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THE HULL NUMBER OF POWERS OF CYCLES

Hasan Al-Ezeh, Manal Ghanem, and Jameel Rwalah

Abstract. Let C_n be the cycle graph of order n on the vertices v_0, $v_1, \ldots, v_n$ and C_n^k be the k-th power of C_n. In this article we determine the hull-number of C_n^k.

1. Introduction

Given a finite simple connected graph G, let u and v be two vertices of G. The distance between u and v is the length of a shortest path between u and v, we denote it by $d_G(u, v)$. A shortest path between u and v is called a $u - v$ geodesic. The set of all vertices in G that lie on a $u - v$ geodesic is denoted by $I[u, v]$. The closed interval $I[u, v]$ consists of all vertices that lie on a uv geodesic of G. For $A \subseteq V(G)$, let the closed interval $I[A]$ be the union of all sets $I[u, v]$ for $u, v \in A$, then A is called a convex set if $I[A] = A$. The convex hull of A, denoted by $[A]$, is the smallest convex set containing A. If $[A] = V(G)$, then A is called a hull set of G. The cardinality of a minimum hull set of G is called the hull number of G, and it is denoted by $h(G)$. If $I[A] = V(G)$, then A is called a geodetic set of G. The minimum cardinality of a geodetic set in G is named the geodetic number of G and it is denoted by $g(G)$. Certainly, $h(G) \le g(G)$.

The process of rebuilding a network modelled by a connected graph is a discrete optimization problem, consisting in finding a subset of vertices of cardinality as small as possible, which would allow us to store and retrieve the whole graph. One way to approach this problem is by using a certain convex operator. This procedure has attracted much attention since it was shown in [9] that every convex subset in a graph is the convex hull of its extreme vertices if and only if the graph is chordal and contains no induced 3-fan. The hull number of a graph was introduced by [8]. They characterized graphs having some particular hull numbers and they obtained a number of bounds for the hull numbers of graphs. Dourado et al. [7] proved that the hull number of

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unit interval graphs, cographs and split graphs can be computed in polynomial time. The hull number of an oriented graph was studied in [5] and [6]. The hull number of the power of paths was determined in [1]. For more results on the subject, see [3], [4] and [5]. For any positive integer k and a connected graph G the k-th power graph G^k of G has $V(G^k) = V(G)$ and the distinct vertices x and y are adjacent in G^k if $d(x, y) \leq k$. Circulant graphs have been extensively studied and have a vast number of applications to multicomputer networks and distributed computation (see [2] and [10]). One type of the circulant graphs is the k-th power of the n-cycle C_n^k. Our aim in this paper is to find the hull number of the graph C_n^k.

2. The hull number of C_n^k

For positive integers n and k, we denote by C_n^k the graph with vertex set $\{\upsilon_0, \upsilon_1, \ldots, \upsilon_{n-1}\}$ and edge set $\{\upsilon_i\upsilon_j : i - j \equiv \pm m \bmod n, 1 \leq m \leq k\}$. The graph C_n^k is the k-th power of the n-cycle C_n.

In this section, we will determine the hull number of the k-th power of the n-cycle C_n^k. The hull number of a connected graph G of order n is n if and only if G is the complete graph of order n, [8]. It is clear that C_n^k is the complete graph of order n when $k \geq \lfloor \frac{n}{2} \rfloor$ and hence its hull number equal n. So next we will only consider C_n^k when $1 < k < \lfloor \frac{n}{2} \rfloor$.

Let G be a graph. Given a vertex v, denote by $N(v)$ the set of neighbors of v. And denote, the subgraph of G induced by the set $B, B \subseteq V(G)$ by $G[B]$. We say that v is a simplicial vertex of G if $N(v)$ induces a complete subgraph. It is clear that every hull set of a graph G contains the set of all simplicial vertices of G. In this section, we characterize the hull number of the graph C_n^k in the following sequences of lemmas.

First, we start by the following lemma that determines the geodetic number of the k-th power of a path with $qk + 2$ vertices.

Lemma 1 ([1]). *Let P_{n+1} be the path of order $n+1$ and P_{n+1}^k be the k-th power of P_{n+1}. Suppose that $n = qk + r$ where q is a positive integer and $0 \leq r < k$, then $g(P_{n+1}^k) = 2$ if and only if $n = qk + 1$.*

In the following lemma, we show that 3 is an upper bound for $h(C_n^k)$.

Lemma 2. *The hull number of any power of cycle graph is at most* 3.

Proof. Let C_n^k be the k-th power of the cycle graph with n vertices. Then there are two cases of n.

Case 1: n is even. Use division algorithm, to write $n = 2qk + 2r$, where q is a positive integer, $0 \leq 2r < 2k$, and $V(C_n^k) = \{\upsilon_0, \upsilon_1, \ldots, \upsilon_k, \ldots, \upsilon_{qk}, \ldots, \upsilon_{qk+r}, \ldots, \upsilon_{qk+2r}, \ldots, \upsilon_{(q+1)k+2r}, \ldots, \upsilon_{(2q-1)k+2r}, \ldots, \upsilon_{2qk+2r-1}\}$. Now, assume that the hull number of C_n^k is not equal 2 and let $A = \{\upsilon_0, \upsilon_1, \upsilon_{qk+r}\}$, we claim that A is a hull-set of C_n^k. To show this, first note that $\{\upsilon_0, \upsilon_{qk+r}\} \subseteq A$ gives $A_1 = \{\upsilon_0, \upsilon_k, \upsilon_{2k}, \ldots, \upsilon_{qk}, \upsilon_{qk+r}\}$ and $A_2 = \{\upsilon_{qk+r}, \upsilon_{qk+2r}, \upsilon_{(q+1)k+2r}, \ldots$

$v_{(2q-1)k+2r}, v_0\}$ are subsets of $[A]$, since their elements lie on geodesics between v_0 and v_{qk+r}. And observe that, $\{v_1, v_{qk}, v_{qk+2r}\} \subseteq [A]$ and the induced subgraph of C_n^k with the set of vertices $A_3 = \{v_1, v_0, v_{2qk+2r-1}, \ldots, v_{(q+1)k+2r}, \ldots, v_{qk+2r}\}$ is isomorphic to P_{m+1}^k, where $m = qk+1$. Use Lemma 1, to get $A_3 \subseteq [A]$. Therefore, $v_{2qk+2r-1} \in [A]$. Take $A_4 = \{v_{2qk+2r-1}, v_0, \ldots, v_k, \ldots, v_{qk}\}$, then $C_n^k[A_4] \cong P_{qk+2}^k$, and hence $A_4 \subseteq [A]$. Moreover,

$$A_5 = \{v_{(q+1)k+r+1}, v_{(q+1)k+r}, \ldots, v_{qk+2r}, \ldots, v_{qk+r}\} \subseteq [A] \text{ and}$$
$$A_6 = \{v_{qk+r}, v_{qk+r-1}, \ldots, v_{qk}, \ldots, v_{(q-1)k+r}, v_{(q-1)k+r-1}\} \subseteq [A],$$

since $\{v_{(q+1)k+r+1}, v_{qk+r}, v_{(q-1)k+r-1}\} \subseteq [A]$ and $C_n^k[A_i] \cong P_{k+2}^k$ for $i = 5, 6$. So, $V(C_n^k) = \bigcup_{i=1}^6 A_i = [A]$ and hence A is a hull set of C_n^k.

Case 2: n is odd. By division algorithm $n - 1 = 2qk + 2r$, where q is a positive integer, $0 \le 2r < 2k$ and $V(C_n^k) = \{v_0, v_1, \ldots, v_k, \ldots, v_{qk}, \ldots, v_{qk+r}, \ldots, v_{qk+2r+1}, \ldots, v_{(q+1)k+2r+1}, \ldots, v_{(2q-1)k+2r+1}, \ldots, v_{2qk+2r}\}$. If $r = 0$, then $A = \{v_0, v_{(q-1)k}, v_{qk+1}\}$ is a hull set of C_n^k. To show this set $A_1 = \{v_{(q-1)k}, v_{(q-1)k+1}, \ldots, v_{qk+1}\}$, $A_2 = \{v_0, v_1, \ldots, v_{(q-1)k+1}\}$, $A_3 = \{v_{qk+1}, v_{(q+1)k+1}, \ldots, v_{(2q-1)k+1}, v_0\}$, $A_4 = \{v_{qk}, v_{qk+1}, \ldots, v_{(q+1)k+1}\}$, $A_5 = \{v_{(q+1)k+1}, \ldots, v_{(2q-1)k+1}, \ldots, v_0, v_1\}$. By using Lemma 1 and noting that all vertices of A_3 lie on a $v_0 - v_{qk+1}$ geodesic, we can prove respectively that $A_i \subseteq [A]$ for all i. Hence, $V(C_n^k) = \cup_{i=1}^5 A_i = [A]$ and thus $h(C_n^k) = 3$. Now, suppose that $r \ne 0$. We claim that $A = \{v_0, v_1, v_{qk+r}\}$ is a hull set of C_n^k, to prove this claim we mimic the proof of Case 1. First, observe that, there are two paths between v_0 and v_{qk+r}, the first one is $v_0 - v_1 - \cdots - v_k - \cdots - v_{qk} - \cdots - v_{qk+r}$ and the second is $v_{qk+r} - \cdots - v_{qk+2r+1} - \cdots - v_{(q+1)k+2r+1} - \cdots - v_{(2q-1)k+2r+1} - \cdots - v_0$. Since the length of the first path is $qk + r$ and the length of the second path is $qk + r + 1$ and $r + 1 \le k$, we have $v_0 - v_k - \cdots - v_{qk} - v_{qk+r}$ and $v_{qk+r} - v_{qk+2r+1} - v_{(q+1)k+2r+1} - \cdots - v_{(2q-1)k+2r+1} - v_0$ are geodesics between v_0 and v_{qk+r}. Thus, $A_1 = \{v_0, v_k, \ldots, v_{qk}, v_{qk+r}, v_{qk+2r+1}, v_{(q+1)k+2r+1}, \ldots, v_{(2q-1)k+2r+1}\} \subseteq [A]$. Now, take

$$A_2 = \{v_{qk+2r+1}, v_{qk+2r+2}, \ldots, v_{(q+1)k+r+1}, \ldots, v_{2qk+2r}, v_0, v_1\},$$
$$A_3 = \{v_{2qk+2r}, v_0, \ldots, v_k, \ldots, v_{qk}\},$$
$$A_4 = \{v_{qk+r}, v_{qk+r+1}, \ldots v_{qk+2r}, \ldots, v_{(q+1)k+r}, v_{(q+1)k+r+1}\},$$
$$A_5 = \{v_{(q-1)k+r-1}, v_{(q-1)k+r}, \ldots v_{qk}, v_{qk+1}, \ldots, v_{qk+r}\}.$$

Then $C_n^k[A_i] \cong P_{qk+2}^k$ for $i = 2, 3$ and $C_n^k[A_i] \cong P_{k+2}^k$ for $i = 4, 5$. By Lemma 1, we get $\bigcup_{i=2}^5 A_i \subseteq [A]$. So, $V(C_n^k) = [A]$ and hence A is a hull set of C_n^k. $\square$

In the following lemma, we show that 3 is a sharp upper bound of C_n^k.

Lemma 3. *Suppose that $n = 2qk$ where q is a positive integer, then $h(C_n^k) = 3$.*

Proof. Suppose that $A = \{v_0, v_{qk}\}$, we will show that A is not a hull set of C_n^k. Observe that, there are exactly two $v_0 - v_{qk}$ geodesics, the first one is

$v_0 - v_k - \cdots - v_{qk}$ and the second is $v_{qk} - v_{(q+1)k} - v_{(q+2)k} - \cdots - v_{(2q-1)k} - v_0$. So, $[A] \neq V(C_n^k)$ and hence A is not a hull set of C_n^k. Similarly, if we replace v_{qk} in A by any other vertex of C_n^k we can easily show that A is not a hull set. So $h(C_n^k) > 2$. By Lemma 2, we get the result. □

Lemma 4 ([1])**.** *Suppose that $n = qk + r$ where $0 < r < k$, then*

$$h(P_{n+1}^k) = \begin{cases} 2, & \text{if } q > 1, r \neq k; \\ 3, & \text{if } q = 1, r \neq 1; \\ 2, & \text{if } q = 1, r = 1. \end{cases}$$

Lemma 5. *Suppose that $n = 2qk + 2r$ where q is a positive integer and $0 < r < k$, then $h(C_n^k) = 2$.*

Proof. Let

$$A_1 = \{v_0, v_1, \ldots, v_k, \ldots, v_{qk}, \ldots, v_{qk+r}\} \text{ and}$$
$$A_2 = \{v_{qk+r}, \ldots, v_{qk+2r}, \ldots, v_{(q+1)k+2r}, \ldots, v_{2qk+2r-1}, v_0\}.$$

Then $C_n^k[A_i] \cong P_{qk+r+1}^k$ for $i = 1, 2$. By using Lemma 4, we have the following three cases:

Case 1: $q = 1$ and $r = 1$. Since v_0 and v_{qk+r} are simplicial vertices of $C_n^k[A_i]$, the hull set of $C_n^k[A_i]$ is $A = \{v_0, v_{qk+r}\}$ for $i = 1, 2$. But $A_1 \cup A_2 = V(C_n^k)$, so A is a hull set of C_n^k.

Case 2: $q > 1$ and $r \neq k$, then the hull set of $C_n^k[A_i]$ is $A = \{v_0, v_{qk+r}\}$ for $i = 1, 2$. Thus A is a hull set of C_n^k.

Case 3: $q = 1$ and $r \neq 1$. Then the hull number $h(C_n^k[A_i]) = 3$ for $i = 1, 2$. In this case, $A = \{v_0, v_{k+r}\}$ is a hull set of C_n^k. To show this, observe that $v_0 - v_k - v_{k+r}$ and $v_{k+r} - v_{2k+r} - v_0$ are $v_0 - v_{k+r}$ geodesics. Therefore, $\{v_{2k+r}, v_k\} \subseteq [A]$. Since $r < k$, the path $v_{2k+r} - v_1 - v_k$ is a $v_{2k+r} - v_k$ geodesic. So, v_1 belongs to $[A]$ and hence $\{v_0, v_1, v_{k+r}\} \subseteq [A]$. By using the proof of Lemma 2, we have A is a hull set of C_n^k. □

Lemma 6. *Suppose that $n-1 = 2qk$ where q is a positive integer, then $h(C_n^k) = 3$.*

Proof. Assume that $n - 1 = 2qk$, where q is a positive integer, that means the number of the vertices of the graph C_n^k is odd. Set $A = \{v_0, v_{qk}\}$. Clearly, there exists unique $v_0 - v_{qk}$ geodesic which is $v_0 - v_k - \cdots - v_{qk}$. So, A is not a hull set of C_n^k. Similarly, if we replace v_{qk} by any other vertex, we get the same result. By Lemma 2, we conclude that $h(C_n^k) = 3$. □

Lemma 7. *Suppose that $n - 1 = 2qk + 2r$ where q is a positive integer and $0 < r < k$, then $h(C_n^k) = 2$.*

Proof. Let $A = \{v_0, v_{qk+r}\}$. Then A is a hull set of C_n^k. To show this it is enough to show that v_1 belongs to $[A]$ (see the proof of Lemma 2). Since $v_0 - v_k - v_{k+r} - v_{2k+r} - \cdots - v_{qk+r}, v_0 - v_r - v_{k+r} - v_{2k+r} - \cdots - v_{qk+r}$ and $v_0 - v_{(2q-1)k+2r+1} - v_{(2q-2)k+2r+1} - \cdots - v_{qk+2r+1} - v_{qk+r}$ are $v_0 -$

υ_{qk+r} geodesics, we have $\{\upsilon_{(2q-1)k+2r+1}, \upsilon_r, \upsilon_k\} \subseteq [A]$. But $\upsilon_{(2q-1)k+2r+1} - \upsilon_{2qk+2r} - \upsilon_r$ is a $\upsilon_{(2q-1)k+2r+1} - \upsilon_r$ geodesic, so $\upsilon_{2qk+2r} \in [A]$. Now, let $B = \{\upsilon_{2qk+2r}, \upsilon_0, \ldots, \upsilon_k\}$, then $C_n^k[B] \cong P_{k+2}^k$. By Lemma 1, we get $\upsilon_1 \in [A]$ and hence the result holds. □

We can summarize the above in the following theorem.

Theorem 8. *If $n = 2qk + 2r$ or $n - 1 = 2qk + 2r$ where q is a positive integer and $0 \le r < k$, then*

$$h(C_n^k) = \begin{cases} 2, & \text{if} \quad 0 < r < k; \\ 3, & \text{if} \quad r = 0. \end{cases}$$

For interested readers one might try to find the hull number of some other types of circulant graphs.

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Hasan Al-Ezeh
Department of Mathematics
School of Science
The University of Jordan Amman 11942, Jordan
E-mail address: alezehh@ju.edu.jo

Manal Ghanem
Department of Mathematics
School of Science
The University of Jordan Amman 11942, Jordan
E-mail address: m.ghanem@ju.edu.jo

Jameel Rwalah
Department of Mathematics
School of Science
The University of Jordan Amman 11942, Jordan
E-mail address: jameel_31@yahoo.com