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EXPONENTIAL STABILITY OF A CLASS OF NONLINEAR DIFFERENCE EQUATIONS IN BANACH SPACES

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Abstract. The problems of global and local exponential stability analysis of a class of nonlinear non-autonomous difference equations in Banach spaces are studied in this paper. By a novel comparison technique, new explicit exponential stability conditions are derived. Numerical examples are given to illustrate the effectiveness of the obtained results.

1. Introduction

It is well-known that delay differential equations (DDEs) play an essential role in modeling a wide range of phenomena in the vivid world [9,25]. Typical examples are chemical and physical processes, data transmission lines, biological models, robotic control or communication networks. These practical systems are usually modeled in the form of nonlinear continuous-time systems with time-varying delays, which are much more complicated than corresponding ordinary differential equations [12,20]. Today, with a dramatical development of computer-based computational techniques, difference equations are found to be much more appropriate for computer simulation, experiment and computation, which play an important role in realistic applications. By a discretization process, discrete-time systems described by difference equations inherit the similar dynamical behavior of the continuous-time ones [12]. Considerable attention from researchers has been devoted to develop efficient numerical methods and enrich theoretical aspects for this type of systems [7,23,24,28]. Particularly, the problem of stability analysis of difference equations with or without delays has received extensive attention recently [4,11,12,16,17,27].

In most of existing results, which concern with stability of difference equations with delay, the Lyapunov–Krasovskii functional (LKF) method is widely used to derive sufficient stability conditions. However, this approach, on one hand, relies heavily on how to choose an appropriate LKF candidate which usually leads to serious difficulties and, on the other hand, is hard to apply to

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nonlinear non-autonomous equations [14]. Another effective approach is the use of discrete-type inequalities such as Gronwall inequality or Halanay inequalities [1, 2, 12, 19, 26–28, 30].

On the other hand, a numerical approximation process of a variety of practical systems leads to difference equations in infinite-dimensional spaces [4, 7, 29]. When considering the applicability of numerical methods for dynamical systems, it is important to analyze whether or not numerical methods inherit dynamical properties of the underlying systems. Therefore, it is relevant and important to study asymptotic behavior of difference equations in a Banach spaces [3, 5, 6, 10, 21, 29].

Motivated by the above discussion, in this paper, we consider the problem of exponential stability analysis of a class of higher-order nonlinear non-autonomous difference equations in Banach spaces. Based on a novel comparison technique, we establish explicit conditions that ensure exponential stability of the equation with general growth condition of the nonlinear part. Specifically, the derived conditions ensure both global exponential stability and local exponential stability of the equation when the nonlinear part satisfies a sub-linear condition and a superlinear condition, respectively. Numerical examples are given to illustrate the effectiveness of the obtained results.

Notation. Throughout this paper, we let $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the set of integers and positive integers, respectively. For $r \in \mathbb{Z}$, we denote $\mathbb{Z}_r = \{m \in \mathbb{Z} : m \geq r\}$ and for $r_1, r_2 \in \mathbb{Z}, r_1 \leq r_2$, the set $\{r_1, r_1 + 1, \dots, r_2\}$ is denoted as $\mathbb{Z}[r_1, r_2]$. We also use the notation $\underline{m}$ to denote the set $\mathbb{Z}[1, m]$ for $m \in \mathbb{Z}^+$.

2. Preliminaries

Consider the following functional equation

$$x'(t) = -ax(t) + b(t)f(t, x_t), \quad x_0 = \phi, \tag{1}$$

where $a > 0$ is a constant, $b \in L^\infty(\mathbb{R}, \mathbb{R})$, $f : \mathbb{R} \times C([-\tau, 0], \mathbb{R}) \to \mathbb{R}$, $\tau > 0$, is a continuous function, $\phi \in C([-\tau, 0], \mathbb{R})$ is the initial condition and $x_t \in C([-\tau, 0], \mathbb{R})$ defined by $x_t(s) = x(t + s), s \in [-\tau, 0]$.

By employing continuous Halanay inequality, it was proved in [15] that if $|f(t, \phi)| \leq \|\phi\|$ for all $\phi \in C([-\tau, 0], \mathbb{R})$ and $b_\infty = \text{esssup}|b(t)| < a$, then all solutions of (1) converge exponentially to zero. Later, by a discrete Halanay inequality, it was proved in [18] the delay-independent condition $b_\infty < a$ is preserved under a numerical scheme. More precisely, let r be a positive integer (the number of subintervals) and define $h = \tau/r$ as the discretization step. Denote $t_0 = 0, t_{n+i} = t_n + ih$, $x_n = x(t_n), x_{n+i} = x(t_n + ih)$, $i \in \mathbb{Z}$, then (1) can be approximated by the following scheme:

Step 1 Calculate $x_n = \phi(t_n)$ for $n \in \mathbb{Z}[-r, 0]$.

Step 2 Define $\psi_n : [t_{n-r}, t_n] \to \mathbb{R}$ as the piecewise linear function connecting the $r + 1$ points $p_{n-r} = (t_{n-r}, x_{n-r}), \dots, p_n = (t_n, x_n)$ and $\varphi_n(t) = \psi_n(t - t_n)$.

Step 3 Calculate $\bar{f}(n, x_n, \ldots, x_{n-r}) = b(t_n)f(t_n, \varphi_n)$.

By using the explicit Euler discretization method to approximate (1), we obtain

$$x_{n+1} = (1 - ah)x_n + h\bar{f}(n, x_n, \ldots, x_{n-r}), \quad n > 0. \tag{2}$$

Theorem 2 in [18] ensures that equation (2) is asymptotically stable if $b_\infty < a$ and $h \leq \frac{1}{a}$.

It should be pointed out that the asymptotic stability of (1) is preserved in (2) only for sufficiently small sizes of discretization step. Now, we slightly modify the approximation of (1). By applying the backward Euler method into the linear part, equation (1) is approximated as

$$\frac{x_{n+1} - x_n}{h} = -ax_{n+1} + \bar{f}(n, x_n, \ldots, x_{n-r}) \tag{3}$$

which can be written in the form

$$x_{n+1} = \left(1 - \frac{ah}{1 + ah}\right)x_n + \frac{h}{1 + ah}\bar{f}(n, x_n, \ldots, x_{n-r}). \tag{4}$$

It can be verified by Theorem 2 in [18] that the asymptotic stability of (1) is preserved for difference equation (4) without any restriction on the step size $h > 0$.

Now, we adapt the discretization scheme in deriving (4) to the following generalized nonlinear non-autonomous delay differential equation

$$x'(t) = -a(t)x(t) + b(t)G(t, x(t - \tau_1), x(t - \tau_2), \ldots, x(t - \tau_m)),\ t \geq 0, \tag{5}$$

where $a, b \in C(\mathbb{R}^+, \mathbb{R})$ are continuous functions, $a(t) \geq 0$, $x(t) \in \mathbb{X}$, a Banach space with the norm $\|\cdot\|$, $G : \mathbb{R}^+ \times \mathbb{X}^m \to \mathbb{X}$ is a continuous function, $G(t, 0) = 0$, τ_i, $i \in \underline{m} = \{1, 2, \ldots, m\}$, are positive numbers involving time-delay. We assume that $0 < \tau_1 < \tau_2 < \cdots < \tau_m$ and $\tau_i, i \in \underline{m}$, have a positive common factor, namely h, that means there are integers $r_1, r_2, \ldots, r_m$ such that $\tau_i = hr_i, i \in \underline{m}$. By the same discretization scheme used in deriving (4) with step size h, it follows from (5) that

$$x_{n+1} = \frac{1}{1 + a(n)}x_n + \frac{b(n)}{1 + a(n)}F(n, x_{n-r_1}, x_{n-r_2}, \ldots, x_{n-r_m}), \quad n \geq 0, \tag{6}$$

where $F(n, x_{n-r_1}, \ldots, x_{n-r_m}) = G(t_n, \varphi_{n1}, \ldots, \varphi_{nm})$, $a(n) = ha(t_{n+1})$ and $b(n) = hb(t_n)$. In equation (6), each initial condition is defined by a sequence, also denoted by ϕ, on $\mathbb{Z}[-r_m, 0]$, that is $x(n) = \phi(n), n \in \mathbb{Z}[-r_m, 0]$. For convenience, we denote $\|\phi\| = \max_{n \in \mathbb{Z}[-r_m, 0]} \|\phi(n)\|$.

First, we recall here that equation (6) is said to be globally exponentially stable if there exist positive scalars β, γ such that any solution x_n of (6) with initial condition ϕ satisfies the inequality

$$\|x_n\| \leq \beta\|\phi\|e^{-\gamma n},\ n \in \mathbb{Z}^+. \tag{7}$$

If there exists a scalar $\delta > 0$ such that inequality (7) holds for any initial condition ϕ satisfying $\|\phi\| < \delta$, then equation (6) is said to be locally exponentially stable.

The main objective of this paper is to derive exponential stability conditions for equation (6) subject to general growth conditions of the nonlinear function F. Inspired by the comparison techniques of discrete Halanay inequalities [12,27,30], we derive explicit conditions that ensure global exponential stability of equation (6) when the nonlinear function F satisfies a sublinear condition. Different from existing results in the literature, in this paper, we will prove that if the nonlinear function F satisfies a type of superlinear conditions, then our derived conditions guarantee a local exponential estimate for (6). Moreover, as shown by numerical examples, our conditions are competitive and less conservative than existing results in the literature.

3. Main results

3.1. Global exponential stability

We make the following assumptions.

(H1) $a(n), |b(n)|$ are bounded functions satisfying

$$0 \le a_+ \le a(n) \le a^+, \quad b_+ \le |b(n)| \le b^+, \ \forall n \in \mathbb{Z}^+,$$

where, for a bounded function $g(n)$ on $\mathbb{Z}^+$, we denote $g_+ = \inf_{n\in\mathbb{Z}^+} g(n)$ and $g^+ = \sup_{n\in\mathbb{Z}^+} g(n)$.

(H2) There exist a positive integer p, nonnegative bounded functions $\lambda_i(n)$, $i \in \underline{p}$, and nonnegative scalars α_{ij} such that $\sum_{j=1}^m \alpha_{ij} = 1, \forall i \in \underline{p}$, and

$$\|F(n, x_1, x_2, \ldots, x_m)\| \le \sum_{i=1}^p \lambda_i(n) \prod_{j=1}^m \|x_j\|^{\alpha_{ij}}, \forall (x_1, x_2, \ldots, x_m) \in \mathbb{X}^m. \tag{8}$$

Remark 3.1. It follows from (H2) that $\alpha_{ij} \le 1$ for all $i \in \underline{p}, j \in \underline{m}$, and thus condition (8) is referred to a sublinear condition of nonlinear function F.

We are now in a position to present our global exponential stability conditions for equation (6) as given in the following theorem.

Theorem 3.2. *Let assumptions* (H1) *and* (H2) *hold. If*

$$\sigma \triangleq \liminf_{n\to\infty} \Big\{ a(n) - |b(n)| \sum_{i=1}^p \lambda_i(n) \Big\} > 0, \tag{9}$$

then equation (6) *is globally exponentially stable. More precisely, there exist positive constants β, γ such that any solution x_n of* (6) *with initial condition ϕ satisfies*

$$\|x_n\| \le \beta \|\phi\| e^{-\gamma n}, \ n \in \mathbb{Z}^+. \tag{10}$$

Proof. Note at first that, by (8) and (9), $x = 0$ is the unique equilibrium point of (6). We now prove that this equilibrium point is globally exponentially stable.

By (9), there exists an integer $N_0 \geq 1$ such that

$$\inf_{n\geq N_0}\left\{a(n)-|b(n)|\sum_{i=1}^{p}\lambda_i(n)\right\}\geq\frac{\sigma}{2}. \tag{11}$$

Therefore,

$$\begin{aligned}\frac{1+|b(n)|\sum_{i=1}^{p}\lambda_i(n)}{1+a(n)} &= 1-\frac{a(n)-|b(n)|\sum_{i=1}^{p}\lambda_i(n)}{1+a(n)}\\ &\leq 1-\frac{1}{1+a^+}\inf_{n\geq N_0}\left\{a(n)-|b(n)|\sum_{i=1}^{p}\lambda_i(n)\right\}\\ &\leq 1-\frac{\sigma}{2(1+a^+)}\triangleq\rho\in(0,1),\ \forall n\geq N_0. \end{aligned}\tag{12}$$

From (6) we have

$$\begin{aligned}\|x_{n+1}\| &\leq \frac{1}{1+a(n)}\|x_n\|+\frac{|b(n)|}{1+a(n)}\|F(n,x_{n-r_1},\ldots,x_{n-r_m})\|\\ &\leq \frac{1}{1+a(n)}\|x_n\|+\frac{|b(n)|}{1+a(n)}\sum_{i=1}^{p}\lambda_i(n)\prod_{j=1}^{m}\|x_{n-r_j}\|^{\alpha_{ij}}.\end{aligned}\tag{13}$$

For any initial condition ϕ, $\|x_n\|\leq\|\phi\|$, $\forall n\in\mathbb{Z}[-r_m,0]$. Thus, it follows from (13) that $\|x_1\|\leq\frac{1+b^+\lambda^*}{1+a_+}\|\phi\|$, where $\lambda^*=\sum_{j=1}^{p}\lambda_j^+$. By induction, from (13) we readily obtain

$$\|x_n\|\leq\left(\frac{1+b^+\lambda^*}{1+a_+}\right)^n\|\phi\|,\ n\geq 0. \tag{14}$$

Suppose $\frac{1+b^+\lambda^*}{1+a_+}>1$. Then, by (14), $\|x_n\|\leq\beta_0\|\phi\|$, $\forall n\in\mathbb{Z}[-r_m,N_0-1]$, where $\beta_0=\max\left\{\left(\frac{1+b^+\lambda^*}{1+a_+}\right)^{N_0-1},1\right\}$. Note that the above estimate obviously holds if $\frac{1+b^+\lambda^*}{1+a_+}\leq 1$. This, in regard to (13)-(14), leads to

$$\|x_n\|\leq\beta_0\rho\|\phi\|,\ \forall n\geq N_0. \tag{15}$$

Now, inspired by the Archimède algorithm, we rescale $\mathbb{Z}_{N_0}$ by the following sets

$$I_k=\{N_0+(k-1)r_m+s : s\in\mathbb{Z}[0,r_m-1]\},\ k\geq 1, \tag{16}$$

then it can be seen that

$$\mathbb{Z}_{N_0}=\{n\in\mathbb{Z} : n\geq N_0\}=\cup_{k=1}^{\infty}I_k.$$

We will prove by induction that

$$\|x_n\|\leq\beta_0\rho^k\|\phi\|,\ \forall n\in I_k,\ \forall k\in\mathbb{Z}^+. \tag{17}$$

Clearly, (17) holds for $k = 1$. Suppose (17) holds for some $k \in \mathbb{Z}^+$. Then, for $n = N_0 + kr_m - 1$, we have $n - r_j = N_0 + (k-1)r_m + r_m - 1 - r_j \in I_k$. Thus,

$$\begin{aligned}\|x_{n+1}\| &\le \frac{1}{1+a(n)}\|x_n\| + \frac{|b(n)|}{1+a(n)}\sum_{i=1}^{p}\lambda_i(n)\prod_{j=1}^{m}\|x_{n-r_j}\|^{\alpha_{ij}}\\ &\le \frac{1}{1+a(n)}\beta_0\rho^k\|\phi\| + \frac{|b(n)|}{1+a(n)}\sum_{i=1}^{p}\lambda_i(n)\prod_{j=1}^{m}\left(\beta_0\rho^k\|\phi\|\right)^{\alpha_{ij}}\\ &\le \frac{1+|b(n)|\sum_{i=1}^{m}\lambda_i(n)}{1+a(n)}\beta_0\rho^k\|\phi\|\\ &\le \beta_0\rho^{k+1}\|\phi\|.\end{aligned}$$

Now, for any $n \in I_{k+1}$, we write $n = N_0 + kr_m + s, s \in \mathbb{Z}[0, r_m - 1]$, then by the same arguments we have

$$\begin{aligned}\|x_{n+1}\| &\le \frac{1}{1+a(n)}\|x_n\| + \frac{|b(n)|}{1+a(n)}\sum_{i=1}^{p}\lambda_i(n)\prod_{j=1}^{m}\|x_{n-r_j}\|^{\alpha_{ij}}\\ &\le \frac{1}{1+a(n)}\|x_n\| + \frac{|b(n)|}{1+a(n)}\sum_{i=1}^{p}\lambda_i(k)\prod_{s<r_j}\left(\|x_{n-r_j}\|\right)^{\alpha_{ij}}\prod_{r_j\le s}\left(\|x_{n-r_j}\|\right)^{\alpha_{ij}}\\ &\le \frac{1}{1+a(n)}\beta_0\rho^{k+1}\|\phi\| + \frac{|b(n)|}{1+a(n)}\sum_{i=1}^{p}\lambda_i(k)\prod_{s<r_j}\left(\beta_0\rho^k\|\phi\|\right)^{\alpha_{ij}}\\ &\quad\times\prod_{r_j\le s}\left(\beta_0\rho^{k+1}\|\phi\|\right)^{\alpha_{ij}}\\ &\le \frac{1+|b(n)|\sum_{i=1}^{m}\lambda_i(n)}{1+a(n)}\beta_0\rho^k\|\phi\|\\ &\le \beta_0\rho^{k+1}\|\phi\|.\end{aligned}$$

This shows that (17) holds for $k + 1$. By induction, (17) holds for all $k \in \mathbb{Z}^+$.

Finally, to get a global exponential estimate, we define $\gamma = -\frac{1}{r_m}\log(\rho)$ and $\beta_1 = \rho^{\frac{1-N_0}{r_m}}\beta_0$. For any $n \ge N_0$ there exists a unique $k \ge 1$ such that $n = N_0 + (k-1)r_m + s$, $s \in \mathbb{Z}[0, r_m - 1]$, and thus, $n \le kr_m + N_0 - 1$ which yields $\rho^k \le \left(\sqrt[r_m]{\rho}\right)^{n+1-N_0}$. Therefore, by (17), we have

$$\begin{aligned}\|x_n\| &\le \beta_0\rho^k\|\phi\| \le \beta_0\left(\sqrt[r_m]{\rho}\right)^{n+1-N_0}\|\phi\|\\ &= \beta_1\|\phi\|e^{-\gamma n},\ n \ge N_0.\end{aligned}\tag{18}$$

In addition,

$$\|x_n\| \le \beta_0\|\phi\| \le \beta_0 e^{\gamma(N_0-1)}\|\phi\|e^{-\gamma n},\ \forall n \in \mathbb{Z}[0, N_0 - 1].\tag{19}$$

A combination of (18) and (19) gives

$$\|x_n\| \le \beta\|\phi\|e^{-\gamma n},\ \forall n \ge 0,$$

where $\beta = \max\left\{\beta_0 e^{\gamma(N_0-1)}, \beta_1\right\}$. This shows the global exponential stability of (6). The proof is completed. □

Remark 3.3. If condition $a(n) - |b(n)|\sum_{i=1}^{p}\lambda_i(n) \geq \sigma > 0$ holds for all $n \in \mathbb{Z}^+$, then $\beta = 1$ and any solution x_n of (6) satisfies

$$\|x_n\| \leq \sup_{n\in\mathbb{Z}[-r_m,0]} \|\phi(n)\| e^{-\tilde{\gamma}n},\ \forall n \geq 0, \tag{20}$$

where $\tilde{\gamma} = -\frac{1}{r_m}\log\left(1 - \frac{\sigma}{1+a^+}\right)$.

By the same arguments used in the proof of Theorem 3.2 we obtain the following result [22].

Corollary 3.4 ([22], Thm. 3.1)**.** *Let $x(n)$ be a nonnegative sequence satisfying*

$$x(n+1) \leq \frac{1}{1+\tilde{a}(n)h}x(n) + \frac{\tilde{b}(n)h}{1+\tilde{a}(n)h}\sup_{n-k(n)\leq j\leq n} x(j),\ n \geq n_0, \tag{21a}$$

$$x(n) = |\varphi(n)|,\ n \in [n_0 - k^*, n_0], \tag{21b}$$

where $h > 0$, $k(n)$, $n \in \mathbb{Z}$, is a nonnegative bounded sequence of integers, $k^* = \sup k(n)$ is a positive integer, $\varphi(n)$, $n \in [n_0 - k^*, n_0]$, is a real valued sequence and $\tilde{a}(n)$, $\tilde{b}(n)$ are nonnegative bounded sequences of real numbers. Suppose that

$$\tilde{\sigma} = \inf_{n\in\mathbb{Z}}(\tilde{a}(n) - \tilde{b}(n)) > 0. \tag{22}$$

Then, there exists a real number $\tilde{\lambda} > 1$ such that

$$x(n) \leq \sup_{n_0-k^*\leq j\leq n_0} x(j)\tilde{\lambda}^{-(n-n_0)},\ n \geq n_0. \tag{23}$$

Proof. The proof is straightforward from Theorem 3.2 and Remark 3.3 by setting $a(n) = h\tilde{a}(n)$, $b(n) = h\tilde{b}(n)$ and $\sigma = h\tilde{\sigma}$. Then, $\tilde{\lambda}$ is defined as $\tilde{\lambda} = \left(1 - \frac{h\tilde{\sigma}}{1+h\tilde{a}^+}\right)^{-\frac{1}{k^*}}$. □

Remark 3.5. In [22], by using the original technique of continuous Halanay inequality, the existence of $\tilde{\lambda}$ was defined by the continuity of the function

$$F(\lambda) = \sup_{n\in\mathbb{Z}}\left\{\frac{\lambda}{1+\tilde{a}(n)h} + \frac{\tilde{b}(n)h}{1+\tilde{a}(n)h}\lambda^{k^*+1} - 1\right\}$$

and condition $F(\tilde{\lambda}) \leq -\frac{\tilde{\sigma}h}{1+\tilde{a}^+h}$. In contrast, in Corollary 3.4 we give an explicit convergent rate $\tilde{\lambda}$.

It is also worth to note that Theorem 3.2 in this paper encompasses some recent results found in the literature, for instance, [17, 26, 27].

Corollary 3.6 ([27])**.** *The nonlinear difference equation*

$$x_{n+1} = (1-p)x_n + f(n, x_n, x_{n-h_1}, \dots, x_{n-h_m}), \tag{24}$$

where $h_i \in \mathbb{Z}^+$, $p > 0$, *is globally exponentially stable if there exist* $q_i \geq 0$, $q_m > 0$ *satisfying* $\sum_{i=0}^m q_i < p < 1$ *such that*

$$|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_m})| \leq \sum_{i=0}^m q_i |x_{n-h_i}|. \tag{25}$$

Proof. We apply Theorem 3.2 for sequences

$$a(n) = \frac{p + q_0}{1 - p - q_0}, \quad b(n) = 1, \ \forall n \in \mathbb{Z}^+$$

and $p = m, \lambda_i(n) = \frac{q_i}{1-p-q_0}$, $\forall n$, $\alpha_{ij} = 1$ if $j = i$ and $\alpha_{ij} = 0$ for $j \neq i$, then it can be found that assumptions (H1) and (H2) are satisfied. Condition $\sum_{i=0}^m q_i < p$ implies that

$$a(n) - b(n) \sum_{i=1}^m \lambda_i(n) \geq \sigma := \frac{p - \sum_{i=1}^m q_i}{1 - p} > 0, \ \forall n.$$

By Remark 3.3, equation (24) is globally exponentially stable. □

As an example, let us consider a linear non-autonomous difference equation in Banach space $\mathbb{X}$

$$x_{n+1} = \sum_{i=0}^k A_i(n) x_{n-i}, \ n \geq 0, \tag{26}$$

where $x_n \in \mathbb{X}$ and $A_i(n) \in L(\mathbb{X})$ are linear continuous operators for all $n \geq 0$. The following result generalizes Corollary 2.6 in [17].

Corollary 3.7. *Assume that*

$$\limsup_{n \to \infty} \sum_{i=0}^k \|A_i(n)\| = \alpha < 1. \tag{27}$$

Then, equation (26) *is globally exponentially stable.*

Proof. Denote $F(n, x_1, \dots, x_k) = \sum_{i=1}^k A_i(n) x_i$, then

$$\|F(n, x_1, \dots, x_k)\| \leq \sum_{i=1}^k \|A_i(n)\| \|x_i\| = \sum_{i=1}^k \lambda_i(n) \|x_i\|,$$

where $\lambda_i(n) = \|A_i(n)\|$. Let $\epsilon = \frac{1-\alpha}{2}$, $a(n) = \frac{1 - \|A_0(n)\| - \epsilon}{\|A_0(n)\| + \epsilon}$ and $b(n) = \frac{1}{\|A_0(n)\| + \epsilon}$, then from (27) we have

$$\|x_{n+1}\| \leq \frac{1}{1 + a(n)} \|x_n\| + \frac{b(n)}{1 + a(n)} \sum_{i=1}^k \lambda_i(n) \|x_{n-i}\|. \tag{28}$$

It is easy to verify that

$$\liminf_{n\to\infty}\Big\{a(n)-|b(n)|\sum_{i=1}^{k}\lambda_i(n)\Big\}=\liminf_{n\to\infty}\frac{1-\epsilon-\sum_{i=0}^{k}\|A_i(n)\|}{\|A_0(n)\|+\epsilon}$$
$$\geq\frac{1-\alpha}{1+\alpha}=\sigma>0.$$

Thus, by Theorem 3.2, equation (26) is globally exponentially stable. □

3.2. Local exponential stability

As discussed in [17], in many important practical models, function F does not satisfy the sublinear condition (H2). In this section, we introduce the following assumption including the case of superlinear functions.

(H3) There exist positive integers p, q, nonnegative bounded functions $\lambda_i(n)$, $\mu_k(n)$ and nonnegative scalars α_{ij}, β_{kj} such that $\sum_{j=1}^{m}\alpha_{ij}=1$, $i\in\underline{p}$, $\sum_{j=1}^{m}\beta_{kj}>1$, $k\in\underline{q}$, and the following condition holds

$$\|F(n,x_1,x_2,\dots,x_m)\|\leq\sum_{i=1}^{p}\lambda_i(n)\prod_{j=1}^{m}\|x_j\|^{\alpha_{ij}}+\sum_{k=1}^{q}\mu_k(n)\prod_{j=1}^{m}\|x_j\|^{\beta_{kj}} \tag{29}$$

for all $n\in\mathbb{Z}^+$, $(x_1,x_2,\dots,x_m)\in\mathbb{X}^m$.

Remark 3.8. Assumption (H3) is obviously weaker than (H2). More precisely, if nonlinear function F satisfies (H2), then F also satisfies (H3). In the following theorem we will show that under weaker assumption (H3), condition (9) ensures local exponential stability of (6) but does not guarantee global exponential stability of (6).

Theorem 3.9. *Under assumptions* (H1) *and* (H3)*, equation* (6) *is locally exponentially stable provided that condition* (9) *holds.*

Proof. Let condition (9) hold. Then, estimates (11) and (12) are still valid. Similar to (13), we have

$$\|x_{n+1}\|\leq\frac{1}{1+a^+}\Big\{\|x_n\|+b^+\Big(\sum_{i=1}^{p}\lambda_i^+\prod_{j=1}^{m}\|x_{n-r_j}\|^{\alpha_{ij}}+\sum_{k=1}^{q}\mu_k^+\prod_{j=1}^{m}\|x_{n-r_j}\|^{\beta_{kj}}\Big)\Big\}. \tag{30}$$

For a fixed $\delta_0>0$ and $\|\phi\|<\delta_0$, by induction, it can be found from (30) that $\|x_n\|\leq\delta_n$, $n\in\mathbb{Z}[0,N_0]$, where (δ_n), $n\in\mathbb{Z}[0,N_0]$, is a sequence of positive numbers defined by

$$\delta_{n+1}=\frac{1}{1+a^+}\Big(1+b^+\lambda^*+b^+\sum_{k=1}^{q}\mu_k^+\delta_n^{-1+\sum_{j=1}^{m}\beta_{kj}}\Big)\delta_n. \tag{31}$$

Since $-1+\sum_{j=1}^{m}\beta_{kj}>0$ for all $k\in\underline{q}$, it can be seen from (31) that $\delta\triangleq\sup_{n\in\mathbb{Z}[0,N_0]}\|x_n\|\leq 1$ for sufficiently small δ_0.

For $n = N_0$, we have

$$\begin{aligned}\|x_{n+1}\| &\le \frac{1}{1+a(n)}\|x_n\| + \frac{b(n)}{1+a(n)}\Big(\sum_{i=1}^p \lambda_i(n)\prod_{j=1}^m \|x_{n-r_j}\|^{\alpha_{ij}} \\ &\qquad + \sum_{k=1}^q \mu_i(n)\prod_{j=1}^m \|x_{n-r_j}\|^{\beta_{ij}}\Big) \\ &\le \frac{1}{1+a(n)}\Big[1 + b(n)\Big(\sum_{i=1}^p \lambda_i(n) + \sum_{k=1}^q \mu_k(n)\delta^{-1+\sum_{j=1}^m \beta_{kj}}\Big)\Big]\delta \\ &\le \Big(1 - \frac{\sigma}{2(1+a^+)} + \frac{b^+}{1+a_+}\sum_{k=1}^q \mu_k^+\delta^{-1+\sum_{j=1}^m \beta_{kj}}\Big)\delta.\end{aligned}$$

Since $0 < 1 - \frac{\sigma}{2(1+a^+)} < 1$ and $-1 + \sum_{j=1}^m \beta_{kj} > 0$ for all $k \in \underline{q}$, there exists a $\hat{\delta} \in (0,1]$ such that if $\delta \le \hat{\delta}$, then

$$\rho = 1 - \frac{\sigma}{2(1+a^+)} + \frac{b^+}{1+a_+}\sum_{k=1}^q \mu_k^+\delta^{-1+\sum_{j=1}^m \beta_{kj}} \in (0,1).$$

Therefore,

$$\sup_{n\ge N_0} \frac{1}{1+a(n)}\Big\{1 + b(n)\Big(\sum_{i=1}^p \lambda_i(n) + \sum_{k=1}^q \mu_i(n)\delta^{-1+\sum_{j=1}^m \beta_{kj}}\Big)\Big\} \le \rho. \tag{32}$$

Moreover, by the induction method, we also have

$$\|x_n\| \le \delta,\ \forall n \ge N_0. \tag{33}$$

Now, we define a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as follow

(34)

$$\begin{aligned}\varphi(\eta) = \sup_{n\ge N_0}\Big(&\frac{\eta}{1+a(n)} + \frac{b(n)}{1+a(n)}\sum_{i=1}^p \lambda_i(n)\eta^{1+\sum_{j=1}^m \alpha_{ij}r_j} \\ &+ \frac{b(n)}{1+a(n)}\sum_{k=1}^q \mu_i(n)\delta^{-1+\sum_{j=1}^m \beta_{kj}}\eta^{1+\sum_{j=1}^m \beta_{kj}r_j}\Big).\end{aligned}$$

It is obvious that $\varphi(\cdot)$ is a continuous function on $\mathbb{R}^+$, $\varphi(1) \le \rho < \frac{1+\rho}{2} < 1$ and $\varphi(\eta) \to \infty$ as η tends to infinity. Hence, there exists an $\eta_0 > 1$ such that $\varphi(\eta_0) < \hat{\rho} = \frac{1+\rho}{2}$.

By (33) we have $\|x_{N_0}\| \le \delta$ and thus $\eta_0\|x_{N_0+1}\| \le \varphi(\eta_0)\delta \le \hat{\rho}\delta < \delta$. By induction, once again, we readily obtain $\eta_0^n\|x_{N_0+n}\| < \delta$, $\forall n \ge 0$, which yields

$$\|x_{N_0+n}\| < \delta\eta_0^{-n},\ \forall n \ge 0. \tag{35}$$

This shows that the zero solution of (6) is locally exponentially stable. The proof is completed. $\square$

Remark 3.10. The results of Theorem 3.2 and Theorem 3.9 can be easily extended to the case of time-varying delays $r_j = r_j(k)$ satisfying $r_j(k) \leq r_j^+$, $j \in \underline{m}$. The proof is then straightforward from the proof of Theorems 3.2 and 3.9 and thus let us omit it here.

4. Illustrative examples

In this section, three numerical examples are given to illustrate the effectiveness of the obtained results presented in the preceding section.

Example 4.1. Let $\mathbb{X}$ be a finite-dimensional Banach space with basic $\{e_k\}_{k=1}^N$. Consider the following difference equation

(36)
$$x_{n+1} = \frac{1}{1+a(n)}x_n + \frac{b(n)}{1+a(n)}\Big(x_{n-1} + \lambda\|x_{n-1}\|^{\frac{1}{3}}\|x_{n-2}\|^{\frac{2}{3}}e_1 + \mu\|x_{n-1}\|x_{n-3}\Big),$$

where $a(n) = 2 + \frac{1}{2^n}$, $b(n) = 1 + \frac{1}{3^{n-1}}$ and λ, μ are real parameters.

We have $a^+ = 3, a_+ = 2$, $b^+ = 4, b_+ = 1$ and

$$a(n) - b(n) = 1 + \frac{1}{2^n} - \frac{1}{3^{n-1}} \leq 2 - \frac{1}{3^{n-1}}.$$

(a) For $\lambda = \mu = 0$, note that $\inf_{n\geq 0}\{a(n) - b(n)\} \leq -1$, the proposed conditions in [17, 19, 22, 26] are not satisfied. However, it can be seen that $\liminf_{n\to\infty}\{a(n) - b(n)\} = 1$. By Theorem 3.2 in this paper, equation (36) with $\lambda = \mu = 0$ is globally exponentially stable.

(b) Let $\mu = 0$ then $\liminf_{n\to\infty}\{a(n) - b(n)\sum_{i=1}^p \lambda_i(n)\} = 1 - |\lambda|$. In this case, Theorem 3.2 ensures the global exponential stability of equation (36) for $|\lambda| < 1$.

(c) For $\mu \neq 0$ and $|\lambda| < 1$, by Theorem 3.9, equation (36) is locally exponentially stable. However, in this case, (36) is not globally exponentially stable. For instance, let $\mathbb{X} = \mathbb{R}^2$, $\lambda = 0$, $\mu = 1$ and initial sequence $\phi(n) = [\delta \quad 0]^T \in \mathbb{R}^2, n \in \mathbb{Z}[-3, 0]$, where $\delta > 1$ is a constant. It is found from (36) that the corresponding solution of (36) satisfies $\|x(n)\| > 1$ for all $n \geq 0$. This shows that (36) is not globally exponentially stable.

Example 4.2. Let $\mathbb{X} = l^2$ be the Hilbert space of real sequences endowed with Cauchy product $(\cdot,\cdot)_{l^2}$ and induced norm $\|\cdot\|_{l^2}$ defined as

$$\|u\|_{l^2} = \Big(\sum_{j=1}^{\infty}|u_j|^2\Big)^{\frac{1}{2}}, \quad u = (u_j)_{j=1}^{\infty} \in l^2.$$

Consider the following equation in l^2

$$x_{k+1,j} = \frac{1}{2-e^{-k}}x_{k,j} + \frac{q}{4^{j-1}}x_{k-1,j}^{\alpha}x_{k-2,j}^{\beta},\ j \geq 1, \tag{37}$$

where $q \in \mathbb{R}$, $\alpha > 0, \beta > 0$ are real parameters.

Note that equation (37) can be written in the form of (6) with $a(k) = 1-e^{-k}$ and $b(k) = q(2 - e^{-k})$. The nonlinear function in the right-hand side of (37) satisfies

$$\|F(k,u,v)\|_{l^2} \le \Big(\sum_{j=1}^{\infty} 4^{1-j}\Big)^{\frac{1}{2}} \|u\|_{l^2}^{\alpha}\|v\|_{l^2}^{\beta} = \frac{2}{\sqrt{3}}\|u\|_{l^2}^{\alpha}\|v\|_{l^2}^{\beta},\ \forall u,v \in l^2. \tag{38}$$

Therefore,

(a) when $\alpha + \beta \le 1$, condition (10) is satisfied if $|q| < \frac{\sqrt{3}}{4}$. By Theorem 3.2, equation (37) is globally exponentially stable.
(b) when $\alpha+\beta > 1$, by Theorem 3.9, equation (37) is locally exponentially stable for all $q \in \mathbb{R}$.

Example 4.3. Consider the following non-autonomous equation with delays

$$x'(t) = -\alpha(t)x(t) + \beta(t)x(t-\tau(t))e^{-\gamma(t)x(t-\tau(t))}, \tag{39}$$

where α, β, γ and τ are continuous functions on $\mathbb{R}^+$ satisfying $\alpha(t) > 0, \gamma(t) > 0$ and $\beta(t) \ge 0$ for all $t \in \mathbb{R}^+$. Equation (39) represents a well-known biological model, namely Nicholson's blowflies model (see, [13] and the references therein). It was proved in [13] that for any initial function $\varphi \in C([-\tau^+, 0], \mathbb{R}^+)$, there exists a unique nonnegative solution $x(t,\varphi)$ of (39) on $[0,\infty)$. Assume that there are positive numbers α_+, η such that

$$\alpha(t) \ge \alpha_+, \quad 0 \le \frac{\beta(t)}{\gamma(t)} \le \eta,\ \forall t \ge 0.$$

Now, let $h > 0$ be a given step of discretization such that $r = \frac{\tau^+}{h} \in \mathbb{Z}^+$. Then (39) is discretized as follows

$$x_{n+1} = \frac{1}{1+h\alpha(t_{n+1})}x_n + \frac{h\beta(t_n)}{1+h\alpha(t_{n+1})}x_{n-r}e^{-\gamma(t_n)x_{n-r}}. \tag{40}$$

In this case we have $|F(n, x_{n-r})| = x_{n-r}e^{-\gamma(t_n)x_{n-r}} \le \frac{1}{\gamma(t_n)e}$. Therefore, (40) satisfies (9) with $\lambda_1(n) = \frac{1}{\gamma(t_n)e}, \lambda_2(n) = 0$ and hence

$$a(n) - b(n)\sum_{i=1}^{2}\lambda_i(n) = h\Big(\alpha(t_{n+1}) - \frac{\beta(t_n)}{\gamma(t_n)e}\Big) \ge h(\alpha_+ - \eta/e).$$

If $\alpha_+ > \frac{\eta}{e}$, then $\sigma = h\Big(\alpha_+ - \frac{\eta}{e}\Big) > 0$ and condition (10) is satisfied. By Theorem 3.2, equation (40) is globally exponentially stable. For illustrative purpose, let $\alpha(t) = 1.5 + |\sin(t)|$, $\beta(t) = 3|\cos(2t)|$ and $\gamma(t) = 1 + |\cos(2t)|$. We have $\alpha_+ = \inf_{t\ge0}\alpha(t) = 1.5$, $\alpha^+ = \sup_{t\ge0}\alpha(t) = 2.5$, $\beta^+ = \sup_{t\ge0}\beta(t) = 3$ and $\eta = \sup_{t\ge0}\frac{\beta(t)}{\gamma(t)} = 3/2$. Thus, $\alpha_+ - \frac{\eta}{e} = 3/2(1-1/e) > 0$ and equation (41) is globally exponentially stable. This means that the discretization of (39) is exponentially convergent for any bounded delay $\tau(t)$. It is worth noting that, for this example, stability conditions proposed in [8] (Theorem 5) ensure that all solutions of (39) converge to zero provided that

$$\frac{\alpha_+}{\beta^+} e^{-\alpha^+\tau^+} > \ln\left(\frac{\beta^{+2} + \alpha_+\beta^+}{\beta^{+2} + \alpha_+^2}\right) \tag{41}$$

which yields $\tau^+ < 0.2042$. This shows the effectiveness of our derived conditions.

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