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APPLICATIONS ON THE BESSEL-STRUVE-TYPE FOCK SPACE

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Abstract. In this work, we establish Heisenberg-type uncertainty principle for the Bessel-Struve Fock space $\mathbb{F}_\nu$ associated to the Airy operator L_ν. Next, we give an application of the theory of extremal function and reproducing kernel of Hilbert space, to establish the extremal function associated to a bounded linear operator $T : \mathbb{F}_\nu \to H$, where H be a Hilbert space. Furthermore, we come up with some results regarding the extremal functions, when T are difference operators.

1. Introduction

Fock space $\mathbb{F}$ (called also Segal-Bargmann space [3, 4]) is the Hilbert space of entire functions on $\mathbb{C}$ with inner product given by

$$\langle f, g\rangle := \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^2}\mathrm{d}x\mathrm{d}y, \quad z = x + iy.$$

This space was introduced by Bargmann [2] and it was the aim of many works [3, 4, 21, 23, 24]. The study of several generalizations of the classical Fock spaces has a long and rich history in many different settings [5, 9, 19, 20, 22].

In this paper we consider the Bessel-Struve kernel S_ν, $\nu > -1/2$:

$$S_\nu(z) := j_\nu(iz) - i\, h_\nu(iz), \quad z \in \mathbb{C},$$

where

$$j_\nu(z) := 2^\nu\Gamma(\nu + 1)\frac{J_\nu(z)}{z^\nu} \text{ and } h_\nu(z) := 2^\nu\Gamma(\nu + 1)\frac{\mathbf{H}_\nu(z)}{z^\nu}.$$

Here J_ν and $\mathbf{H}_\nu$ are the Bessel and the Struve functions [6, 25].

The kernel S_ν is analytic and it can be expanded in the form

$$S_\nu(z) = \sum_{n=0}^{\infty}\frac{z^n}{c_n(\nu)}, \quad c_n(\nu) = \frac{\sqrt{\pi}\, n!\,\Gamma(\frac{n}{2} + \nu + 1)}{\Gamma(\nu + 1)\Gamma(\frac{n+1}{2})}, \tag{1.1}$$

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and possesses the following integral representation

$$S_\nu(z) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{zt}\mathrm{d}t.$$

The Bessel-Struve kernel $S_\nu(z)$, $z \in \mathbb{C}$, solves the equation

$$L_\nu u(z) = u(z), \quad u(0) = 1,$$

where L_ν is the Bessel-Struve operator given by

$$L_\nu u(z) := \frac{\mathrm{d}^2}{\mathrm{d}z^2}u(z) + \frac{2\nu+1}{z}\left[\frac{\mathrm{d}}{\mathrm{d}z}u(z) - \frac{\mathrm{d}}{\mathrm{d}z}u(0)\right].$$

During the last years, the Bessel-Struve operator have gained considerable interest in various field of mathematics [1, 8, 11–14] and in certain parts of quantum calculus [22]. The results of this work will be useful when discussing the Fock space associated to this operator. This space is the background of some applications in this contribution. Especially,

- we study the Bessel-Struve operator and its adjoint operator on the Bessel-Struve-type Fock space;

- we establish Heisenberg-type uncertainty principle for the Bessel-Struve-type Fock space;

- we give an application of the theory of extremal function and reproducing kernel of Hilbert space, to establish the extremal function associated to a bounded linear operator $T : \mathbb{F}_\nu \to H$, where H be a Hilbert space;

- we come up with some results regarding the extremal functions associated to the difference operators $Tf(z) := \frac{1}{z^2}(f(z) - zf'(0) - f(0))$ and $Tf(z) := \frac{1}{2z^2}(f(z) + f(-z) - 2f(0))$.

The contents of the paper are as follows. In Section 2, we establish Heisenberg-type uncertainty principle for the Bessel-Struve-type Fock space $\mathbb{F}_\nu$. In Section 3, we give an application of the theory of reproducing kernels to the Tikhonov regularization problem for bounded linear operator $T : \mathbb{F}_\nu \to H$, where H be a Hilbert space. Next, we come up with some results regarding the Tikhonov regularization problem for the difference operators given above.

2. Uncertainty principle for $\mathbb{F}_\nu$

We denote by

• m_ν, $\nu > -1/2$, the measure defined on $\mathbb{C}$ by

$$\mathrm{d}m_\nu(z) := \frac{1}{\pi 2^\nu \Gamma(\nu+1)}|z|^{2\nu+2}K_\nu(|z|^2)\mathrm{d}x\mathrm{d}y, \quad z = x + iy,$$

where K_ν is the Macdonald function [6].

• $L^2(m_\nu)$, the Hilbert space of measurable functions on $\mathbb{C}$, for which

$$\|f\|_{L^2(m_\nu)} := \left[\int_{\mathbb{C}} |f(z)|^2 \mathrm{d}m_\nu(z)\right]^{1/2} < \infty.$$

• $H(\mathbb{C})$, the space of entire functions on $\mathbb{C}$.

Let $\nu > -1/2$. We define the Bessel-Struve-type Fock space $\mathbb{F}_\nu$ as

$$\mathbb{F}_\nu := L^2(m_\nu) \cap H(\mathbb{C}).$$

The space $\mathbb{F}_\nu$ is equipped with the norm $\|f\|_{\mathbb{F}_\nu} := \|f\|_{L^2(m_\nu)}$.

The space $\mathbb{F}_{\nu,e}$ of even functions of $\mathbb{F}_\nu$ is just the generalized Fock space associated with the Bessel operator (see [5]).

Theorem 2.1 (See [9])**.** *Let $f, g \in \mathbb{F}_\nu$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. One has*

$$\langle f, g\rangle_{\mathbb{F}_\nu} = \sum_{n=0}^{\infty} a_n \overline{b_n}\, c_n(\nu),$$

where $c_n(\nu)$ are the constants given by (1.1).

Theorem 2.2. (i) *The function k_ν given for $w, z \in \mathbb{C}$ by*

$$k_\nu(z, w) = S_\nu(\bar{z}w), \tag{2.1}$$

is a reproducing kernel for the Bessel-Struve-type Fock space $\mathbb{F}_\nu$. That is $k_\nu(z, \cdot) \in \mathbb{F}_\nu$, and for all $f \in \mathbb{F}_\nu$, one has $\langle f, k_\nu(z, \cdot)\rangle_{\mathbb{F}_\nu} = f(z)$.

(ii) *If $f \in \mathbb{F}_\nu$, then $|f(z)| \le e^{|z|^2/2} \|f\|_{\mathbb{F}_\nu}$, $z \in \mathbb{C}$.*

(iii) *The space $\mathbb{F}_\nu$ equipped with the inner product $\langle \cdot, \cdot\rangle_{\mathbb{F}_\nu}$ is a Hilbert space; and the set $\left\{\frac{z^n}{\sqrt{c_n(\nu)}}\right\}_{n\in\mathbb{N}}$ forms a Hilbert's basis for the space $\mathbb{F}_\nu$.*

Proof. (i) See [9].

(ii) Let $f \in \mathbb{F}_\nu$ and $z \in \mathbb{C}$. From (i), we have

$$|f(z)| \le \|k_\nu(z, \cdot)\|_{\mathbb{F}_\nu} \|f\|_{\mathbb{F}_\nu}.$$

Using the fact that

$$\|k_\nu(z, \cdot)\|^2_{\mathbb{F}_\nu} = k_\nu(z, z) = S_\nu(|z|^2) \le e^{|z|^2}, \tag{2.2}$$

we deduce the result.

(iii) Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{F}_\nu$. We put $f = \lim_{n\to\infty} f_n$, in $L^2(m_\nu)$. From Theorem 2.2(ii), we have $|f_{n+p}(z) - f_n(z)| \le e^{|z|^2/2} \|f_{n+p} - f_n\|_{\mathbb{F}_\nu}$. This inequality shows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to f. Since the function $z \to e^{|z|^2/2}$ is continuous on $\mathbb{C}$, then $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on all compact set of $\mathbb{C}$. Consequently, f is an entire function on $\mathbb{C}$, then f belongs to the space $\mathbb{F}_\nu$. On the other hand, from Theorem 2.1, we get $\langle z^n, z^m\rangle_{\mathbb{F}_\nu} = c_n(\nu)\delta_{n,m}$. This shows that the family $\left\{\frac{z^n}{\sqrt{c_n(\nu)}}\right\}_{n\in\mathbb{N}}$ is an orthonormal set in $\mathbb{F}_\nu$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an element of $\mathbb{F}_\nu$ such that $\langle f, z^n\rangle_{\mathbb{F}_\nu} = 0$ for all $n \in \mathbb{N}$. From Theorem 2.1, we deduce that $a_n = 0$ for all $n \in \mathbb{N}$. This completes the proof. □

We consider the space $\mathbb{U}_\nu$ defined as the space of entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=0}^{\infty} n^2|a_n|^2 c_n(\nu) < \infty$. This space is a subspace of the Bessel-Struve-type Fock space $\mathbb{F}_\nu$. For $f \in \mathbb{U}_\nu$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ one has

$$L_\nu f(z) = \sum_{n=0}^{\infty} (n+2)(n+2\nu+2) a_{n+2} z^n,$$

and

$$z^2 f(z) = \sum_{n=2}^{\infty} a_{n-2} z^n.$$

Using the fact that $c_{n+2}(\nu) = (n+2)(n+2\nu+2)c_n(\nu)$ one has

$$\|L_\nu f\|^2_{\mathbb{F}_\nu} \leq (1+2\nu) \sum_{n=0}^{\infty} n^4 |a_n|^2 c_n(\nu),$$

and

$$\|z^2 f\|^2_{\mathbb{F}_\nu} \leq 4(\nu+1)|a_0|^2 + 3(2\nu+3) \sum_{n=0}^{\infty} n^2 |a_n|^2 c_n(\nu).$$

Therefore $L_\nu f$ and $z^2 f$ belong to $\mathbb{F}_\nu$.

The Bessel-Struve operator L_ν satisfies the following properties (see [9]).

(i) For $f, g \in \mathbb{U}_\nu$, $\langle L_\nu f, g\rangle_{\mathbb{F}_\nu} = \langle f, L_\nu^* g\rangle_{\mathbb{F}_\nu}$, where $L_\nu^* g(z) = z^2 g(z)$.

(ii) For $f \in \mathbb{U}_\nu$, $[L_\nu, L_\nu^*]f(z) := (L_\nu L_\nu^* - L_\nu^* L_\nu)f(z) = 4(\nu+1)f(z) + W_\nu f(z)$, where

$$W_\nu f(z) = 4z \frac{\mathrm{d}}{\mathrm{d}z} f(z) + (2\nu+1) z \frac{\mathrm{d}}{\mathrm{d}z} f(0).$$

(iii) If $f \in \mathbb{U}_\nu$, then $W_\nu f \in \mathbb{F}_\nu$ and

$$\|L_\nu^* f\|^2_{\mathbb{F}_\nu} = \|L_\nu f\|^2_{\mathbb{F}_\nu} + 4(\nu+1)\|f\|^2_{\mathbb{F}_\nu} + \langle W_\nu f, f\rangle_{\mathbb{F}_\nu}. \tag{2.3}$$

By applying the previous properties of L_ν and the following result of functional analysis.

Theorem 2.3 (See [7, 10])**.** *Let A and B be self-adjoint operators on a Hilbert space H. One has*

$$\|(A-a)f\|_H \|(B-b)f\|_H \geq \frac{1}{2} |\langle [A,B]f, f\rangle_H|$$

for all $f \in Dom(AB) \cap Dom(BA)$, and all $a, b \in \mathbb{R}$.

We obtain the following Heisenberg-type uncertainty principle for the Bessel-Struve-type Fock space $\mathbb{F}_\nu$.

Theorem 2.4. *Let $f \in \mathbb{U}_\nu$. For all $a, b \in \mathbb{R}$, one has*

$$\|(L_\nu + z^2 - a)f\|_{\mathbb{F}_\nu} \|(L_\nu - z^2 + ib)f\|_{\mathbb{F}_\nu} \geq \Big| \|z^2 f\|^2_{\mathbb{F}_\nu} - \|L_\nu f\|^2_{\mathbb{F}_\nu} \Big|, \tag{2.4}$$

where i is the imaginary unit.

Proof. Let us consider the following two operators on $\mathbb{U}_\nu$ by

$$A = L_\nu + z^2, \quad B = i(L_\nu - z^2). \tag{2.5}$$

It follows that, for a function $f \in \mathbb{U}_\nu$, we have $L_\nu f \in \mathbb{F}_\nu$ and $z^2 f \in \mathbb{F}_\nu$. Therefore Af and Bf are in $\mathbb{F}_\nu$. The operators A, B are self-adjoint on $\mathbb{F}_\nu$ and $[A, B] = -2i(4(\nu+1)I + W_\nu)$. Thus the inequality (2.4) follows from Theorem 2.3 and (2.3). □

The Heisenberg-type uncertainty principle of Theorem 2.4 can be written as the following.

Theorem 2.5. *Let $f \in \mathbb{U}_\nu$. Then*

$$\Delta_\nu^+(f)\Delta_\nu^-(f) \geq \|f\|^4_{\mathbb{F}_\nu} \left(\|z^2 f\|^2_{\mathbb{F}_\nu} - \|L_\nu f\|^2_{\mathbb{F}_\nu}\right)^2, \tag{2.6}$$

where

$$\Delta_\nu^\pm(f) = \|f\|^2_{\mathbb{F}_\nu}\|(L_\nu \pm z^2)f\|^2_{\mathbb{F}_\nu} - |\langle (L_\nu \pm z^2)f, f\rangle_{\mathbb{F}_\nu}|^2.$$

Proof. Let $f \in \mathbb{U}_\nu$. The operator A given by (2.5) is self-adjoint, then for any real a we have

$$\|(A-a)f\|^2_{\mathbb{F}_\nu} = \|Af\|^2_{\mathbb{F}_\nu} + a^2\|f\|^2_{\mathbb{F}_\nu} - 2a\langle Af, f\rangle_{\mathbb{F}_\nu}.$$

This shows that

$$\min_{a\in\mathbb{R}} \|(A-a)f\|^2_{\mathbb{F}_\nu} = \|Af\|^2_{\mathbb{F}_\nu} - \frac{|\langle Af, f\rangle_{\mathbb{F}_\nu}|^2}{\|f\|^2_{\mathbb{F}_\nu}},$$

and the minimum is attained when $a = \frac{\langle Af, f\rangle_{\mathbb{F}_\nu}}{\|f\|^2_{\mathbb{F}_\nu}}$. In other words, we have

$$\min_{a\in\mathbb{R}} \|(L_\nu + z^2 - a)f\|^2_{\mathbb{F}_\nu} = \|(L_\nu + z^2)f\|^2_{\mathbb{F}_\nu} - \frac{|\langle (L_\nu + z^2)f, f\rangle_{\mathbb{F}_\nu}|^2}{\|f\|^2_{\mathbb{F}_\nu}}, \tag{2.7}$$

and the minimum is attained when $a = \frac{\langle (L_\nu + z^2)f, f\rangle_{\mathbb{F}_\nu}}{\|f\|^2_{\mathbb{F}_\nu}}$. Similarly, we have

$$\min_{b\in\mathbb{R}} \|(L_\nu - z^2 + ib)f\|^2_{\mathbb{F}_\nu} = \|(L_\nu - z^2)f\|^2_{\mathbb{F}_\nu} - \frac{|\langle (L_\nu - z^2)f, f\rangle_{\mathbb{F}_\nu}|^2}{\|f\|^2_{\mathbb{F}_\nu}}, \tag{2.8}$$

and the minimum is attained when $b = i\frac{\langle (L_\nu - z^2)f, f\rangle_{\mathbb{F}_\nu}}{\|f\|^2_{\mathbb{F}_\nu}}$.

Then by (2.4), (2.7) and (2.8) we deduce the inequality (2.6). □

3. Extremal functions on $\mathbb{F}_\nu$

Let $\lambda > 0$ and let $T : \mathbb{F}_\nu \to H$ be a bounded linear operator from $\mathbb{F}_\nu$ into a Hilbert H. We denote by $\langle \cdot, \cdot \rangle_{T,\lambda}$ the inner product defined on the space $\mathbb{F}_\nu$ by

$$\langle f, g\rangle_{T,\lambda} := \lambda\langle f, g\rangle_{\mathbb{F}_\nu} + \langle Tf, Tg\rangle_H,$$

and the norm $\|f\|_{T,\lambda} := \sqrt{\langle f, f\rangle_{T,\lambda}}$.

By using the theory reproducing kernels of Hilbert space and building on the ideas of Saitoh [15,18] we examine the extremal functions associated to the operator T on the Airy-type Fock space $\mathbb{F}_\nu$.

Theorem 3.1. *Let $\lambda > 0$. The Fock space $(\mathbb{F}_\nu, \langle\cdot,\cdot\rangle_{T,\lambda})$ possesses a reproducing kernel $k_{T,\lambda}(z,w)$; $z,w \in \mathbb{C}$ satisfying the equation $(\lambda I + T^*T)k_{T,\lambda}(z,\cdot) = k_\nu(z,\cdot)$, where k_ν is the kernel given by (2.1). Moreover the kernel $k_{T,\lambda}$ satisfies*

$$\|Tk_{T,\lambda}(z,.)\|_H \leq \frac{e^{|z|^2/2}}{\sqrt{2\lambda}}. \tag{3.1}$$

Proof. Let $f \in \mathbb{F}_\nu$. From Theorem 2.2(ii), we have $|f(z)| \leq \frac{e^{|z|^2/2}}{\sqrt{\lambda}}\|f\|_{T,\lambda}$. Then, the map $f \to f(z)$, $z \in \mathbb{C}$ is a continuous linear functional on $(\mathbb{F}_\nu, \langle\cdot,\cdot\rangle_{T,\lambda})$. Thus $(\mathbb{F}_\nu, \langle\cdot,\cdot\rangle_{T,\lambda})$ has a reproducing kernel denoted by $k_{T,\lambda}(z,w)$. On the other hand, one has

$$\begin{aligned} f(z) &= \lambda\langle f, k_{T,\lambda}(z,\cdot)\rangle_{\mathbb{F}_\nu} + \langle Tf, Tk_{T,\lambda}(z,\cdot)\rangle_H \\ &= \langle f, (\lambda I + T^*T)k_{T,\lambda}(z,\cdot)\rangle_{\mathbb{F}_\nu}. \end{aligned}$$

Thus $(\lambda I + T^*T)k_{T,\lambda}(z,\cdot) = k_\nu(z,\cdot)$. Furthermore the precedent relation implies that

$$\lambda^2\|k_{T,\lambda}(z,\cdot)\|^2_{\mathbb{F}_\nu} + 2\lambda\|Tk_{T,\lambda}(z,\cdot)\|^2_H + \|T^*Tk_{T,\lambda}(z,\cdot)\|^2_{\mathbb{F}_\nu} = \|k_\nu(z,\cdot)\|^2_{\mathbb{F}_\nu}.$$

From this relation and using (2.2) we obtain (3.1). □

The main result of this section can then be stated as follows.

Theorem 3.2. *For any $h \in H$ and for any $\lambda > 0$, there exists a unique function $f^*_{\lambda,h}$, where the infimum*

$$\inf_{f\in\mathbb{F}_\nu}\left\{\lambda\|f\|^2_{\mathbb{F}_\nu} + \|h - Tf\|^2_H\right\} \tag{3.2}$$

is attained. Moreover, the extremal function $f^*_{\lambda,h}$ is given by

$$f^*_{\lambda,h}(z) = \langle h, Tk_{T,\lambda}(z,\cdot)\rangle_H, \tag{3.3}$$

and satisfies the following inequality $|f^*_{\lambda,h}(z)| \leq \frac{e^{|z|^2/2}}{\sqrt{2\lambda}}\|h\|_H$.

Proof. The existence and unicity of the extremal function $f^*_{\lambda,h}$ satisfying (3.2) is obtained in [16–18]. Especially, $f^*_{\lambda,h}$ is given by the reproducing kernel of $\mathbb{F}_\nu$ with $\|\cdot\|_{T,\lambda}$ norm as

$$f^*_{\lambda,h}(z) = \langle h, Tk_{T,\lambda}(z,\cdot)\rangle_H.$$

This clearly yields the result. On the other hand, from (3.1) and (3.3), one has

$$|f^*_{\lambda,h}(z)| \leq \|h\|_H\|Tk_{T,\lambda}(z,\cdot)\|_H \leq \frac{e^{|z|^2/2}}{\sqrt{2\lambda}}\|h\|_H,$$

which completes the proof of the theorem. □

Application 3.3. Let H be the prehilbertian space of entire functions, equipped with the inner product

$$\langle f, g\rangle_H := \int_{\mathbb{C}} f(z)\overline{g(z)}|z|^4 \mathrm{d}m_\nu(z).$$

If $f, g \in H$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g\rangle_H = \sum_{n=0}^{\infty} a_n \overline{b_n} c_{n+2}(\nu), \quad \|f\|_H^2 = \sum_{n=0}^{\infty} |a_n|^2 c_{n+2}(\nu).$$

The space H is a Hilbert space with Hilbert's basis $\left\{\frac{z^n}{\sqrt{c_{n+2}(\nu)}}\right\}_{n\in\mathbb{N}}$ and reproducing kernel

$$s_\nu(z, w) = \sum_{n=0}^{\infty} \frac{(\overline{z}w)^n}{c_{n+2}(\nu)} = \frac{1}{(\overline{z}w)^2}\Big(S_\nu(\overline{z}w) - \frac{\overline{z}w}{c_1(\nu)} - 1\Big).$$

1) Let T be the difference operator defined on $\mathbb{F}_\nu$ by

$$Tf(z) := \frac{1}{z^2}(f(z) - zf'(0) - f(0)).$$

Then the operator T maps continuously from $\mathbb{F}_\nu$ into H, and $\|Tf\|_H \le \|f\|_{\mathbb{F}_\nu}$. If $f, g \in \mathbb{F}_\nu$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, one has

$$\langle f, g\rangle_{T,\lambda} = \lambda a_0 \overline{b_0} + \lambda a_1 \overline{b_1} c_1(\nu) + (\lambda + 1) \sum_{n=2}^{\infty} a_n \overline{b_n} c_n(\nu).$$

Thus, for $z, w \in \mathbb{C}$ one has

$$k_{T,\lambda}(z, w) = \Big(\frac{1}{\lambda} - \frac{1}{\lambda+1}\Big)\Big(1 + \frac{\overline{z}w}{c_1(\nu)}\Big) + \frac{1}{\lambda+1} S_\nu(\overline{z}w),$$

$$Tk_{T,\lambda}(z, \cdot)(w) = \frac{1}{(\lambda+1)w^2}\Big(S_\nu(\overline{z}w) - \frac{\overline{z}w}{c_1(\nu)} - 1\Big),$$

and for all $h \in H$ we deduce that

$$f^*_{\lambda,h}(z) = \frac{1}{\lambda+1} z^2 h(z).$$

2) Let T be the difference operator defined on $\mathbb{F}_\nu$ by

$$Tf(z) := \frac{1}{2z^2}(f(z) + f(-z) - 2f(0)),$$

then the operator T maps continuously from $\mathbb{F}_\nu$ into H, and $\|Tf\|_H \le \|f\|_{\mathbb{F}_\nu}$. If $f, g \in \mathbb{F}_\nu$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, one has

$$\langle f, g\rangle_{T,\lambda} = \lambda a_0 \overline{b_0} + \sum_{n=1}^{\infty} \Big[\lambda + \frac{1}{2}(1 + (-1)^n)\Big] a_n \overline{b_n} c_n(\nu).$$

Thus, for $z, w \in \mathbb{C}$ one has

$$k_{T,\lambda}(z, w) = \frac{1}{\lambda} + \frac{1}{\lambda+1} \sum_{n=1}^{\infty} \frac{(\overline{z}w)^{2n}}{c_{2n}(\nu)} + \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(\overline{z}w)^{2n+1}}{c_{2n+1}(\nu)},$$

$$Tk_{T,\lambda}(z, \cdot)(w) = \frac{1}{\lambda+1} \sum_{n=0}^{\infty} \frac{(\overline{z})^{2n+2}}{c_{2n+2}(\nu)} w^{2n},$$

and for all $h \in H$ we deduce that

$$f^*_{\lambda,h}(z) = \frac{1}{2(\lambda+1)} z^2[h(z) + h(-z)].$$

Remark 3.4. Theorem 3.2 can be applied to other various operators. In paper [23], the author compute the extremal functions associated to operators written by means of Fourier transform and Segal-Bargmann transform. For thus, we need more details about Fourier transform and Segal-Bargmann transform associated to the Bessel-Struve operator.

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