Commun. Korean Math. Soc. **32** (2017), No. 4, pp. 885–898
https://doi.org/10.4134/CKMS.c160220
pISSN: 1225-1763 / eISSN: 2234-3024

CERTAIN NEW WP-BAILEY PAIRS AND BASIC HYPERGEOMETRIC SERIES IDENTITIES

S. Ahmad Ali and Sayyad Nadeem Hasan Rizvi

Abstract. The Bailey lemma has been a powerful tool in the discovery of identities of Rogers-Ramanujan type and also ordinary and basic hypergeometric series identities. The mechanism of Bailey lemma has also led to the concepts of Bailey pair and Bailey chain. In the present work certain new WP-Bailey pairs have been established. We also have deduced a number of basic hypergeometric series identities as an application of new WP-Bailey pairs.

1. Introduction

For $|q| < 1$, we define

$$(a;q)_n = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}); n \in \mathbb{N} \\ 1; n = 0 \end{cases}$$

or equivalently,

$$(a;q)_n = \prod_{j=0}^{\infty} \frac{(1-aq^j)}{(1-aq^{n+j})} \equiv \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

where a is real or complex. A basic hypergeometric series is defined as

$$\begin{aligned} &{}_r\varphi_s(a_1, a_2, a_3, \ldots, a_r; b_1, b_2, b_3, \ldots, b_s; q, z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n (b_2;q)_n \cdots (b_s;q)_n} [(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r} z^n. \end{aligned}$$

For $0 < |q| < 1$, the series converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s+1$. This series also converges absolutely if $|q| > 1$ and $|z| < |b_1 b_2 \cdots b_s| / |a_1 a_2 \cdots a_r|$.

Received October 20, 2016; Revised June 29, 2017; Accepted July 18, 2017.
2010 *Mathematics Subject Classification.* 33D15.
Key words and phrases. Bailey's lemma, Bailey pairs, Bailey chain, WP-Bailey pairs, basic hypergeometric series, q-series.

In 1944, Bailey [7] introduced a very useful and simple identity known as Bailey's lemma. The Bailey's lemma states that, if

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} \upsilon_{n+r} \tag{1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} \upsilon_{n+r}, \tag{2}$$

then under the suitable convergence conditions and if change in the order of summations is allowed

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{3}$$

where α_r, δ_r, u_r and υ_r are functions of r such that β_n and γ_n exist. The proof of the lemma is trivial.

During last seven decades Bailey lemma and its various generalizations have proved to be a powerful tool in the discoveries of Rogers-Ramanujan type of identities, transformations and summations theorems of ordinary and basic hypergeometric series. Slater [17,18] used Bailey lemma to discover the famous list of 130 identities of Rogers-Ramanujan type. Using the same tool more identities of Rogers-Ramanujan type and basic hypergeometric series have been given by Andrew [4], Foda and Quano [11], Denis [9], Denis and Singh [10], Singh [14,15] and Ali and Rizvi [1].

In (1), if we choose $u_r = 1/(q;q)_r$ and $\upsilon_r = 1/(aq;q)_r$, we get

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}. \tag{4}$$

The pair of sequence (α_n, β_n) that satisfies (4) is called a Bailey pair relative to the parameter a. The concept of Bailey pairs has been generalized in the works of Bressoud [8] and Singh [16]. The most elegant generalizations of Bailey pair have been given by Andrews [5] which is

$$\beta_n = \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}} \alpha_r. \tag{5}$$

The pair (α_n, β_n) satisfying (5) is termed as WP-Bailey pair. It is easy to see that (5) follows by setting $u_r = \frac{(k/a;q)_r}{(q;q)_r}$ and $\upsilon_r = \frac{(k;q)_r}{(aq;q)_r}$ in (1). For $k = 0$ in (5), we get the standard Bailey pair (4).

Andrew et al. [2–4, 6] have exploited very effective mechanism of Bailey lemma in the form of Bailey chain. In fact Andrews [5] described that the process may be iterated to produce a chain of WP-Bailey pairs constructing new WP-bailey pairs from existing initial WP-Bailey pair and introduced the concept of Bailey chain. The aforesaid technique developed by Andrews have been effectively used to discover new WP-Bailey pairs. The idea of WP-Bailey pairs and chain has further been generalized by Liu and Ma [13] and Spiridonov [19]. For more details on Bailey lemma and its various generalizations and applications one is referred the recent and a very elegant survey of Warnaar [21].

In the present work, we have established some new WP-Bailey pairs using the following results of Andrews [5] and Warnaar [22]. We, further, have used the new WP-Bailey pairs to produce a number of new transformations of basic hypergeometric series.

Theorem 1 ([5])**.** *If $(\alpha_n(a,k;q),\beta_n(a,k;q))$ is a WP-Bailey pair and satisfies (5), then so is the pair $(\alpha_n^{'}(a,k;q),\beta_n^{'}(a,k;q))$ given by*

$$\alpha_n^{'}(a,k;q)=\frac{(\rho_1,\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2;q)_n}\Big(\frac{k}{m}\Big)^n\alpha_n(a,m;q), \tag{6}$$

$$\beta_n^{'}(a,k;q)=\frac{(mq/\rho_1,mq/\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2;q)_n}\sum_{r=0}^{n}\frac{(1-mq^{2r})(\rho_1,\rho_2;q)_r(k/m;q)_{n-r}}{(1-m)(mq/\rho_1,mq/\rho_2;q)_r(q;q)_{n-r}} \times\frac{(k;q)_{n+r}}{(mq;q)_{n+r}}\Big(\frac{k}{m}\Big)^r\beta_r(a,m;q), \tag{7}$$

where $m=k\rho_1\rho_2/aq$.

Theorem 2 ([5])**.** *If $(\alpha_n(a,k;q),\beta_n(a,k;q))$ is a WP-Bailey pair, then so is the pair $(\alpha_n^{'}(a,k;q),\beta_n^{'}(a,k;q))$ given by*

$$\alpha_n^{'}(a,k;q)=\frac{(m;q)_{2n}}{(k;q)_{2n}}\Big(\frac{k}{m}\Big)^n\alpha_n(a,m;q), \tag{8}$$

$$\beta_n^{'}(a,k;q)=\sum_{r=0}^{n}\frac{(k/m;q)_{n-r}}{(q;q)_{n-r}}\Big(\frac{k}{m}\Big)^r\beta_r(a,m;q), \tag{9}$$

where $m=a^2q/k$.

Theorem 3 ([5])**.** *If $(\alpha_n(a,k;q),\beta_n(a,k;q))$ is a WP-Bailey pair, then so is the pair $(\alpha_n^{'}(a,k;q),\beta_n^{'}(a,k;q))$ given by*

$$\alpha_n^{'}(a,k;q)=\frac{(qa^2/k;q)_{2n}}{(k;q)_{2n}}\Big(\frac{k^2}{qa^2}\Big)^n\alpha_n(a,qa^2/k;q), \tag{10}$$

$$\beta_n^{'}(a,k;q)=\sum_{r=0}^{n}\frac{(k^2/qa^2;q)_{n-r}}{(q;q)_{n-r}}\Big(\frac{k^2}{qa^2}\Big)^r\beta_r(a,qa^2/k;q). \tag{11}$$

Theorem 4 ([22])**.** *If $(\alpha_n(a,k;q),\beta_n(a,k;q))$ is a WP-Bailey pair, then so is the pair $(\alpha_n^{'}(a,k;q),\beta_n^{'}(a,k;q))$ given by*

$$\alpha_n^{'}(a,k;q)=\frac{(1-\sigma\sqrt{k})(1+\sigma\sqrt{m}q^n)(m;q)_{2n}}{(1-\sigma\sqrt{k}q^n)(1+\sigma\sqrt{m})(k;q)_{2n}}\Big(\frac{k}{m}\Big)^n\alpha_n(a,m;q), \tag{12}$$

(13)
$$\beta_n^{'}(a,k;q)=\frac{(1-\sigma\sqrt{k})}{(1-\sigma\sqrt{k}q^n)}\sum_{r=0}^{n}\frac{(1+\sigma\sqrt{m}q^r)(k/m;q)_{n-r}}{(1+\sigma\sqrt{m})(q;q)_{n-r}}\left(\frac{k}{m}\right)^r\beta_r(a,m;q),$$

where $m=a^2/k$ *and* $\sigma\in(-1,1)$.

Theorem 5 ([22])**.** *If* $(\alpha_n(a,k;q),\beta_n(a,k;q))$ *is a WP-Bailey pair, then so is the pair* $(\alpha_n^{'}(a,k;q),\beta_n^{'}(a,k;q))$ *given by*

$$\alpha_n^{'}(a^2,k;q^2)=\alpha_n(a,m;q), \tag{14}$$

(15)
$$\beta_n^{'}(a^2,k;q^2)$$
$$=\frac{(-mq;q)_{2n}}{(-aq;q)_{2n}}\sum_{r=0}^{n}\frac{(1-mq^{2r})(k/m^2;q^2)_{n-r}(k;q^2)_{n+r}}{(1-m)(q^2;q^2)_{n-r}(m^2q^2;q^2)_{n+r}}\left(\frac{m}{a}\right)^{n-r}\beta_r(a,m;q),$$

where $m=k/aq$.

Theorem 6 ([22])**.** *If* $(\alpha_n(a,k;q),\beta_n(a,k;q))$ *is a WP-Bailey pair, then so is the pair* $(\alpha_n^{'}(a,k;q),\beta_n^{'}(a,k;q))$ *given by*

$$\alpha_n^{'}(a^2,k;q^2)=q^{-n}\frac{(1+aq^{2n})}{(1+a)}\alpha_n(a,m;q), \tag{16}$$

(17)
$$\beta_n^{'}(a^2,k;q^2)$$
$$=\frac{(-mq;q)_{2n}q^{-n}}{(-a;q)_{2n}}\sum_{r=0}^{n}\frac{(1-mq^{2r})(k/m^2;q^2)_{n-r}(k;q^2)_{n+r}}{(1-m)(q^2;q^2)_{n-r}(m^2q^2;q^2)_{n+r}}\left(\frac{m}{a}\right)^{n-r}\beta_r(a,m;q),$$

where $m=k/a$.

Theorem 7 ([22])**.** *If* $(\alpha_n(a,k;q),\beta_n(a,k;q))$ *is a WP-Bailey pair, then so is the pair* $(\alpha_n^{'}(a,k;q),\beta_n^{'}(a,k;q))$ *given by*

$$\alpha_{2n}^{'}(a,k;q)=\alpha_n(a,m;q^2);\quad \alpha_{2n+1}^{'}(a,k;q)=0, \tag{18}$$

(19)
$$\beta_n^{'}(a,k;q)$$
$$=\frac{(mq;q^2)_n}{(aq;q^2)_n}\sum_{r=0}^{[n/2]}\frac{(1-mq^{4r})(k/m;q)_{n-2r}(k;q)_{n+2r}}{(1-m)(q;q)_{n-2r}(mq;q)_{n+2r}}\left(\frac{-k}{a}\right)^{n-2r}\beta_r(a,m;q^2),$$

where $m=k^2/a$.

In the next section, we shall also require the following identities.

$${}_4\varphi_3(a,-q\sqrt{a},b,q^{-n};-\sqrt{a},aq/b,aq^{n+1};q,q^{n+1}a^{1/2}/b)=\frac{(aq,q\sqrt{a}/b;q)_n}{(q\sqrt{a},aq/b;q)_n}, \tag{20}$$

([12]; II.14)

for $b = kq^n$ in (20), we get

$$ {}_4\varphi_3(a, -q\sqrt{a}, kq^n, q^{-n}; -\sqrt{a}, aq^{1-n}/k, aq^{n+1}; q, qa^{1/2}/k) = \frac{(aq, k/\sqrt{a}; q)_n}{(q\sqrt{a}, k/a; q)_n}\Big(\frac{1}{\sqrt{a}}\Big)^n, \tag{21} $$

$$ {}_4\varphi_3(a, c, aq^{n+1/2}/c, q^{-n}; aq/c, cq^{1/2-n}, aq^{n+1}; q, q^2) = \frac{(1+\sqrt{a})(aq, \sqrt{q}, \sqrt{aq}/c, q\sqrt{a}/c; q)_n}{2\sqrt{a}(aq/c, \sqrt{q}/c, \sqrt{aq}, q\sqrt{a}; q)_n} - \frac{(1-\sqrt{a})(aq, \sqrt{q}, \sqrt{aq}/c, -q\sqrt{a}/c; q)_n}{2\sqrt{a}(aq/c, -\sqrt{q}/c, \sqrt{aq}, -q\sqrt{a}; q)_n}, \tag{22} $$

([20]; 44)

for $c = a\sqrt{q}/k$ in (22), we get

$$ {}_4\varphi_3(a, a\sqrt{q}/k, kq^n, q^{-n}; k\sqrt{q}, aq^{1-n}/k, aq^{n+1}; q, q^2) = \frac{(1+\sqrt{a})(aq, \sqrt{q}, k/\sqrt{a}, k\sqrt{q}/\sqrt{a}; q)_n}{2\sqrt{a}(k\sqrt{q}, k/a, \sqrt{aq}, q\sqrt{a}; q)_n} - \frac{(1-\sqrt{a})(aq, \sqrt{q}, k/\sqrt{a}, -k\sqrt{q}/\sqrt{a}; q)_n}{2\sqrt{a}(k\sqrt{q}, -k/a, \sqrt{aq}, -q\sqrt{a}; q)_n}. \tag{23} $$

2. New WP-Bailey pairs

If $(\alpha_n(a,k;q), \beta_n(a,k;q))$ is a WP-Bailey pair, then so is the pair $(\alpha'_n(a,k;q), \beta'_n(a,k;q))$ given by

$$ \alpha'_n(a,k;q) = \frac{(a, \rho_1, \rho_2, -q\sqrt{a}; q)_n}{(q, -\sqrt{a}, aq/\rho_1, aq/\rho_2; q)_n}\Big(\frac{k}{m\sqrt{a}}\Big)^n, \tag{24} $$

$$ \beta'_n(a,k;q) = \frac{(mq/\rho_1, mq/\rho_2, k/m, k; q)_n}{(q, aq/\rho_1, aq/\rho_2, mq; q)_n} \times\sum_{r=0}^{n}\frac{(q\sqrt{m}, -q\sqrt{m}, \rho_1, \rho_2, q^{-n}, kq^n, m, m/\sqrt{a}; q)_r}{(q, q\sqrt{a}, \sqrt{m}, -\sqrt{m}, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{n+1}; q)_r}\Big(\frac{q}{\sqrt{a}}\Big)^r, \tag{25} $$

where $m = k\rho_1\rho_2/aq$.

Proof of (24)-(25). Let us choose

$$ \alpha_r(a,k;q) = \frac{(a, -q\sqrt{a}; q)_r}{(q, -\sqrt{a}; q)_r}\Big(\frac{1}{\sqrt{a}}\Big)^r \tag{26} $$

in (5), we obtain

$$ \beta_n(a,k;q) = \frac{(k, k/a; q)_n}{(q, aq; q)_n}\sum_{r=0}^{n}\frac{(kq^n, q^{-n}, a, -q\sqrt{a}; q)_r}{(q, aq^{1-n}/k, aq^{n+1}, -\sqrt{a}; q)_r}\Big(\frac{q\sqrt{a}}{k}\Big)^r, $$

by making the use (21), we have

$$ \beta_n(a,k;q) = \frac{(k, k/\sqrt{a}; q)_n}{(q, q\sqrt{a}; q)_n}\Big(\frac{1}{\sqrt{a}}\Big)^n. \tag{27} $$

We obtain new WP-Bailey pair (26), (27). Now using WP-Bailey pair (26) and (27) in (6) and (7), we get (24), (25).

$$\alpha_n^{'}(a,k;q) = \frac{(m;q)_{2n}(a,-q\sqrt{a};q)_n}{(k;q)_{2n}(q,-\sqrt{a};q)_n}\Big(\frac{k}{m\sqrt{a}}\Big)^n, \tag{28}$$

$$\beta_n^{'}(a,k;q) = \frac{(k/m;q)_n}{(q;q)_n}\sum_{r=0}^{n}\frac{(m,q^{-n},m/\sqrt{a};q)_r}{(q,q\sqrt{a},mq^{1-n}/k;q)_r}\Big(\frac{q}{\sqrt{a}}\Big)^r, \tag{29}$$

where $m = qa^2/k$. □

Proof of (28)-(29). Using WP-Bailey pair (26), (27) in (8) and (9), we obtain (28) and (29).

$$\alpha_n^{'}(a,k;q) = \frac{(qa^2/k;q)_{2n}}{(k;q)_{2n}}\frac{(a,-q\sqrt{a};q)_n}{(q,-\sqrt{a};q)_n}\Big(\frac{k^2}{qa^{5/2}}\Big)^n, \tag{30}$$

$$\beta_n^{'}(a,k;q) = \frac{(k^2/qa^2;q)_n}{(q;q)_n}\sum_{r=0}^{n}\frac{(q^{-n},qa^2/k,a^{3/2}q/k;q)_r}{(q,q\sqrt{a},a^2q^{2-n}/k^2;q)_r}\Big(\frac{q}{\sqrt{a}}\Big)^r. \tag{31}$$

□

Proof of (30)-(31). By using WP-Bailey pair (26), (27) in (10) and (11), we get (30) and (31).

$$\alpha_n^{'}(a,k;q) = \frac{(\sigma\sqrt{k},-\sigma q\sqrt{m},a,-q\sqrt{a};q)_n(m;q)_{2n}}{(q,-\sqrt{a},\sigma q\sqrt{k},-\sigma\sqrt{m};q)_n(k;q)_{2n}}\Big(\frac{k}{m\sqrt{a}}\Big)^n, \tag{32}$$

$$\beta_n^{'}(a,k;q) = \frac{(k/m,\sigma\sqrt{k};q)_n}{(q,q\sigma\sqrt{k};q)_n}\sum_{r=0}^{n}\frac{(m,q^{-n},m/\sqrt{a},-q\sigma\sqrt{m};q)_r}{(q,q\sqrt{a},mq^{1-n}/k,-\sigma\sqrt{m};q)_r}\Big(\frac{q}{\sqrt{a}}\Big)^r, \tag{33}$$

where $m = a^2/k, \sigma \in (-1,1)$. □

Proof of (32)-(33). By making the use of WP-Bailey pair (26), (27) in (12) and (13), we have (32), (33).

$$\alpha_n^{'}(a^2,k;q^2) = \frac{(a,-q\sqrt{a};q)_n}{(q,-\sqrt{a};q)_n}\Big(\frac{1}{\sqrt{a}}\Big)^n, \tag{34}$$

(35)

$$\begin{aligned}\beta_n^{'}(a^2,k;q^2) = {}& \frac{(-mq;q)_{2n}(k,k/m^2;q^2)_n}{(-aq;q)_{2n}(q^2,m^2q^2;q^2)_n}\Big(\frac{m}{a}\Big)^n \\ & \times\sum_{r=0}^{n}\frac{(m,m/\sqrt{a};q)_r(q^{-2n},kq^{2n},mq^2;q^2)_r}{(q,q\sqrt{a};q)_r(m,m^2q^{2+2n},m^2q^{2-2n}/k;q^2)_r}\Big(\frac{mq^2\sqrt{a}}{k}\Big)^r,\end{aligned}$$

where $m = k/aq$. □

Proof of (34)-(35). Now using WP-Bailey pair (26), (27) in (14) and (15), we get (34) and (35).

$$\alpha_n^{'}(a^2,k;q^2) = \frac{(1+aq^{2n})(a,-q\sqrt{a};q)_n}{(1+a)(q,-\sqrt{a};q)_n}\Big(\frac{1}{q\sqrt{a}}\Big)^n, \tag{36}$$

(37)

$$\beta_n^{'}(a^2,k;q^2) = \frac{(-mq;q)_{2n}(k,k/m^2;q^2)_n}{(-a;q)_{2n}(q^2,m^2q^2;q^2)_n}\Big(\frac{m}{aq}\Big)^n \times \sum_{r=0}^{n}\frac{(m,m/\sqrt{a};q)_r(q^{-2n},kq^{2n},mq^2;q^2)_r}{(q,q\sqrt{a};q)_r(m^2q^{2+2n},m^2q^{2-2n}/k,m;q^2)_r}\Big(\frac{mq^2\sqrt{a}}{k}\Big)^r,$$

where $m = k/a$. □

Proof of (36)-(37). Using WP-Bailey pair (26), (27) in (16) and (17), we obtain (36) and (37).

$$\alpha_{2n}^{'}(a,k;q) = \frac{(a,-q^2\sqrt{a};q^2)_n}{(-\sqrt{a},q^2;q^2)_n}\Big(\frac{1}{\sqrt{a}}\Big)^n; \quad \alpha_{2n+1}^{'}(a,k;q) = 0, \tag{38}$$

(39)

$$\beta_n^{'}(a,k;q) = \frac{(\sqrt{mq},-\sqrt{mq},k,k/m;q)_n}{(\sqrt{aq},-\sqrt{aq},q,mq;q)_n}\Big(\frac{-k}{a}\Big)^n \times \sum_{r=0}^{[n/2]}\frac{(1-mq^{4r})(q^{-n},q^{-n+1},kq^n,kq^{n+1},m,m/\sqrt{a};q^2)_r}{(1-m)(mq^{1-n}/k,mq^{2-n}/k,mq^{n+1},mq^{n+2},q^2,q^2\sqrt{a};q^2)_r}\Big(\frac{m^2a^{3/2}q^2}{k^4}\Big)^r,$$

where $m = k^2/a$. □

Proof of (38)-(39). By using WP-Bailey pair (26), (27) in (18) and (19), we get (38), (39).

$$\alpha_n^{'}(a,k;q) = \frac{(\rho_1,\rho_2,a,a\sqrt{q}/m;q)_n}{(q,m\sqrt{q},aq/\rho_1,aq/\rho_2;q)_n}\Big(\frac{kq}{a}\Big)^n, \tag{40}$$

(41)

$$\begin{aligned}\beta_n^{'}(a,k;q) = {} & \frac{(1+\sqrt{a})(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{2\sqrt{a}\quad(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \\ & \times\sum_{r=0}^{n}\frac{(q\sqrt{m},-q\sqrt{m},\rho_1,\rho_2,q^{-n},kq^n,m,\sqrt{q},m/\sqrt{a},m\sqrt{q}/\sqrt{a};q)_r\quad q^r}{(\sqrt{m},-\sqrt{m},mq/\rho_1,mq/\rho_2,mq^{1-n}/k,mq^{n+1},q,m\sqrt{q},\sqrt{aq},q\sqrt{a};q)_r} \\ & -\frac{(1-\sqrt{a})(k,k/m,mq/\rho_1,mq/\rho_2;q)_n}{2\sqrt{a}\quad(q,mq,aq/\rho_1,aq/\rho_2;q)_n} \\ & \times\sum_{r=0}^{n}\frac{(q\sqrt{m},-q\sqrt{m},\rho_1,\rho_2,q^{-n},kq^n,m,m/a,\sqrt{q},m/\sqrt{a},-m\sqrt{q}/\sqrt{a};q)_r\quad q^r}{(\sqrt{m},-\sqrt{m},mq/\rho_1,mq/\rho_2,mq^{1-n}/k,mq^{n+1},q,m\sqrt{q},-m/a,\sqrt{aq},-q\sqrt{a};q)_r},\end{aligned}$$

where $m = k\rho_1\rho_2/aq$. □

Proof of (40)-(41). Let us choose

$$\alpha_r(a,k;q) = \frac{(a, a\sqrt{q}/k;q)_r}{(q, k\sqrt{q};q)_r}\Big(\frac{kq}{a}\Big)^r \tag{42}$$

in (5) and by using (23), we obtain

$$\begin{aligned}\beta_n(a,k;q) = {} & \frac{(1+\sqrt{a})}{2\sqrt{a}}\frac{(k, \sqrt{q}, k/\sqrt{a}, k\sqrt{q}/\sqrt{a};q)_n}{(q, k\sqrt{q}, \sqrt{aq}, q\sqrt{a};q)_n} \\ & - \frac{(1-\sqrt{a})}{2\sqrt{a}}\frac{(k, k/a, \sqrt{q}, k/\sqrt{a}, -k\sqrt{q}/\sqrt{a};q)_n}{(q, k\sqrt{q}, -k/a, \sqrt{aq}, -q\sqrt{a};q)_n}.\end{aligned} \tag{43}$$

We obtain new WP-Bailey pair (42) and (43), use (42), (43) in (6) and (7), we get (40) and (41).

$$\alpha_n^{'}(a,k;q) = \frac{(a, a\sqrt{q}/m;q)_n (m;q)_{2n}}{(q, m\sqrt{q};q)_n (k;q)_{2n}}\Big(\frac{kq}{a}\Big)^n, \tag{44}$$

$$\begin{aligned}&\beta_n^{'}(a,k;q) \\ &= \frac{(1+\sqrt{a})(k/m;q)_n}{2\sqrt{a}\quad (q;q)_n}\sum_{r=0}^{n}\frac{(m, \sqrt{q}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}, q^{-n};q)_r \quad q^r}{(q, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}, mq^{1-n}/k;q)_r} \\ &\quad - \frac{(1-\sqrt{a})(k/m;q)_n}{2\sqrt{a}\quad (q;q)_n}\sum_{r=0}^{n}\frac{(m, m/a, \sqrt{q}, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}, q^{-n};q)_r \quad q^r}{(q, m\sqrt{q}, -m/a, \sqrt{aq}, -q\sqrt{a}, mq^{1-n}/k;q)_r},\end{aligned} \tag{45}$$

where $m = qa^2/k$. □

Proof of (44)-(45). By making the use (42), (43) in (8) and (9), we obtain (44), (45).

$$\alpha_n^{'}(a,k;q) = \frac{(qa^2/k;q)_{2n}}{(k;q)_{2n}}\frac{(a, k/a\sqrt{q};q)_n}{(q, a^2q^{3/2}/k;q)_n}\Big(\frac{kq}{a}\Big)^n, \tag{46}$$

$$\begin{aligned}\beta_n^{'}(a,k;q) = {} & \frac{(1+\sqrt{a})(k^2/qa^2;q)_n}{2\sqrt{a}\quad (q;q)_n}\sum_{r=0}^{n}\frac{(q^{-n}, \sqrt{q}, qa^2/k, a^{3/2}q/k, a^{3/2}q^{3/2}/k;q)_r q^r}{(q, q\sqrt{a}, a^2q^{3/2}/k, \sqrt{aq}, a^2q^{2-n}/k^2;q)_r} \\ & - \frac{(1-\sqrt{a})(k^2/qa^2;q)_n}{2\sqrt{a}\quad (q;q)_n}\sum_{r=0}^{n}\frac{(q^{-n}, \sqrt{q}, aq/k, qa^2/k, a^{3/2}q/k, -a^{3/2}q^{3/2}/k;q)_r \quad q^r}{(q, -q\sqrt{a}, -aq/k, a^2q^{3/2}/k, \sqrt{aq}, a^2q^{2-n}/k^2;q)_r}.\end{aligned} \tag{47}$$

□

Proof of (46)-(47). By using (42), (43) in (10) and (11), we deduce (46) and (47).

$$\alpha_n^{'}(a,k;q) = \frac{(1-\sigma\sqrt{k})(1+\sigma\sqrt{m}q^n)(a, a\sqrt{q}/m;q)_n (m;q)_{2n}}{(1-\sigma q^n\sqrt{k})(1+\sigma\sqrt{m})(q, m\sqrt{q};q)_n (k;q)_{2n}}\Big(\frac{kq}{a}\Big)^n, \tag{48}$$

(49)

$$\begin{aligned}\beta_n'(a,k;q) = {} & \frac{(1+\sqrt{a})(\sigma\sqrt{k},k/m;q)_n}{2\sqrt{a}(\sigma q\sqrt{k},q;q)_n} \\ & \times \sum_{r=0}^{n} \frac{(m,\sqrt{q},m/\sqrt{a},m\sqrt{q}/\sqrt{a},-\sigma q\sqrt{m},q^{-n};q)_r \quad q^r}{(q,m\sqrt{q},\sqrt{aq},q\sqrt{a},-\sigma\sqrt{m},mq^{1-n}/k;q)_r} \\ & - \frac{(1-\sigma\sqrt{k})(1-\sqrt{a})(k/m;q)_n}{2\sqrt{a} \quad (1-\sigma q^n\sqrt{k})(q;q)_n} \\ & \times \sum_{r=0}^{n} \frac{(m,m/a,\sqrt{q},m/\sqrt{a},-m\sqrt{q}/\sqrt{a},-\sigma q\sqrt{m},q^{-n};q)_r \quad q^r}{(q,m\sqrt{q},-m/a,\sqrt{aq},-q\sqrt{a},-\sigma\sqrt{m},mq^{1-n}/k;q)_r},\end{aligned}$$

where $m = a^2/k, \sigma \in (-1,1)$. □

Proof of (48)-(49). Making the use (42), (43) in (12) and (13), we obtain (48) and (49).

(50)
$$\alpha_n'(a^2,k;q^2) = \frac{(a,a\sqrt{q}/m;q)_n}{(q,m\sqrt{q};q)_n}\Big(\frac{mq}{a}\Big)^n,$$

(51)

$$\begin{aligned}\beta_n'(a^2,k;q^2) = {} & \frac{(1+\sqrt{a})(-mq,-mq^2,k/m^2,k;q^2)_n}{2\sqrt{a} \quad (-aq,-aq^2,q^2,m^2q^2;q^2)_n}\Big(\frac{m}{a}\Big)^n \\ & \sum_{r=0}^{n}\frac{(m,\sqrt{q},m/\sqrt{a},m\sqrt{q}/\sqrt{a},q\sqrt{m},-q\sqrt{m},q^{-n},-q^{-n},q^n\sqrt{k},-q^n\sqrt{k};q)_r}{(q,m\sqrt{q},\sqrt{aq},q\sqrt{a},\sqrt{m},-\sqrt{m},mq^{1-n}/\sqrt{k},-mq^{1-n}/\sqrt{k},mq^{1+n},-mq^{1+n};q)_r} \\ & \times\Big(\frac{amq^2}{k}\Big)^r - \frac{(1-\sqrt{a})(-mq,-mq^2,k/m^2,k;q^2)_n}{2\sqrt{a} \quad (-aq,-aq^2,q^2,m^2q^2;q^2)_n}\Big(\frac{m}{a}\Big)^n\sum_{r=0}^{n}\frac{(m,m/a,m/\sqrt{a},}{(q,m\sqrt{q},-m/a,\sqrt{aq},} \\ & \times\frac{\sqrt{q},-m\sqrt{q}/\sqrt{a},q\sqrt{m},-q\sqrt{m},q^{-n},-q^{-n},q^n\sqrt{k},-q^n\sqrt{k};q)_r}{-q\sqrt{a},\sqrt{m},-\sqrt{m},mq^{1-n}/\sqrt{k},-mq^{1-n}/\sqrt{k},mq^{1+n},-mq^{1+n};q)_r}\Big(\frac{amq^2}{k}\Big)^r,\end{aligned}$$

where $m = k/aq$. □

Proof of (50)-(51). Using (42), (43) in (14) and (15), we get (50), (51).

(52)
$$\alpha_n'(a^2,k;q^2) = \frac{(a,a\sqrt{q}/m,iq\sqrt{a},-iq\sqrt{a};q)_n}{(q,m\sqrt{q},i\sqrt{a},-i\sqrt{a};q)_n}\Big(\frac{m}{a}\Big)^n,$$

(53)

$$\begin{aligned}\beta_n'(a^2,k;q^2) = {} & \frac{(1+\sqrt{a})}{2\sqrt{a}}\frac{(-mq,-mq^2,k/m^2,k;q^2)_n}{(-a,-aq,q^2,m^2q^2;q^2)_n}\Big(\frac{m}{aq}\Big)^n \\ & \times\sum_{r=0}^{n}\frac{(m,\sqrt{q},m/\sqrt{a},m\sqrt{q}/\sqrt{a},q\sqrt{m},-q\sqrt{m},q^{-n},-q^{-n},q^n\sqrt{k},-q^n\sqrt{k};q)_r}{(q,m\sqrt{q},\sqrt{aq},q\sqrt{a},\sqrt{m},-\sqrt{m},mq^{1-n}/\sqrt{k},-mq^{1-n}/\sqrt{k},mq^{1+n},-mq^{1+n};q)_r}\Big(\frac{amq^2}{k}\Big)^r \\ & -\frac{(1-\sqrt{a})}{2\sqrt{a}}\frac{(-mq,-mq^2,k/m^2,k;q^2)_n}{(-a,-aq,q^2,m^2q^2;q^2)_n}\Big(\frac{m}{aq}\Big)^n \\ & \times\sum_{r=0}^{n}\frac{(m,m/a,\sqrt{q},m/\sqrt{a},-m\sqrt{q}/\sqrt{a},q\sqrt{m},-q\sqrt{m},q^{-n},-q^{-n},q^n\sqrt{k},-q^n\sqrt{k};q)_r}{(q,-m/a,m\sqrt{q},\sqrt{aq},-q\sqrt{a},\sqrt{m},-\sqrt{m},mq^{1-n}/\sqrt{k},-mq^{1-n}/\sqrt{k},mq^{1+n},-mq^{1+n};q)_r}\Big(\frac{amq^2}{k}\Big)^r,\end{aligned}$$

where $m = k/a$. □

Proof of (52)-(53). By using (42), (43) in (16) and (17), we obtain (52) and (53).

(54) $$\alpha'_{2n}(a,k;q)=\frac{(a,aq/m;q^2)_n}{(q^2,mq;q^2)_n}\Big(\frac{mq^2}{a}\Big)^n;\quad \alpha'_{2n+1}(a,k;q)=0,$$

(55)

$$\beta'_n(a,k;q)=\frac{(1+\sqrt{a})}{2\sqrt{a}}\frac{(mq;q^2)_n}{(aq;q^2)_n}\frac{(k,k/m;q)_n}{(q,mq;q)_n}\Big(\frac{-k}{a}\Big)^n$$
$$\times\sum_{r=0}^{[n/2]}\frac{(1-mq^{4r})(q^{-n},q^{-n+1},kq^n,kq^{n+1},m,q,m/\sqrt{a},mq/\sqrt{a};q^2)_r}{(1-m)(mq^{1-n}/k,mq^{2-n}/k,mq^{n+1},mq^{n+2},q^2,mq,q\sqrt{a},q^2\sqrt{a};q^2)_r}\Big(\frac{amq}{k^2}\Big)^{2r}$$
$$-\frac{(1-\sqrt{a})}{2\sqrt{a}}\frac{(mq;q^2)_n}{(aq;q^2)_n}\frac{(k,k/m;q)_n}{(q,mq;q)_n}\Big(\frac{-k}{a}\Big)^n\sum_{r=0}^{[n/2]}\frac{(1-mq^{4r})}{(1-m)}$$
$$\times\frac{(q^{-n},q^{-n+1},kq^n,kq^{n+1},m,m/a,q,m/\sqrt{a},-mq/\sqrt{a};q^2)_r}{(mq^{1-n}/k,mq^{2-n}/k,mq^{n+1},mq^{n+2},q^2,mq,-m/a,q\sqrt{a},-q^2\sqrt{a};q^2)_r}\Big(\frac{amq}{k^2}\Big)^{2r},$$

where $m=k^2/a$. □

Proof of (54)-(55). By making the use (42), (43) in (18) and (19), we get (54) and (55). □

3. Applications

As an application of the new WP-Bailey pairs established in the previous section, we obtain a number of basic hypergeometric series identities in this section. If we use the WP-Bailey pairs (24)-(55) in (5), after some simplification, we obtain the following presumably new transformations.

(56) $$ {}_8\varphi_7(q\sqrt{m},-q\sqrt{m},\rho_1,\rho_2,q^{-n},kq^n,m,m/\sqrt{a};\sqrt{m},-\sqrt{m},mq/\rho_1,mq/\rho_2,$$
$$mq^{1-n}/k,mq^{n+1},q\sqrt{a};q,q/\sqrt{a})$$
$$=\frac{(k/a,mq,aq/\rho_1,aq/\rho_2;q)_n}{(aq,mq/\rho_1,mq/\rho_2,k/m;q)_n}\,{}_6\varphi_5(a,-q\sqrt{a},\rho_1,\rho_2,kq^n,q^{-n};-\sqrt{a},$$
$$aq/\rho_1,aq/\rho_2,aq^{1-n}/k,aq^{n+1};q,q\sqrt{a}/m),$$

where $m=k\rho_1\rho_2/aq$, $|q/\sqrt{a}|<1$ and $|q\sqrt{a}/m|<1$.

(57) $${}_8\varphi_7(a,-q\sqrt{a},\sqrt{m},-\sqrt{m},\sqrt{mq},-\sqrt{mq},kq^n,q^{-n};-\sqrt{a},\sqrt{k},-\sqrt{k},\sqrt{kq},$$
$$-\sqrt{kq},aq^{1-n}/k,aq^{1+n};q,q\sqrt{a}/m)$$
$$=\frac{(aq,k/m;q)_n}{(k/a,k;q)_n}\,{}_3\varphi_2(m,m/\sqrt{a},q^{-n};q\sqrt{a},mq^{1-n}/k;q,q/\sqrt{a}),$$

where $m=qa^2/k$, $|q\sqrt{a}/m|<1$ and $|q/\sqrt{a}|<1$.

(58) $${}_8\varphi_7(a,-q\sqrt{a},q^{-n},kq^n,a\sqrt{q/k},-a\sqrt{q/k},aq/\sqrt{k},-aq/\sqrt{k};-\sqrt{a},aq^{n+1},$$
$$aq^{1-n}/k,\sqrt{k},-\sqrt{k},\sqrt{kq},-\sqrt{kq};q,k/a^{3/2})$$

$$= \frac{(aq, k^2/qa^2; q)_n}{(k/a, k; q)_n}\ {}_3\varphi_2(qa^2/k, qa^{3/2}/k, q^{-n}; q\sqrt{a}, a^2q^{2-n}/k^2; q, q/\sqrt{a}),$$

$|k/a^{3/2}| < 1$ and $|q/\sqrt{a}| < 1$.

(59)

$$\begin{aligned}&{}_{10}\varphi_9(\sigma\sqrt{k}, -q\sigma\sqrt{m}, \sqrt{m}, -\sqrt{m}, \sqrt{mq}, -\sqrt{mq}, a, -q\sqrt{a}, kq^n, q^{-n}; q\sigma\sqrt{k},\\ &\qquad -\sigma\sqrt{m}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, -\sqrt{a}, aq^{1-n}/k, aq^{n+1}; q, q\sqrt{a}/m)\\ &= \frac{(aq, k/m, \sigma\sqrt{k}; q)_n}{(k/a, k, \sigma q\sqrt{k}; q)_n}\\ &\quad \times {}_4\varphi_3(m, m/\sqrt{a}, q^{-n}, -\sigma q\sqrt{m}; q\sqrt{a}, mq^{1-n}/k, -\sigma\sqrt{m}; q, q/\sqrt{a}),\end{aligned}$$

where $m = a^2/k, \sigma \in (-1, 1)$, $|q\sqrt{a}/m| < 1$ and $|q/\sqrt{a}| < 1$.

(60)

$$\begin{aligned}&{}_8\varphi_7(q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}, m, m/\sqrt{a}; \sqrt{m}, -\sqrt{m}, q\sqrt{a},\\ &\qquad mq^{1-n}/\sqrt{k}, -mq^{1-n}/\sqrt{k}, mq^{1+n}, -mq^{1+n}; q, mq^2\sqrt{a}/k)\\ &= \frac{(m^2q^2, -aq, -aq^2, q^2; q^2)_n (k, k/a; q)_n}{(k, -mq, -mq^2, k/m^2; q^2)_n (q, aq; q)_n}\left(\frac{a}{m}\right)^n {}_6\varphi_5(a, -q\sqrt{a}, q^{-n}, -q^{-n},\\ &\quad q^n\sqrt{k}, -q^n\sqrt{k}; -\sqrt{a}, aq^{1-n}/\sqrt{k}, -aq^{1-n}/\sqrt{k}, aq^{n+1}, -aq^{n+1}; q, q^2a^{3/2}/k),\end{aligned}$$

where $m = k/aq$, $|mq^2\sqrt{a}/k| < 1$ and $|q^2a^{3/2}/k| < 1$.

(61)

$$\begin{aligned}&{}_8\varphi_7(q\sqrt{m}, -q\sqrt{m}, m, m/\sqrt{a}, q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k} : \sqrt{m}, -\sqrt{m},\\ &\qquad mq^{1-n}/\sqrt{k}, -mq^{1-n}/\sqrt{k}, mq^{1+n}, -mq^{1+n}, q\sqrt{a}; q, mq^2\sqrt{a}/k)\\ &= \frac{(m^2q^2, -aq, -a, q^2; q^2)_n (k, k/a; q)_n}{(k, -mq, -mq^2, k/m^2; q^2)_n (q, aq; q)_n}\left(\frac{aq}{m}\right)^n {}_8\varphi_7(iq\sqrt{a}, -iq\sqrt{a}, a, -q\sqrt{a},\\ &\quad q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}; i\sqrt{a}, -i\sqrt{a}, -\sqrt{a}, aq^{1-n}/\sqrt{k}, -aq^{1-n}/\sqrt{k},\\ &\quad aq^{n+1}, -aq^{n+1}; q, a^{3/2}q/k),\end{aligned}$$

where $m = k/a$, $|mq^2\sqrt{a}/k| < 1$ and $|a^{3/2}q/k| < 1$.

(62)

$$\begin{aligned}&\frac{(1+\sqrt{a})}{2\sqrt{a}}\ {}_{10}\varphi_9(q\sqrt{m}, -q\sqrt{m}, \rho_1, \rho_2, q^{-n}, kq^n, \sqrt{q}, m, m/\sqrt{a}, m\sqrt{q}/\sqrt{a};\\ &\sqrt{m}, -\sqrt{m}, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{n+1}, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}; q, q)\\ &- \frac{(1-\sqrt{a})}{2\sqrt{a}}\ {}_{11}\varphi_{10}(q\sqrt{m}, -q\sqrt{m}, \rho_1, \rho_2, q^{-n}, kq^n, \sqrt{q}, m, m/a, m/\sqrt{a},\\ &- m\sqrt{q}/\sqrt{a}; \sqrt{m}, -\sqrt{m}, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{n+1}, m\sqrt{q}, \sqrt{aq},\end{aligned}$$

$$- m/a, -q\sqrt{a}; q; q)$$
$$= \frac{(aq/\rho_1, aq/\rho_2, mq, k/a; q)_n}{(mq/\rho_1, mq/\rho_2, k/m, aq; q)_n}$$
$$\times_6 \varphi_5(\rho_1, \rho_2, a, a\sqrt{q}/m, q^{-n}, kq^n; aq/\rho_1, aq/\rho_2, m\sqrt{q}, aq^{1-n}/k, aq^{n+1}; q, q^2),$$

where $m = k\sigma_1\sigma_2/aq$.

(63)
$$\frac{(1+\sqrt{a})}{2\sqrt{a}}\ {}_5\varphi_4(m, \sqrt{q}, q^{-n}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}; m\sqrt{q}, \sqrt{aq}, q\sqrt{a}, mq^{1-n}/k; q, q)$$
$$- \frac{(1-\sqrt{a})}{2\sqrt{a}}\ {}_6\varphi_5(m, \sqrt{q}, m/a, q^{-n}, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}; m\sqrt{q}, -m/a,$$
$$\sqrt{aq}, -q\sqrt{a}, mq^{1-n}/k; q, q)$$
$$= \frac{(k, k/a; q)_n}{(aq, k/m; q)_n}\ {}_8\varphi_7(\sqrt{m}, -\sqrt{m}, \sqrt{mq}, -\sqrt{mq}, a, a\sqrt{q}/m, kq^n, q^{-n};$$
$$\sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, m\sqrt{q}, aq^{1+n}, aq^{1-n}/k; q, q^2),$$

where $m = a^2q/k$.

(64)
$${}_8\varphi_7(a, q^{-n}, kq^n, k/a\sqrt{q}, a\sqrt{q/k}, -a\sqrt{q/k}, aq/\sqrt{k}, -aq/\sqrt{k}; aq^{1-n}/k,$$
$$aq^{1+n}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, a^2q^{3/2}/k; q, q^2)$$
$$= \frac{(aq, k^2/qa^2; q)_n(1+\sqrt{a})}{(k, k/a; q)_n(2\sqrt{a})}\ {}_5\varphi_4(\sqrt{q}, qa^2/k, qa^{3/2}/k, q^{3/2}a^{3/2}/k,$$
$$q^{-n}; \sqrt{aq}, q\sqrt{a}, a^2q^{2-n}/k^2, a^2q^{3/2}/k; q, q)$$
$$- \frac{(aq, k^2/qa^2; q)_n(1-\sqrt{a})}{(k, k/a; q)_n(2\sqrt{a})}\ {}_6\varphi_5(\sqrt{q}, qa^2/k, aq/k, -q^{3/2}a^{3/2}/k,$$
$$qa^{3/2}/k, q^{-n}; -aq/k, \sqrt{aq}, -q\sqrt{a}, a^2q^{2-n}/k^2, a^2q^{3/2}/k; q, q),$$

(65)
$$\frac{(1+\sqrt{a})}{2\sqrt{a}}\ {}_6\varphi_5(-\sigma q\sqrt{m}, q^{-n}, m, \sqrt{q}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}; -\sigma\sqrt{m}, m\sqrt{q},$$
$$\sqrt{aq}, q\sqrt{a}, mq^{1-n}/k; q, q)$$
$$- \frac{(1-\sqrt{a})}{2\sqrt{a}}\ {}_7\varphi_6(-\sigma q\sqrt{m}, q^{-n}, m, m/a, \sqrt{q}, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}; -\sigma\sqrt{m},$$
$$m\sqrt{q}, -m/a, \sqrt{aq}, -q\sqrt{a}, mq^{1-n}/k; q, q)$$
$$= \frac{(\sigma q\sqrt{k}, k/a, k; q)_n}{(\sigma\sqrt{k}, k/m, aq; q)_n}\ {}_{10}\varphi_9(\sigma\sqrt{k}, -\sigma q\sqrt{m}, \sqrt{m}, -\sqrt{m}, \sqrt{mq}, -\sqrt{mq},$$
$$a, a\sqrt{q}/m, q^{-n}, kq^n; \sigma q\sqrt{k}, -\sigma\sqrt{m}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, m\sqrt{q},$$
$$aq^{1-n}/k, aq^{n+1}; q, q^2),$$

where $m = a^2/k, \sigma \in (-1, 1)$.

(66)

$$\frac{(1+\sqrt{a})}{2\sqrt{a}}\ {}_{10}\varphi_9(q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^n, q^n\sqrt{k}, -q^n\sqrt{k}, m, \sqrt{q}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}; \sqrt{m}, -\sqrt{m}, -mq^{1-n}/\sqrt{k}, mq^{1-n}/\sqrt{k}, mq^{n+1}, -mq^{n+1}, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}; q, amq^2/k)$$
$$-\frac{(1-\sqrt{a})}{2\sqrt{a}}\ {}_{11}\varphi_{10}(q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^n, q^n\sqrt{k}, -q^n\sqrt{k}, m, m/a, m/\sqrt{a}, \sqrt{q}, -m\sqrt{q}/\sqrt{a}; \sqrt{m}, -\sqrt{m}, -mq^{1-n}/\sqrt{k}, mq^{1-n}/\sqrt{k}, mq^{n+1}, -mq^{n+1}, -m/a, m\sqrt{q}, \sqrt{aq}, -q\sqrt{a}; q, amq^2/k)$$
$$= \frac{(-aq, -aq^2, q^2, m^2q^2; q^2)_n(k, k/a; q)_n}{(-mq, -mq^2, k/m^2, k; q^2)_n(q, aq; q)_n}\left(\frac{a}{m}\right)^n {}_6\varphi_5(a, a\sqrt{q}/m, q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}; m\sqrt{q}, aq^{1-n}/\sqrt{k}, -aq^{1-n}/\sqrt{k}, aq^{n+1}, -aq^{n+1}; q, amq^3/k),$$

where $m = k/aq$ and $|amq^2/k| < 1$.

(67)

$$\frac{(1+\sqrt{a})}{2\sqrt{a}}\ {}_{10}\varphi_9(q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^n, q^n\sqrt{k}, -q^n\sqrt{k}, m, \sqrt{q}, m/\sqrt{a}, \sqrt{q}/\sqrt{a}; \sqrt{m}, -\sqrt{m}, -mq^{1-n}/\sqrt{k}, mq^{1-n}/\sqrt{k}, mq^{n+1}, -mq^{n+1}, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}; q, amq^2/k)$$
$$-\frac{(1-\sqrt{a})}{2\sqrt{a}}\ {}_{11}\varphi_{10}(q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^n, q^n\sqrt{k}, -q^n\sqrt{k}, m, \sqrt{q}, m/a, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}; \sqrt{m}, -\sqrt{m}, -mq^{1-n}/\sqrt{k}, mq^{1-n}/\sqrt{k}, mq^{n+1}, -mq^{n+1}, -m/a, m\sqrt{q}, \sqrt{aq}, -q\sqrt{a}; q, amq^2/k)$$
$$= \frac{(-a, -aq, q^2, m^2q^2; q^2)_n(k, k/a; q)_n}{(-mq, -mq^2, k/m^2, k; q^2)_n(q, aq; q)_n}\left(\frac{aq}{m}\right)^n {}_8\varphi_7(iq\sqrt{a}, -iq\sqrt{a}, a, a\sqrt{q}/m, q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}; i\sqrt{a}, -i\sqrt{a}, m\sqrt{q}, aq^{1-n}/\sqrt{k}, -aq^{1-n}/\sqrt{k}, aq^{n+1}, -aq^{n+1}; q, amq^2/k),$$

where $m = k/a$ and $|amq^2/k| < 1$.

References

[1] S. Ahmad Ali and S. Nadeem Hasan Rizvi, *Certain transformations and summations of basic hypergeometric series*, J. Math. Comput. Sci. **5** (2015), no 1, 25–33.

[2] G. E. Andrews, *A Genaral theory of identities of the Rogers-Ramanujan type*, Bull. Amer. Math. Soc. **80** (1974), 1033–1052.

[3] ———, *An analytic generalization of the Rogers-Ramanujan identities for odd moduli*, Proc. Nat. Acad. Sci. USA **71** (1974), 4082–4085.
[4] ———, *Multiple series Rogers-Ramanujan type identities*, Pacific J. Math. **114** (1984), no. 2, 267–283.
[5] ———, *Bailey's transform, lemma, chains and tree*, Special functions 2000: current perspective and future directions (Tempe, AZ), 1–22, NATO Sci. Ser. II Math. Phys. Chem., 30, Kluwer Acad. Publ., Dordrecht, 2001
[6] G. E. Andrews and A. Berkovich, *The WP-Bailey tree and its implications*, J. Londan Math. Soc. (2) **66** (2002), no. 3, 529–549.
[7] W. N. Bailey, *Identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. **50** (1949), 1–10.
[8] D. M. Bressoud, *Some identities for terminating q-series*, Math. Proc. Cambridge Philos. Soc. **89** (1981), no. 2, 211–223.
[9] R. Y. Denis, *On certain summation of q-series and identities of Rogers-Ramanujan type*, J. Math. Phys. Sci. **22** (1988), no. 1, 87–99.
[10] R. Y. Denis, S. N. Singh, and S. P. Singh, *Certain transformation and summation formulae for q-series*, Ital. J. Pure Appl. Math. **27** (2010), 179–190.
[11] Q. Foda and Y. H. Quano, *Polynomial identities of the Rogers-Ramanujan type*, Internat. J. Modern. Phys. A **10** (1995), no. 16, 2291–2315.
[12] G. Gasper and R. Mizan, *Basic Hypergeometric Series*, Second Edition. Encyclopedia of Mathematics and Its Applications, 96. Cambridge University Press, Cambridge, 2004.
[13] Q. Liu and X. Ma, *On the characteristic equation of well-poised Bailey chains*, Ramanujan J. **18** (2009), no. 3, 351–370.
[14] S. N. Singh, *Certain partition theorems of Rogers-Ramanujan type*, J. Indian Math. Soc. (N.S.) **62** (1996), no. 1-4, 113–120.
[15] ———, *On certain summation and transformation formulas for basic hypergeometric series*, Int. J. Math. Arch. **2** (2011), no. 12, 2670–2677.
[16] U. B. Singh, *A note on a transformation of Bailey*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 177, 111–116.
[17] L. J. Slater, *A new proof of Rogers's transformations of infinite series*, Proc. London Math. Soc. (2) **53** (1951), 460–475.
[18] ———, *Futher identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. **54** (1952), 147–167.
[19] V. P. Spiridonov, *An elliptic incarnation of the Bailey chain*, Int. Math. Res. Not. **37** (2002), no. 37, 1945–1977.
[20] A. Verma and V. K. Jain, *Certain summation formulae for q-series*, J. Indian Math. Soc. **47** (1983), no. 1-4, 71–85.
[21] S. O. Warnaar, *50 years of Bailey's lemma*, Algebraic combinatorics and applications (Gweinstein, 1999), 333–347, Springer, Berlin, 2001.
[22] ———, *Extensions of the well-poised and elliptic well-poised Bailey lemma*, Indag. Math. New Series **14** (2003), no. 3-4, 571–588.

S. Ahmad Ali
Department of Mathematics
Babu Banarasi Das University
Lucknow 226 028, (U.P) India
E-mail address: ali.sahmad@yahoo.com

Sayyad Nadeem Hasan Rizvi
Department of Mathematics
Babu Banarasi Das University
Lucknow 226 028, (U.P) India
E-mail address: snhrizvi110@gmail.com