Commun. Korean Math. Soc. **32** (2017), No. 4, pp. 827–833
https://doi.org/10.4134/CKMS.c170019
pISSN: 1225-1763 / eISSN: 2234-3024

# ON A LIE RING OF GENERALIZED INNER DERIVATIONS

Neşet Aydin and Selin Türkmen

Abstract. In this paper, we define a set including of all $f_a$ with $a \in R$ generalized derivations of $R$ and is denoted by $f_R$. It is proved that (i) the mapping $g : L(R) \to f_R$ given by $g(a) = f_{-a}$ for all $a \in R$ is a Lie epimorphism with kernel $N_{\sigma,\tau}$; (ii) if $R$ is a semiprime ring and $\sigma$ is an epimorphism of $R$, the mapping $h : f_R \to I(R)$ given by $h(f_a) = i_{\sigma(-a)}$ is a Lie epimorphism with kernel $l(f_R)$; (iii) if $f_R$ is a prime Lie ring and $A, B$ are Lie ideals of $R$, then $[f_A, f_B] = (0)$ implies that either $f_A = (0)$ or $f_B = (0)$.

## 1. Introduction

Let $R$ be an associative ring with center $Z(R)$ and $\sigma, \tau : R \to R$ be two mappings. $R$ is said to be 2-*torsion free* if $2x = 0$ with $x \in R$, then $x = 0$. A ring $R$ is called a *semiprime ring* if $a \in R$ and $aRa = (0)$ implies that $a = 0$. $R$ is called a *prime ring* if $a, b \in R$ and $aRb = (0)$ implies that $a = 0$ or $b = 0$. For $x, y \in R$, $xy - yx$ is denoted by $[x, y]$ and $x\sigma(y) - \tau(y)x$ is denoted by $[x, y]_{\sigma,\tau}$. An additive subgroup $U$ of $R$ is said to be a *Lie ideal* of $R$ if $[u, r] \in U$ for all $u \in U$ and for all $r \in R$. A Lie ideal $P$ of a Lie ring $L$ is a *prime Lie ideal* if $A, B$ are two Lie ideals of $L$ such that $[A, B] \subset P$ implies that either $A \subset P$ or $B \subset P$. Similarly, a Lie ideal $P$ of $L$ is a *semiprime Lie ideal* if $A$ is a Lie ideal of $L$ such that $[A, A] \subset P$ implies that $A \subset P$ (see [3]). $L$ is a *prime Lie ring* if $[A, B] \neq (0)$ for any two nonzero Lie ideals $A, B$ of $L$. Similarly, $L$ is a *semiprime Lie ring* if $[A, A] \neq (0)$ for any nonzero Lie ideal $A$ of $L$ (see [3]). An additive mapping $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For a fixed $a \in R$, the mapping $i_a : R \to R$ given by $i_a(x) = [a, x]$ is called an *inner derivation* determinated by $a$. In [5], the set of all inner derivations $i_a$ of $R$ is denoted by $I(R)$ which is a Lie ring with the product $[i_a, i_b] = i_{[a,b]}$ for all $i_a, i_b \in I(R)$. In [2], a generalized derivation in rings is defined as follow: An additive mapping $f : R \to R$ is called a *generalized derivation* if there exists

Received January 19, 2017; Revised May 17, 2017; Accepted July 18, 2017.
2010 *Mathematics Subject Classification.* Primary 16N60; Secondary 16U80, 16W25.
*Key words and phrases.* semiprime ring, semiprime Lie ring, prime Lie ring, generalized derivation.
The material in this work is a part of second author's Ph. Dissertation under the supervision of Professor Neşet Aydin.

©2017 Korean Mathematical Society

a derivation $d$ of $R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. In [1], the set of all generalized derivations of $R$ is denoted by $Gder(R)$ which is a Lie ring with the product $[f_1, f_2] = f_1 f_2 - f_2 f_1$ for all $f_1,\ f_2 \in Gder(R)$. For fixed $a, b \in R$, mappings of the form $f(x) = ax + xb$ are called *a generalized inner derivation.* For $a \in R,\ f_a : R \to R$ defined as $f_a(x) = [x, a]_{\sigma,\tau}$ is a generalized inner derivation such that $f_a(x) = (-\tau(a))x + x\sigma(a)$.

By $L(R)$, it is denoted the associated Lie ring of $R$ and the product in $L(R)$ is given by $[r, s]$ for all $r, s \in R$. In [4], N. Jacobson proved that the mapping $\theta : L(R) \to I(R)$ given by $\theta(r) = i_r$ for all $r \in R$ is a Lie epimorphism with kernel $Z(R)$. In [5], C. R. Jordan and D. A. Jordan proved by using the isomorphism in [4] if $R$ is a semiprime (resp. prime) 2-torsion free ring, then $I(R)$ is a semiprime (resp. prime) Lie ring.

In this paper, we define a Lie subring of $Gder(R)$ including of all generalized derivations $f_a$ with $a \in R$ and denoted by $f_R$. The isomorphism in [4] is generalized and some properties of $f_R$ are investigated.

Throughout the present paper, $R$ is a ring, $Z(R)$ is the center of $R$, $L(R)$ is the Lie ring of $R$ and $\sigma, \tau$ are homomorphisms of $R$. We use the basic commutator identities:

- $[x, y] = -[y, x] = [-y, x] = [y, -x] = [-x, -y]$
- $[xy, z]_{\sigma,\tau} = [x, z]_{\sigma,\tau}\, y + x\,[y, \sigma(z)]$
- $\left[[x, y]_{\sigma,\tau}, z\right]_{\sigma,\tau} - \left[[x, z]_{\sigma,\tau}, y\right]_{\sigma,\tau} = [x, [y, z]]_{\sigma,\tau}$

## 2. Results

**Lemma 2.1** ([4, Theorem 3])**.** *The mapping $\theta : L(R) \to I(R)$ given by $\theta(r) = i_r$ for all $r \in R$ is a Lie epimorphism with kernel $Z$. Consequently $Z$ is an ideal of $L(R)$ and $L(R)/Z \cong I(R)$.*

**Lemma 2.2** ([5, Lemma 4])**.** *Let $R$ be a prime ring of characteristic not 2. Then $Z$ is a prime ideal of $L(R)$. Equivalently, $I(R)$ is a prime Lie ring.*

**Lemma 2.3** ([5, Lemma 6])**.** *Let $R$ be a semiprime 2-torsion free ring. Then $Z$ is a semiprime ideal of $L(R)$. Equivalently, $I(R)$ is a semiprime Lie ring.*

**Lemma 2.4.** *For a fixed $a \in R,\ f_a : R \to R$ such that $f_a(x) = [x, a]_{\sigma,\tau}$ is a generalized derivation associated with inner derivation $i_{\sigma(-a)}$ of $R$.*

*Proof.* For all $x, y \in R$, it holds

$$f_a(x + y) = [x + y, a]_{\sigma,\tau} = [x, a]_{\sigma,\tau} + [y, a]_{\sigma,\tau} = f_a(x) + f_a(y)$$

and

$$\begin{aligned} f_a(xy) = [xy, a]_{\sigma,\tau} &= [x, a]_{\sigma,\tau}\, y + x\,[y, \sigma(a)] \\ &= [x, a]_{\sigma,\tau}\, y + x\,[\sigma(-a), y] \\ &= f_a(x)\, y + x i_{\sigma(-a)}(y). \end{aligned}$$

So, $f_a$ is a generalized derivation associated with inner derivation $i_{\sigma(-a)}$ of $R$. □

**Example 2.5.** Let $R = \{\left[\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right] \mid x, y, z \in \mathbb{Z}\}$ be a ring, $\sigma, \tau : R \to R$ defined as $\sigma\left(\left[\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\tau\left(\left[\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & z \end{smallmatrix}\right]$ be endomorphisms. For a fixed $a = \left[\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right] \in R$, $i_{\sigma(-a)} : R \to R$ defined by $i_{\sigma(-a)}\left(\left[\begin{smallmatrix} b & c \\ 0 & d \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 & -cx \\ 0 & 0 \end{smallmatrix}\right]$ is an inner derivation determinated by $\sigma(-a)$ and $f_a : R \to R$ defined by $f_a\left(\left[\begin{smallmatrix} b & c \\ 0 & d \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} bx & 0 \\ 0 & -dz \end{smallmatrix}\right]$ is a generalized derivation associated with inner derivation $i_{\sigma(-a)}$.

**Lemma 2.6.** *$N_{\sigma,\tau} = \left\{a \in R \mid [r, a]_{\sigma,\tau} = 0 \text{ for all } r \in R\right\}$ is a subring as well as a Lie ideal of $R$. Moreover, if $R$ is a semiprime ring, then*

$$N_{\sigma,\tau} = \{a \in R \mid \sigma(a) = \tau(a) \in Z(R)\}.$$

*Proof.* Let $a, b \in N_{\sigma,\tau}$. For all $r \in R$

$$[r, a - b]_{\sigma,\tau} = [r, a]_{\sigma,\tau} - [r, b]_{\sigma,\tau} = 0$$

and

$$[r, ab]_{\sigma,\tau} = [r, a]_{\sigma,\tau}\,\sigma(b) + \tau(a)\,[r, b]_{\sigma,\tau} = 0$$

are obtained. It yields that $a - b, ab \in N_{\sigma,\tau}$ for all $a, b \in N_{\sigma,\tau}$. Thus $N_{\sigma,\tau}$ is a subring of $R$.

Since $a \in N_{\sigma,\tau}$, it holds that for all $r, s \in R$

$$[r, [s, a]]_{\sigma,\tau} = \Big[[r, s]_{\sigma,\tau}, a\Big]_{\sigma,\tau} - \Big[[r, a]_{\sigma,\tau}, s\Big]_{\sigma,\tau} = 0.$$

This means $[s, a] \in N_{\sigma,\tau}$ for all $s \in R$ and $a \in N_{\sigma,\tau}$. So, $N_{\sigma,\tau}$ is a Lie ideal of $R$.

Assume that $R$ is a semiprime ring. From the definition of $N_{\sigma,\tau}$, it holds $[rs, a]_{\sigma,\tau} = 0$ for all $r, s \in R$ and $a \in N_{\sigma,\tau}$. It implies that

$$0 = [rs, a]_{\sigma,\tau} = [r, a]_{\sigma,\tau}\, s + r\,[s, \sigma(a)] = r\,[s, \sigma(a)]$$

for all $r, s \in R$, $a \in N_{\sigma,\tau}$. Thus it yields

$$R\,[R, \sigma(a)] = (0)$$

for all $a \in N_{\sigma,\tau}$. The semiprimeness of $R$ gives $\sigma(a) \in Z(R)$ for all $a \in N_{\sigma,\tau}$. Using $\sigma(a) \in Z(R)$ and $[r, a]_{\sigma,\tau} = 0$ together, it holds that

$$(\sigma(a) - \tau(a))\,R = (0)$$

for all $a \in N_{\sigma,\tau}$. Since $R$ is a semiprime ring, for all $a \in N_{\sigma,\tau}$

$$\sigma(a) = \tau(a) \in Z(R)$$

is obtained. □

For a fixed $a \in R$, it holds that

$$f_a(x) = [x, a]_{\sigma,\tau} = x\sigma(a) - \tau(a)\,x = x\sigma(a) + (-\tau(a))\,x$$

for all $x \in R$. So, $f_a$ is a generalized inner derivation of $R$ and $f_a \in Gder(R)$. We denote $f_R$, the set of all $f_a$ with $a \in R$ generalized derivations of $R$, i.e.,

$f_R = \left\{ f_a : R \to R \mid f_a(x) = [x,a]_{\sigma,\tau}, a \in R \right\}$. For all $f_a, f_b \in f_R$ and $r \in R$, it holds

$$\begin{aligned}(f_a - f_b)(r) &= f_a(r) - f_b(r) = [r,a]_{\sigma,\tau} - [r,b]_{\sigma,\tau} \\ &= r\sigma(a) - \tau(a) r - (r\sigma(b) - \tau(b) r) \\ &= r\sigma(a-b) - \tau(a-b) r \\ &= [r, a-b]_{\sigma,\tau} = f_{a-b}(r).\end{aligned}$$

Thus

$$f_a - f_b = f_{a-b}, \ \forall f_a, f_b \in f_R \tag{1}$$

is obtained. Since $\left[[x,y]_{\sigma,\tau}, z\right]_{\sigma,\tau} - \left[[x,z]_{\sigma,\tau}, y\right]_{\sigma,\tau} = [x,[y,z]]_{\sigma,\tau}$ for all $x,y,z \in R$, it holds for all $r \in R$

$$\begin{aligned}([f_a, f_b])(r) &= (f_a f_b - f_b f_a)(r) = f_a(f_b(r)) - f_b(f_a(r)) \\ &= f_a\left([r,b]_{\sigma,\tau}\right) - f_b\left([r,a]_{\sigma,\tau}\right) \\ &= \left[[r,b]_{\sigma,\tau}, a\right]_{\sigma,\tau} - \left[[r,a]_{\sigma,\tau}, b\right]_{\sigma,\tau} \\ &= [r,[b,a]]_{\sigma,\tau} = \left(f_{[b,a]}\right)(r).\end{aligned}$$

So, it implies

$$[f_a, f_b] = f_{[b,a]}, \ \forall f_a, f_b \in f_R. \tag{2}$$

**Lemma 2.7.** $f_R = \left\{ f_a : R \to R \mid f_a(x) = [x,a]_{\sigma,\tau}, a \in R \right\}$ *is a Lie subring of* $Gder(R)$.

*Proof.* $f_R$ is obviously a Lie subring of $Gder(R)$ from the (1) and (2). □

**Lemma 2.8.** *For a fixed* $a \in R$, *the followings hold*:

i) $f_{-a} = -f_a$.
ii) $f_a = i_{\sigma(-a)} + l_{(\sigma-\tau)(a)}$, *where* $u \in R$, $l_u : R \to R$ *such that* $l_u(x) = ux$.
iii) $f_a = i_{\tau(-a)} + r_{(\sigma-\tau)(a)}$, *where* $u \in R$, $r_u : R \to R$ *such that* $r_u(x) = xu$.
iv) *If* $\sigma(a) = \tau(a) \in Z(R)$, *then* $f_a = 0$. *Conversely, if* $f_a = 0$ *and* $R$ *is a semiprime ring, then* $\sigma(a) = \tau(a) \in Z(R)$.
v) *If* $A$ *is a Lie ideal of* $R$, *then* $f_A = \{f_a \in f_R \mid a \in A\}$ *is a Lie subring as well as a Lie ideal of* $f_R$.

*Proof.* (i) For an arbitrary $x \in R$,

$$\begin{aligned}f_{-a}(x) &= x\sigma(-a) - \tau(-a)x = -x\sigma(a) - (-\tau(a))x \\ &= -(x\sigma(a) - \tau(a)x) = -[x,a]_{\sigma,\tau} = -(f_a(x)) = (-f_a)(x).\end{aligned}$$

Thus $f_{-a} = -f_a$.

(ii) For any $x \in R$,

$$f_a(x) = x\sigma(a) - \tau(a)x = x\sigma(a) - \tau(a)x \mp \sigma(a)x$$

$$= [x, \sigma(a)] + ((\sigma - \tau)(a))\, x = [\sigma(-a), x] + ((\sigma - \tau)(a))\, x.$$

Let $l_u : R \to R$ such that $l_u(x) = ux$ where $u \in R$. For all $x \in R$

$$f_a(x) = i_{\sigma(-a)}(x) + l_{(\sigma-\tau)(a)}(x) = \left(i_{\sigma(-a)} + l_{(\sigma-\tau)(a)}\right)(x).$$

Thus $f_a = i_{\sigma(-a)} + l_{(\sigma-\tau)(a)}$.

(iii) We can prove by the similar way as in the proof of (ii).

(iv) Since $\sigma(a) = \tau(a) \in Z(R)$, it holds that $f_a(x) = x\sigma(a) - \tau(a)\, x = x(\sigma(a) - \tau(a)) = 0$ for all $x \in R$. Thus $f_a = 0$.

Conversely, if $f_a = 0$, then $a \in N_{\sigma,\tau}$. Since $R$ is a semiprime ring and $a \in N_{\sigma,\tau}$, it follows that $\sigma(a) = \tau(a) \in Z(R)$ from Lemma 2.6.

(v) Since $A$ is a Lie ideal of $R$, it is obviously that $f_a - f_b = f_{a-b} \in f_A$ and $[f_a, f_b] = f_{[b,a]} \in f_A$ for all $f_a, f_b \in f_A$. So, $f_A$ is a Lie subring of $f_R$. Let $f_a \in f_A$ with $a \in A$ and $f_r \in f_R$ with $r \in R$. Since $A$ is a Lie ideal of $R$, $[r, a] \in A$ with $a \in A, r \in R$ which implies that $[f_a, f_r] = f_{[r,a]} \in f_A$. Thus $f_A$ is a Lie ideal of $f_R$. □

**Lemma 2.9.** $l(f_R) = \left\{f_a \in f_R \mid i_{\sigma(a)} = 0\right\}$ *is a subring as well as a Lie ideal of* $f_R$. *Moreover,*

$$l(f_R) = \{f_a \in f_R \,|\, f_a(xy) = f_a(x)\, y \text{ for all } x, y \in R\}.$$

*Proof.* Let $f_a, f_b \in l(f_R)$. Since $i_{\sigma(a)} = 0$ and $i_{\sigma(b)} = 0$, it holds

$$i_{\sigma(a-b)} = i_{\sigma(a)} - i_{\sigma(b)} = 0$$

and

$$i_{\sigma([b,a])} = \left[i_{\sigma(b)}, i_{\sigma(a)}\right] = 0.$$

This means $f_{a-b}, f_{[b,a]} \in l(f_R)$. Thus $l(f_R)$ is a Lie subring of $f_R$. Assume that $f_a \in l(f_R)$ and $f_r \in f_R$.

$$i_{\sigma([r,a])} = \left[i_{\sigma(r)}, i_{\sigma(a)}\right] = \left[i_{\sigma(r)}, 0\right] = 0$$

which implies $[f_a, f_r] = f_{[r,a]} \in l(f_R)$. So, $l(f_R)$ is a Lie ideal of $f_R$.

Moreover, since $i_{\sigma(a)} = 0$

$$f_a(xy) = f_a(x)\, y + x i_{\sigma(-a)}(y) = f_a(x)\, y + x\left(-i_{\sigma(a)}\right)(y) = f_a(x)\, y$$

is obtained for all $x, y \in R$. □

The following theorem is a generalization of Lemma 2.1.

**Theorem 2.10.** *The mapping* $g : L(R) \to f_R$ *given by* $g(a) = f_{-a}$ *for all* $a \in R$ *is a Lie epimorphism with kernel* $N_{\sigma,\tau}$. *Thus* $L(R)/N_{\sigma,\tau} \cong f_R$.

*Proof.* Mapping $g$ holds that for all $a, b \in R$

$$g(a + b) = f_{-(a+b)} = f_{-a} + f_{-b} = g(a) + g(b)$$

and

$$\begin{aligned} g([a, b]) &= f_{-([a,b])} = f_{[b,a]} = [f_a, f_b] = [-f_a, -f_b] \\ &= [f_{-a}, f_{-b}] = [g(a), g(b)]. \end{aligned}$$

So, $g$ is a Lie homomorphism.

$$\begin{aligned}\ker g &= \{a \in R \mid g(a) = f_0\} = \{a \in R \mid f_{-a} = 0\} \\ &= \left\{a \in R \mid [x,a]_{\sigma,\tau} = 0 \text{ for all } x \in R\right\} \\ &= N_{\sigma,\tau}\end{aligned}$$

and it is clear that $g$ is surjective. Thus $L(R)/N_{\sigma,\tau} \cong f_R$. □

**Theorem 2.11.** *Let $R$ be a semiprime ring and $\sigma$ be an epimorphism of $R$. Then the followings hold*:

i) *The mapping $h : f_R \to I(R)$ given by $h(f_a) = i_{\sigma(-a)}$ for all $f_a \in f_R$ is a Lie epimorphism with kernel $l(f_R)$. Thus $f_R/l(f_R) \cong I(R)$.*

ii) *It holds $hg = \theta\sigma$.*

*Proof.* (i) Taking $f_a, f_b \in f_R$ such that $f_a = f_b$. It follows that $[x,a]_{\sigma,\tau} = [x,b]_{\sigma,\tau}$ for all $x \in R$. So, it holds that $a - b \in N_{\sigma,\tau}$. Since $R$ is a semiprime ring, $\sigma(a-b) \in Z(R)$ from Lemma 2.6. This means $[r, \sigma(a)] = [r, \sigma(b)]$ for all $r \in R$. It follows that $i_{\sigma(-a)} = i_{\sigma(-b)}$ which implies that $h(f_a) = h(f_b)$. Thus $h$ is well defined. For all $f_a, f_b \in f_R$

$$\begin{aligned}h(f_a + f_b) &= h(f_{a+b}) = i_{\sigma(-(a+b))} = i_{\sigma(-a)} + i_{\sigma(-b)} \\ &= h(f_a) + h(f_b)\end{aligned}$$

and

$$\begin{aligned}h([f_a, f_b]) &= h\left(f_{[b,a]}\right) = i_{\sigma(-[b,a])} = i_{\sigma([a,b])} = \left[i_{\sigma(a)}, i_{\sigma(b)}\right] \\ &= \left[i_{\sigma(-a)}, i_{\sigma(-b)}\right] = [h(f_a), h(f_b)]\end{aligned}$$

are obtained. So, $h$ is a Lie homomorphism. Let $i_x \in I(R)$ with $x \in R$. Since $\sigma$ is an epimorphism of $R$, there exists an element $y \in R$ which holds $x = \sigma(y)$. In this case, there exists an element $f_{-y} \in f_R$ which holds

$$i_x = i_{\sigma(y)} = i_{\sigma(-(-y))} = h(f_{-y}).$$

This means that $h$ is surjective. Moreover,

$$\ker h = \{f_a \in f_R \mid h(f_a) = 0\} = \left\{f_a \in f_R \mid i_{\sigma(a)} = 0\right\} = l(f_R).$$

Thus $f_R/l(f_R) \cong I(R)$.

(ii) By using the mappings in (i), Lemma 2.1 and Theorem 2.10

$$\begin{array}{ccc} L(R) & & \\ \downarrow_g & \searrow^{\theta\sigma} & \\ f_R & \underset{h}{\rightarrow} & I(R) \end{array}$$

is obtained. Thus $hg = \theta\sigma$. □

**Corollary 2.12.** *Let $R$ be a prime ring of characteristic not 2 and $\sigma$ be an epimorphism of $R$. Then $f_R/l(f_R)$ is a prime Lie ring. So, $l(f_R)$ is a prime Lie ideal of $f_R$.*

*Proof.* $I(R)$ is a prime Lie ring from Lemma 2.2 and it holds that $f_R/l\,(f_R) \cong I\,(R)$ from Theorem 2.11(i). Thus $f_R/l\,(f_R)$ is a prime Lie ring. It follows that $l\,(f_R)$ is a prime Lie ideal of $f_R$. □

**Corollary 2.13.** *Let $R$ be a semiprime 2-torsion free ring and $\sigma$ be an epimorphism of $R$. Then $f_R/l\,(f_R)$ is a semiprime Lie ring. So, $l\,(f_R)$ is a semiprime Lie ideal of $f_R$.*

*Proof.* We can prove by the similar way as in the proof of Corollary 2.12 by using Lemma 2.3 and Theorem 2.11(i). □

**Theorem 2.14.** *Let $A, B$ be Lie ideals of $R$. If $f_R$ is a prime Lie ring, then $[f_A, f_B] = (0)$ implies that either $f_A = (0)$ or $f_B = (0)$.*

*Proof.* Since $f_R$ is a prime Lie ring, $L\,(R)\,/N_{\sigma,\tau}$ is a prime Lie ring from Theorem 2.10. So, $N_{\sigma,\tau}$ is a prime Lie ideal of $L\,(R)$. Since $[f_A, f_B] = (0)$, it holds that $0 = [f_a, f_b] = f_{[b,a]}$ for all $f_a \in f_A$, for all $f_b \in f_B$. Thus $[b, a] \in N_{\sigma,\tau}$ for all $a \in A$, for all $b \in B$ which implies that $[A, B] \subset N_{\sigma,\tau}$. Since $N_{\sigma,\tau}$ is a prime Lie ideal of $L\,(R)$, it follows that either $A \subset N_{\sigma,\tau}$ or $B \subset N_{\sigma,\tau}$. It implies that either $f_A = (0)$ or $f_B = (0)$. □

**Corollary 2.15.** *Let $A$ be a Lie ideal of $R$. If $f_R$ is a semiprime Lie ring, then $[f_A, f_A] = (0)$ implies that $f_A = (0)$.*

**Acknowledgement.** The second author thanks to TÜBİTAK (The Scientific and Technical Research Council of Turkey) for financial support of BİDEB 2211 National Research Fellowship.

## References

[1] F. Ali and M. A. Chaudhry, *On a Lie ring of generalized derivations of non-commutative rings*, Int. J. Algebra **5** (2011), no. 8, 397–402.

[2] M. Bresar, *On the distance of the composition of two derivations to the generalized derivations*, Glasgow Math. J. **33** (1991), no. 1, 89–93.

[3] B. Brown and N. H. McCoy, *Prime ideals in nonassociative rings*, Trans. Amer. Math. Soc. **89** (1958), no. 1, 245–255.

[4] N. Jacobson, *Abstract derivation and Lie algebras*, Trans. Amer. Math. Soc. **42** (1937), no. 2, 206–224.

[5] C. R. Jordan and D. A. Jordan, *Lie rings of derivations of associative rings*, J. London Math. Soc. (2) **17** (1978), no. 1, 33–41.

Neşet Aydin
Department of Mathematics
Çanakkale Onsekiz Mart University
Çanakkale 17020, Turkey
*E-mail address*: neseta@comu.edu.tr

Selin Türkmen
Lapseki Vocational School
Çanakkale Onsekiz Mart University
Çanakkale 17800, Turkey
*E-mail address*: selinvurkac@gmail.com