| + \|T_{k_N} y - T_{k_N} y_N\| \\ &\leq M\|y_N - y\| + \|T_{i_N} y - T_{k_N} y\| + M\|y - y_N\| \\ &< \epsilon/3.\end{aligned}$$

Now, we have

$$\|T_{i_N} x_N - T_{i_N} y_N\| \leq \|T_{i_N} x_N - z\| + \|z - T_{k_N} y_N\| + \|T_{k_N} y_N - T_{i_N} y_N\| < \epsilon.$$

But $\mathcal{T}$ is uniformly bounded below, thus there exists $c > 0$ such that

$$\|x_N - y_N\| \leq c\|T_{i_N} x_N - T_{i_N} y_N\| < c\epsilon.$$

Since

$$\|x - y\| \leq \|x - x_N\| + \|x_N - y_N\| + \|y_N - y\|,$$

we conclude that $x = y$, which is a contradiction.

In the case that $\mathcal{T}$ is a BC-Sequence, since any BC-Sequence is a bounded BSC-Sequence, the result is true for BC-Sequences, but we can also prove it directly: suppose contrary that $z \in J(\mathcal{T}, x) \cap J(\mathcal{T}, y)$, then there exist strictly increasing sequences of positive integers $(i_n)_n$ and $(k_n)_n$ and sequences $(x_n)_n$ and $(y_n)_n$ in $X$ such that $x_n \to x$, $y_n \to y$ and

$$T_{i_n} x_n \to z, \;\; T_{k_n} y_n \to z.$$

For every $\epsilon > 0$ there exists a natural number $N_1$ such that for every $i, k \geq N_1$ we have

$$\|T_i - T_k\| < \epsilon/3,$$

and

$$\|x_k - x\| < \epsilon, \;\; \|y_k - y\| < \epsilon.$$

Since the sequences $(i_n)_n$ and $(k_n)_n$ are strictly increasing, there exists a natural number $N_2$ such that $i_{N_2}$, $k_{N_2} \geq N_1$ and

$$\|T_{i_{N_2}} x_{N_2} - z\| < \epsilon/3, \;\; \|T_{k_{N_2}} y_{N_2} - z\| < \epsilon/3.$$

Now, we have

$$\|T_{i_{N_2}} x_{N_2} - T_{i_{N_2}} y_{N_2}\| \leq \|T_{i_{N_2}} x_{N_2} - z\| + \|z - T_{k_{N_2}} y_{N_2}\|$$

$$+\|T_{k_{N_2}}y_{N_2} - T_{i_{N_2}}y_{N_2}\| < \epsilon.$$

But $\mathcal{T}$ is uniformly bounded below, thus there exists $c > 0$ such that

$$\|x_{N_2} - y_{N_2}\| \le c\|T_{i_{N_2}}x_{N_2} - T_{i_{N_2}}y_{N_2}\| < c\epsilon.$$

Since

$$\|x - y\| \le \|x - x_{N_2}\| + \|x_{N_2} - y_{N_2}\| + \|y_{N_2} - y\|,$$

we conclude that $x = y$. □

*Remark* 2.12. Note that a bounded sequence $\mathcal{T} = (T_n)_n \subseteq B(X)$ is not $J$-class sequence, since otherwise there is some nonzero $x \in X$ such that $J(\mathcal{T}, x) = X$. But by boundedness, if $x_n \to x$, then there exists $M > 0$ such that for any sequence $(k_n)$, $\|T_{k_n}x_n\| < M$. Now, let $y$ be in $X$ with $\|y\| > M$, then no sequence $(T_{k_n}x_n)$ can approximate $y$. Hence, BC-Sequences and bounded BSC-Sequences are not J-class sequence.

Now, we present our main theorems about fixed points of certain sequences:

**Theorem 2.13.** *Let $\mathcal{T} = (T_n)_n \subseteq B(X)$ be a BC-Sequence. A vector $x \in X$ is a fixed point for $\mathcal{T}$ if and only if $0 \in J(\mathcal{T}, x - T_nx)$ for all $n$. The same result holds for bounded BSC-Sequences.*

*Proof.* Suppose that $x \in X$, and for all $n$, $0 \in J(\mathcal{T}, x - T_nx)$. If $n_0 \in \mathbb{N}$ is fixed but arbitrary, then since $0 \in J(\mathcal{T}, x - T_{n_0}x)$, there exist a sequence $(u_k)_k$ in $X$ and a strictly increasing sequence of positive integers $(m_k)_k$ such that $u_k \to x - T_{n_0}x$ and $T_{m_k}u_k \to 0$ as $k \to \infty$. Put, $x_k = u_k + T_{n_0}x$, then $x_k \to x$. Note that $B(X)$ is complete and so $(T_n)_n$ converges in $B(X)$. Since

$$\lim_k T_{m_k}(x_k - T_{n_0}x) = \lim_k T_{m_k}u_k = 0,$$

we have

$$\lim_k T_{m_k}x_k = \lim_k T_{m_k}T_{n_0}x \in J(\mathcal{T}, x) \cap J(\mathcal{T}, T_{n_0}x).$$

By Theorem 2.11, we get $T_{n_0}x = x$. Since $n_0 \in \mathbb{N}$ is arbitrary, the result is obtained. Conversely, if $x$ is a fixed point for $\mathcal{T}$, then we have to show that $0 \in J(\mathcal{T}, 0)$. For any strictly increasing sequence of positive integers $(k_n)_n$, and a sequence $(x_n)_n$ converges to zero, $(T_{k_n}x_n)_n$ tends to zero, which gives the result. By similar method we can proved for bounded BSC-Sequences. □

**Theorem 2.14.** *Let $\mathcal{T} = (T_n)_n$ be a Cauchy sequence in $B(X)$ and suppose that $\mathcal{T}$ is either uniformly bounded below or strongly bounded below. Then a vector $x$ is a fixed point of $\mathcal{T}$ if and only if for all $n$, $0 \in J^{mix}(\mathcal{T}, x - T_nx)$.*

*Proof.* Let $x \in X$ and suppose that $0 \in J^{mix}(\mathcal{T}, x - T_nx)$ for all $n$. For $n_0$ fixed but arbitrary, there exists a sequence $(u_k)_k$ in $X$ such that

$$u_k \to x - T_{n_0}x,\ T_ku_k \to 0$$

as $k$ tends to infinity. Put $x_k = u_k + T_{n_0}x$, then $x_k \to x$. Since $(T_k)_k$ is converges and

$$\lim_k T_k(x_k - T_{n_0}x) = \lim_k T_ku_k = 0,$$

we have

$$\lim_k T_k x_k = \lim_k T_k T_{n_0} x \in J^{mix}(\mathcal{T}, x) \cap J^{mix}(\mathcal{T}, T_{n_0} x).$$

Hence by Theorem 2.9, $T_{n_0} x = x$. Since $n_0$ is arbitrary, we conclude that $x$ is a fixed point of $\mathcal{T}$. The converse can be proved by the same method used in Theorem 2.13. □

**Example 2.15.** Let $\mathcal{S} = (S_n)_n$ be the same defined in Example 2.5. Then $S_n \to B$ as $n \to \infty$, and by Lemma 2.1(a), $J(\mathcal{S}, x) = J^{mix}(\mathcal{S}, x) = \{Bx\}$ for all $x$. Therefore, the only fixed point of $\mathcal{S}$ is the zero vector.

## References


[1] R. Agarwal, M. Meehan, and D. O'Regan, *Fixed point theory Appl.*, Cambridge Tracts in Mathematics, Cambridge Univ. Press, Cambridge, 2004.

[2] S. Banach, *Sur les operations dans les ensembles abstracts ET leur applications aux equations integrals*, Fund. Math., **3** (1922), 133–181.

[3] L. Barbet and K. Nachi, *Sequences of contractions and convergence of fixed points*, Ninth International Conference Zaragoza-Pau on Applied Mathematics and Statistics, 51–58, Monogr. Semin. Mat. Garca Galdeano, 33, Prensas Univ. Zaragoza, Zaragoza, 2006.

[4] F. Bonsall, *Lectures on some fixed point theorem*, Tata Inst. Fundam. Res. Stud. Math, Bombay, 1962.

[5] L. Brouwer, *Über Abbildungen von Mannigfaltigkeiten*, Math. Ann. **70** (1912), 97–115.

[6] L. J. Ciric, *Quasi-contraction in Banach spaces*, Publ. Inst. Math. **21(35)** (1977), 41–48.

[7] G. Costakis and A. Manoussos, *J-class weighted shifts on the space of bounded sequences of complex numbers*, Integral Equations Operator Theory **62** (2008), no. 2, 149–158.

[8] ———, *J-class operators and hypercyclicity*, J. Operator Theory **67** (2012), no. 1, 101–119.

[9] M. Erturk and V. Karakaya, *A note on n-tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces*, J. Inequal. Appl. **2013** (2013), DOI:10.1186/1029-242X-2013-196.

[10] N. S. Feldman, *Perturbations of hypercyclic vectors*, J. Math. Anal. Appl. **273** (2002), no. 1, 67–74.

[11] G. Godefroy and J. H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. **98** (1991), no. 2, 229–269.

[12] K. Grosse and G. Erdmann, *Universal families and hypercyclic vectors*, Bull. Amer. Math. Soc. **36** (1999), no. 3, 345–381.

[13] S. Kakutani, *A generalization of Tychonoff's fixed point theorem*, Duke Math. J. **8** (1968), 457–459.

[14] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **60** (1968), 71–78.

[15] C. Kitai, *Invariant closed sets for linear operators*, Dissertation, Univ. of Toronto, 1982.

[16] A. Manoussos, *Coarse topological transitivity on open cones and coarsely J-class and D-class operators*, J. Math. Anal. Appl. **413** (2014), no. 2, 715–726.

[17] S. Mishral, S. L. Singh, and S. Stofile, *Stability of common fixed points in uniform spaces*, Fixed Point Theory Appl. **2011** (2011), DOI:10.1186/1687-1812-2011-37.

[18] S. B. Nadler, *Sequence of contractions and fixed point*, Pacific J. Math. **27** (1968), no. 3, 579–586.

[19] A. B. Nasseri, *J-class operators on certain Banach spaces*, Dissertation, TU Dortmund, 2013.

[20] H. Poincare, *Surless courbes define barles equations differentiate less*, J. de Math. **2** (1886), 54–65.
[21] B. E. Rhoades, *A fixed point theorem for generalized Metric space*, Internat. J. Math. Sci. **19** (1996), no. 3, 457–460.
[22] B. Sadovskii, *On a fixed point principle*, Funktsional. Anal. i Prilozen. **1** (1967), no. 2, 74–76.
[23] J. H. Shapiro, *Notes on the dynamics of linear operators*, www.joelshapiro.org/Pubvit/Downloads/LinDynamics/LynDynamics.html (2016).
[24] S. Sessa and B. Fisher, *On common fixed points of weakly commuting mappings and set-valued mappings*, internat. J. Math. & Math. Sci. **9** (1986), no. 2, 323–329.
[25] G. Soleimani Rad, S. Shukla, and H. Rahimi, *Some relations between n-tuple fixed point and fixed point results*, RACSAM (2015); Doi:10.1007/s13398-014-0196-0.
[26] A. Soliman, *Results on n-tupled fixed points in metric spaces with uniform normal structure*, Fixed point Theory Appl. **2014** (2014), 168, 16 pp.; DOI:10.1186/1687-1812-2014-168.

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