KYUNGPOOK Math. J. 57(2017), 525-535
https://doi.org/10.5666/KMJ.2017.57.3.525
pISSN 1225-6951 eISSN 0454-8124

Real Hypersurfaces with k-th Generalized Tanaka-Webster Connection in Complex Grassmannians of Rank Two

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Abstract. In this paper, we consider two kinds of derivatives for the shape operator of a real hypersurface in a Kähler manifold which are named the Lie derivative and the covariant derivative with respect to the k-th generalized Tanaka-Webster connection $\widehat{\nabla}^{(k)}$. The purpose of this paper is to study Hopf hypersurfaces in complex Grassmannians of rank two, whose Lie derivative of the shape operator coincides with the covariant derivative of it with respect to $\widehat{\nabla}^{(k)}$ either in direction of any vector field or in direction of Reeb vector field.

1. Introduction

In the class of Hermitian symmetric spaces of rank 2, usually we can give examples of Riemannian symmetric spaces $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ and $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians, respectively (see [9]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$. There are exactly two types of singular tangent vectors X of $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$ which are characterized by the geometric

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Received May 12, 2017; accepted July 18, 2017.
2010 Mathematics Subject Classification: 53C40 and 53C55.
Key words and phrases: real hypersurface, complex two-plane Grassmannian, complex hyperbolic two-plane Grassmannian, Hopf hypersurface, Levi-Civita connection, Lie derivative, k-th generalized Tanaka-Webster connection, shape operator.

properties $JX \in \mathfrak{J}X$ and $JX \perp \mathfrak{J}X$ respectively. Hereafter let $G_2^{m+2}(c)$ be the compact complex Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, and the noncompact complex Grassmannian $G_2^*(\mathbb{C}^{m+2})$, $m \geq 3$, of rank two for $c > 0$ and $c < 0$ respectively, where c is a scaling factor for the Riemannian metric g. The Riemannian curvature tensor $\widetilde{R}$ of $G_2^{m+2}(c)$ is locally given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = & \; c\left\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX\right\} \\ & - c\left\{g(JX,Z)JY + 2g(JX,Y)JZ\right\} \\ & + c\sum_{\nu=1}^{3}\left\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\right\} \\ & + c\sum_{\nu=1}^{3}\left\{g(JJ_\nu Y,Z)JJ_\nu X - g(JJ_\nu X,Z)JJ_\nu Y\right\},\end{aligned}$$

for all X, Y and $Z \in T_xG_2^{m+2}(c)$, $x \in G_2^{m+2}(c)$. Actually, in the previous studies for $G_2^{m+2}(c)$ (e.g. [1], [2], [3], [5], [7], [11] etc.), the scaling factor c was given by 1 and $-\frac{1}{2}$ for $G_2(\mathbb{C}^{m+2})$ and $G_2^*(\mathbb{C}^{m+2})$, respectively.

For real hypersurfaces M in $G_2^{m+2}(c)$, we have the following two natural geometric conditions: the 1-dimensional distribution $\mathcal{C}^\perp = \text{Span}\{\xi\}$ and the 3-dimensional distribution $\mathcal{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M. Here the almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2^{m+2}(c)$. The *almost contact 3-structure* vector fields ξ_1, ξ_2, ξ_3 spanning the 3-dimensional distribution $\mathcal{Q}^\perp$ of M in $G_2^{m+2}(c)$ are defined by $\xi_\nu = -J_\nu N$ ($\nu = 1, 2, 3$), where J_ν denotes a canonical local basis of the quaternionic Kaehler structure $\mathfrak{J}$, such that $T_xM = \mathcal{Q} \oplus \mathcal{Q}^\perp = \mathcal{C} \oplus \mathcal{C}^\perp$, $x \in M$. Under these invariant conditions for two kinds of distributions $\mathcal{C}^\perp$ and $\mathcal{Q}^\perp$ in $T_xG_2^{m+2}(c)$ Berndt and Suh gave the complete classifications for real hypersurfaces in complex Grassmannians $G_2^{m+2}(c)$ of rank 2, respectively (see [3] and [5]).

The Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral curves of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2^{m+2}(c)$ if and only if the Hopf foliation of M is totally geodesic. By the almost contact metric structure (ϕ, ξ, η, g) and the formula $\nabla_X\xi = \phi AX$ for any $X \in TM$, it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf. On the other hand, when the distribution $\mathcal{Q}^\perp$ of a hypersurface M in $G_2^{m+2}(c)$ is invariant under the shape operator, we say that M is a $\mathcal{Q}^\perp$*-invariant hypersurface*. Moreover, we say that the Reeb flow on M in $G_2^{m+2}(c)$ is *isometric*, when the Reeb vector field ξ on M is Killing. This means that the metric tensor g is invariant under the Reeb flow of ξ on M, that is, $\mathcal{L}_\xi g = 0$ where $\mathcal{L}_\xi$ is the Lie derivative along the flow of ξ. For the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ the following result is known.

Theorem A.([4]) *The Reeb flow on a real hypersurface in* $G_2(\mathbb{C}^{m+2})$ *is isometric* *if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Moreover, in [13] Suh has proved:

Theorem B. *Let M be a connected orientable real hypersurface in the complex hyperbolic two-plane Grassmannians $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$, $m \geq 3$. The the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ or a horosphere whose center at infinity is singular.*

Usually, any submanifold in Kaehler manifolds has many kinds of connections. Among them, we consider two connections, namely, Levi-Civita and Tanaka-Webster connections for real hypersurfaces M in $G_2^{m+2}(c)$. In fact, $G_2^{m+2}(c)$ is a Riemannian symmetric space for real hypersurfaces in a Kaehler manifold, we consider an affine connection $\widehat{\nabla}^{(k)}$ which is called by the *k-th generalized Tanaka-Webster connection* (in short, the *g-Tanaka-Webster connection*). It becomes a generalization of the well-known connection defined by Tanno [16]. Besides, it coincides with Tanaka-Webster connection if a real hypersurface in Kaehler manifolds satisfies $\phi A + A\phi = 2k\phi$ for a non-zero real number k. The Tanaka-Webster connection is defined as the canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold ([6], [15] and [17]). Using the k-th generalized Tanaka-Webster connection $\widehat{\nabla}^{(k)}$ defined in such a way that

$$(*) \qquad \widehat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$$

for any X, Y tangent to M, where ∇ denotes the Levi-Civita connection on M and k is a non-zero real number. The latter part of the k-th generalized Tanaka-Webster connection $g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$ is denoted by $\hat{F}^{(k)}_X Y$. Here the operator $\hat{F}^{(k)}_X$ is a kind of (1,1)-type tensor and said to be *Tanaka-Webster operator*. Recently, there are many results for the classification problem of real hypersurfaces in $G_2^{m+2}(c)$ related to the k-th generalized Tanaka-Webster connection $\widehat{\nabla}^{(k)}$. In particular, [7] was given the result about the shape operator as follows:

Theorem C. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M satisfies $(\nabla_\xi A)Y = (\widehat{\nabla}^{(k)}_\xi A)Y$ for all tangent vector field Y on M, then M is locally congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Motivated by this result, in this paper we study a real hypersurface M in $G_2^{m+2}(c)$ whose Lie derivative coincides with k-th generalized Tanaka-Webster derivative for the shape operator of M, that is,

$$\text{(C-1)} \qquad (\mathcal{L}_\xi A)Y = (\widehat{\nabla}^{(k)}_\xi A)Y$$

for arbitrary tangent vector field X and Y on M. Thus we assert the following:

Main Theorem. *Let M be a Hopf hypersurface in complex Grassmannians $G_2^{m+2}(c)$, $c \neq 0$ and $m \geq 3$. If M satisfies* (C-1), *then M is locally congruent one of the following:*

(I) *In case where $G_2^{m+2}(c) = G_2(\mathbb{C}^{m+2})$:*

($\mathcal{T}_A$) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

(II) *In case where $G_2^{m+2}(c) = SU_{2,m}/S(U_2U_m)$:*

($\mathcal{T}_A$) *a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$,*

($\mathcal{H}_A$) *a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$.*

Moreover, from this result we also have:

Corollary. *There does not exist any Hopf hypersurface in complex Grassmannians $G_2^{m+2}(c)$, $c \neq 0$, $m \geq 3$, with*

$$(\mathcal{L}_X A)Y = (\widehat{\nabla}^{(k)}_X A)Y \tag{C-2}$$

for all tangent vector fields X and Y on M.

2. Preliminaries

We use some references [2, 7, 8, 10, 12, 14] to recall the Riemannian geometry of complex Grassmannians of rank two $G_2^{m+2}(c)$, $c \neq 0$, $m \geq 3$, and some fundamental formulas including the Codazzi and Gauss equations for a real hypersurface in $G_2^{m+2}(c)$. We can derive some facts from our assumption that M is a real hypersurface in $G_2^{m+2}(c)$ with geodesic Reeb flow, that is, $A\xi = \alpha\xi$ where $\alpha = g(A\xi, \xi)$. Among them, we introduce a lemma which is induced from the equation of Codazzi [11, 12].

Lemma A. *If M is a connected orientable real hypersurface in $G_2^{m+2}(c)$ with geodesic Reeb flow, then*

$$\operatorname{grad}\alpha = (\xi\alpha)\xi + 4c\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi\xi_\nu \tag{2.1-(i)}$$

and

$$\begin{aligned} &2A\phi AX - \alpha A\phi X - \alpha\phi AX \\ &\quad = 2c\phi X + 2c\sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X\Big\} \\ &\qquad - 4c\sum_{\nu=1}^{3}\Big\{\eta(X)\eta_\nu(\xi)\phi\xi_\nu + \eta_\nu(\phi X)\eta_\nu(\xi)\xi\Big\}, \end{aligned} \tag{2.1-(ii)}$$

for any tangent vector field X on M in $G_2^{m+2}(c)$.

As mentioned in Section 1, the complete classifications of real hypersurfaces in $G_2^{m+2}(c)$, $c \neq 0$, $m \geq 3$, with two kinds of A-invariant for the distributions $\mathcal{C}^\perp = \text{Span}\{\xi\}$ and $\mathcal{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ was given in [3, 5], respectively. Here we introduce these results as follows.

Theorem D.([3]) *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $\mathcal{C}^\perp$ and $\mathcal{Q}^\perp$ are invariant under the shape operator of M if and only if*

(M_A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(M_B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

Theorem E.([5]) *Let M be a connected real hypersurface in $G_2^*(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $\mathcal{C}^\perp$ and $\mathcal{Q}^\perp$ are invariant under the shape operator of M if and only if M is congruent to an open part of one of the following hypersurfaces:*

($\mathcal{T}_A$) *a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$;*

($\mathcal{T}_B$) *a tube around a totally geodesic $\mathbb{H}H^n$ in $SU_{2,m}/S(U_2U_m)$, $m = 2n$;*

($\mathcal{H}_A$) *a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$;*

($\mathcal{H}_B$) *a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JX \perp \mathfrak{J}X$;*
or the following exceptional case holds:

($\mathcal{E}$) *The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$. Moreover, M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = (\mathcal{C} \cap \mathcal{Q})^\perp, \quad T_\gamma = J\mathcal{Q}^\perp, \quad T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then we have $T_\mu \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}$, $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$.

In particular, let us observe the structure of the model spaces, (M_A), ($\mathcal{T}_A$) and ($\mathcal{H}_A$), of Type (A) which are mentioned in Theorems D and E, respectively. In [1, 3], [5] the authors gave the characterization of the singular tangent vector N of M in $G_2^{m+2}(c)$: *There are two types of singular tangent vector, those N for which $JN \perp \mathfrak{J}N$, and those for which $JN \in \mathfrak{J}N$.* In other words, it means that $\xi \in \mathcal{Q}$ or

$\xi \in \mathcal{Q}^\perp$, since $JN = -\xi$ and $\mathfrak{J}N = \text{Span}\{\xi_1, \xi_2, \xi_3\} = \mathcal{Q}^\perp$ where $TM = \mathcal{Q} \oplus \mathcal{Q}^\perp$. The following two propositions tell us that the normal vector field N on these model spaces is singular of type of $JN \in \mathfrak{J}N$, that is, $\xi \in \mathcal{Q}^\perp$.

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathcal{Q} \subset \mathcal{Q}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathcal{Q}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has the following three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures α, β, λ and μ with some $r \in (0, \pi/\sqrt{8})$. Here $\mathbb{H}N$ denotes quaternionic span of the structure vector field ξ.*

Type	Eigenvalues	Eigenspace	Multiplicity
(M_A)	$\alpha=\sqrt{8}\cot(\sqrt{8}r)$	$T_\alpha=\mathcal{C}^\perp$	1
	$\beta=\sqrt{2}\cot(\sqrt{2}r)$	$T_\beta=\mathcal{C}\ominus\mathcal{Q}$	2
	$\lambda=-\sqrt{2}\tan(\sqrt{2}r)$	$T_\lambda=\{X \mid X\perp\mathbb{H}N,\ JX=J_1X\}$	$2m-2$
	$\mu=0$	$T_\mu=\{X \mid X\perp\mathbb{H}N,\ JX=-J_1X\}$	$2m-2$

Proposition B. *Let M be a connected real hypersurface of $G_2^*(\mathbb{C}^{m+2})$. Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \in \mathfrak{J}N$, then one of the following statements holds:*

$(\mathcal{T}_A)$ *M has exactly four distinct constant principal curvatures α, β, λ_1 and λ_2. The principal curvature spaces T_{λ_1} and T_{λ_2} are complex (with respect to J) and totally complex (with respect to $\mathfrak{J}$).*

$(\mathcal{H}_A)$ *M has exactly three distinct constant principal curvatures α, β and λ.*

The eigenvalues and its corresponding eigenspaces and multiplicities are given as follows.

Type	Eigenvalues	Eigenspace	Multiplicity
$(\mathcal{T}_A)$	$\alpha=2\coth(2r)$	$T_\alpha=\mathcal{C}^\perp$	1
	$\beta=\coth(r)$	$T_\beta=\mathcal{C}\ominus\mathcal{Q}$	2
	$\lambda_1=\tanh(r)$	$T_{\lambda_1}=E_{-1}$	$2m-2$
	$\lambda_2=0$	$T_{\lambda_2}=E_{+1}$	$2m-2$
$(\mathcal{H}_A)$	$\alpha=2$	$T_\alpha=\mathcal{C}^\perp$	1
	$\beta=1$	$T_\beta=(\mathcal{C}\ominus\mathcal{Q})\oplus E_{-1}$	$2m$
	$\lambda=0$	$T_\lambda=E_{+1}$	$2m-2$

On $\mathcal{Q}$, we have $(\phi\phi_1)^2 = I$ and $\text{Tr}(\phi\phi_1) = 0$. Let E_{+1} and E_{-1} be the eigenbundles of $\phi\phi_1|_{\mathcal{Q}}$ with respect to the eigenvalues $+1$ and -1, respectively.

3. Proof of Main Theorem

We will prove that on a real hypersurface M satisfying the conditions given in Main Theorem of Section 1, the shape operator A and the structure tensor ϕ commute with each other, that is, the Reeb flow of M becomes isometric. Then by virtue of Theorems A and B we assert our main theorem in Section 1.

In order to do this, first we calculate the squared norm of symmetric operator $(A\phi - \phi A)$ of a real hypersurface M in $G_2^{m+2}(c)$, $c \neq 0$, $m \geq 3$.

Lemma 3.1. *Let M be a real hypersurface in $G_2^{m+2}(c)$, $m \geq 3$. Then the squared norm of a symmetric operator $(A\phi - \phi A)$ of M is given:*

$$\|A\phi - \phi A\|^2 = 2\mathrm{Tr}A^2 + 2\mathrm{Tr}(\phi A\phi A) - 2g(A^2\xi, \xi).$$

Proof. Let $\{e_1, e_2, \cdots, e_{4m-1}\}$ be an orthonormal basis for T_xM where x is any point of M. By direct calculation we have

$$\begin{aligned}
\|\phi A - A\phi\|^2 &= \sum_{i=1}^{4m-1} g\big((\phi A - A\phi)e_i, (\phi A - A\phi)e_i\big) \\
&= \sum_{i=1}^{4m-1} g(\phi Ae_i, \phi Ae_i) - \sum_{i=1}^{4m-1} g(\phi Ae_i, A\phi e_i) \\
&\quad - \sum_{i=1}^{4m-1} g(A\phi e_i, \phi Ae_i) + \sum_{i=1}^{4m-1} g(A\phi e_i, A\phi e_i) \\
&= - \sum_{i=1}^{4m-1} g(A\phi^2 Ae_i, e_i) + \sum_{i=1}^{4m-1} g(\phi A\phi Ae_i, e_i) \\
&\quad + \sum_{i=1}^{4m-1} g(A\phi A\phi e_i, e_i) - \sum_{i=1}^{4m-1} g(\phi A^2\phi e_i, e_i) \\
&= \sum_{i=1}^{4m-1} g(A^2 e_i, e_i) - \sum_{i=1}^{4m-1} \eta(Ae_i) g(A\xi, e_i) \\
&\quad + 2\sum_{i=1}^{4m-1} g(A\phi A\phi e_i, e_i) - \sum_{i=1}^{4m-1} g(\phi A^2\phi e_i, e_i) \\
&= \mathrm{Tr}A^2 - g(A\xi, A\xi) + 2\mathrm{Tr}(A\phi A\phi) - \mathrm{Tr}(\phi A^2\phi) \\
&= \mathrm{Tr}A^2 - g(A\xi, A\xi) + 2\mathrm{Tr}(\phi A\phi A) - \mathrm{Tr}(A^2\phi^2) \\
&= 2\mathrm{Tr}A^2 - 2g(A\xi, A\xi) + 2\mathrm{Tr}(\phi A\phi A),
\end{aligned}$$

where we have used the fact of $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any two matrices A, B with same size. □

From now on, let M be a Hopf hypersurface in $G_2^{m+2}(c)$, $c \neq 0$, $m \geq 3$, satisfying (C-1), that is,

$$(\mathcal{L}_\xi A)Y = (\widehat{\nabla}^{(k)}_\xi A)Y$$

for all tangent vector fields Y on M. From the definition of the Tanaka-Webster operator $\hat{F}^{(k)}_X$, we get $\hat{F}^{(k)}_\xi Y = -k\phi Y$. From this and the basic formula $\nabla_Y \xi = \phi AY$, M satisfies the following condition for new symmetric operator $(A\phi - \phi A)$;

$$(A\phi - \phi A)AY = k(A\phi - \phi A)Y, \tag{3.1}$$

where we have used

$$\begin{aligned}(\mathcal{L}_\xi A)Y &= \mathcal{L}_\xi(AY) - A(\mathcal{L}_\xi Y)\\ &= [\xi, AY] - A[\xi, Y]\\ &= \nabla_\xi(AY) - \nabla_{AY}\xi - A(\nabla_\xi Y) + A(\nabla_Y \xi)\\ &= (\nabla_\xi A)Y - \nabla_{AY}\xi + A(\nabla_Y \xi)\\ &= (\nabla_\xi A)Y - \phi A^2 Y + A\phi AY\end{aligned}$$

and

$$\begin{aligned}(\widehat{\nabla}^{(k)}_\xi A)Y &= \widehat{\nabla}^{(k)}_\xi(AY) - A(\widehat{\nabla}^{(k)}_\xi Y)\\ &= \nabla_\xi(AY) + \hat{F}^{(k)}_\xi AY - A(\nabla_\xi Y + \hat{F}^{(k)}_\xi Y)\\ &= (\nabla_\xi A)Y + \hat{F}^{(k)}_\xi AY - A\hat{F}^{(k)}_\xi Y\\ &= (\nabla_\xi A)Y - k\phi AY + kA\phi Y.\end{aligned}$$

Taking the structure tensor ϕ to (3.1), it gives us

$$\begin{aligned}\phi A\phi AY &= \phi^2 A^2 Y + k\phi A\phi Y - k\phi^2 AY\\ &= -A^2 Y + \eta(A^2 Y)\xi + k\phi A\phi Y + kAY - k\eta(AY)\xi,\end{aligned} \tag{3.2}$$

for all tangent vector fields Y on M. From this and $A\xi = \alpha\xi$, the trace of $\phi A\phi A$ is given by

$$\begin{aligned}\mathrm{Tr}(\phi A\phi A) &= \sum_{i=1}^{4m-1} g(\phi A\phi Ae_i, e_i)\\ &= \sum_{i=1}^{4m-1} g(-A^2 e_i + \eta(A^2 e_i)\xi + k\phi A\phi e_i + kAe_i - k\eta(Ae_i)\xi, e_i)\\ &= -\mathrm{Tr}(A^2) + \alpha^2 + k\mathrm{Tr}(\phi A\phi) + k\mathrm{Tr}(A) - k\alpha.\end{aligned}$$

Moreover, since the matrices in a trace of a product can be switched without changing the result, we have also

$$\begin{aligned}\mathrm{Tr}(\phi A\phi) = \mathrm{Tr}(\phi^2 A) &= \sum_{i=1}^{4m-1} g(\phi^2 Ae_i, e_i) \\ &= \sum_{i=1}^{4m-1} g(-Ae_i + \eta(Ae_i)\xi, e_i) = -\mathrm{Tr}A + \alpha.\end{aligned}$$

Hence it follows that

$$\mathrm{Tr}(\phi A\phi A) = -\mathrm{Tr}(A^2) + \alpha^2. \tag{3.3}$$

By virtue of Lemma 3.1 and (3.3), we see that the squared norm of $(A\phi - \phi A)$ vanishes on M, which implies the symmetric operator $(A\phi - \phi A)$ is identically zero on M. □

Summing up these observations we assert:

Lemma 3.2. *Let M be a Hopf hypersurface in $G_2^{m+2}(c)$, $c \neq 0$, $m \geq 3$. The condition $(\mathcal{L}_\xi A)Y = (\widehat{\nabla}_\xi^{(k)} A)Y$ for all tangent vector fields Y on M is equivalent that the Reeb flow on M is isometric, that is, $A\phi = \phi A$. Furthermore, M is locally congruent to the model space of Type (A).*

4. Proof of Corollary

Hereafter we will give a proof of Corollary introduced in Section 1. From now on, assume that M is a Hopf hypersurface in $G_2^{m+2}(c)$, $c \neq 0$, $m \geq 3$ with

$$(\mathcal{L}_X A)Y = (\widehat{\nabla}_X^{(k)} A)Y \tag{C-2}$$

for all vector fields X and Y are tangent to M. It is trivial that the condition of (C-1) is weaker than (C-2). So, M satisfies the condition (C-1), naturally. Hence we see that if M satisfies our assumptions in Corollary, then M is of Type (A) by virtue of Lemma 3.2.

Now let us check the converse problem: *whether the shape operator of model spaces of Type (A) in $G_2^{m+2}(c)$ satisfies the condition* (C-2) *or not?* In order to do this, suppose that the model spaces of Type (A), that is, (M_A), $(\mathcal{T}_A)$, and $(\mathcal{H}_A)$, in $G_2^{m+2}(c)$ satisfies our conditions given in Corollary. Since

$$\begin{aligned}(\mathcal{L}_X A)Y &= \mathcal{L}_X(AY) - A(\mathcal{L}_X Y) \\ &= [X, AY] - A[X, Y] \\ &= \nabla_X(AY) - \nabla_{AY}X - A(\nabla_X Y) + A(\nabla_Y X) \\ &= (\nabla_X A)Y - \nabla_{AY}X + A(\nabla_Y X)\end{aligned}$$

and

$$\begin{aligned}(\widehat{\nabla}^{(k)}_X A)Y &= \widehat{\nabla}^{(k)}_X (AY) - A(\widehat{\nabla}^{(k)}_X Y) \\ &= \nabla_X (AY) + \hat{F}^{(k)}_X AY - A(\nabla_X Y + \hat{F}^{(k)}_X Y) \\ &= (\nabla_X A)Y + \hat{F}^{(k)}_X AY - A\hat{F}^{(k)}_X Y,\end{aligned}$$

the condition (C-2) can be rewritten as

$$\begin{aligned}(4.1)\qquad -\nabla_{AY}X + A(\nabla_Y X) &= \hat{F}^{(k)}_X (AY) - A(\hat{F}^{(k)}_X Y) \\ &= g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY \\ &\quad - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y,\end{aligned}$$

for all vector fields $X, Y \in TM^*$ where M^* denotes the model space of Type (A). From Propositions A and B we see that $\xi = \xi_1$, furthermore $q_\nu(X) = 2g(A\xi_\nu, X)$ for $\nu = 2, 3$. If we put $X = \xi_2 \in T_\beta$ and $Y = \xi_3 \in T_\beta$ in (4.1), then it becomes

$$(4.2)\qquad -\beta\nabla_{\xi_3}\xi_2 + A(\nabla_{\xi_3}\xi_2) = \beta(\alpha-\beta)\xi \iff 2\beta(\alpha-\beta)\xi = 0,$$

where we have used $\phi\xi_3 = \xi_2$, $\phi\xi_2 = -\xi_3$ and

$$\nabla_{\xi_3}\xi_2 = q_1(\xi_3)\xi_3 - 2g(A\xi_3, \xi_3))\xi_1 + \beta\xi_1 = q_1(\xi_3)\xi_3 - \beta\xi_1.$$

From this, we see that $\beta = 0$ or $\alpha = \beta$.

For the case of (M_A), since $\alpha = \sqrt{8}\cot(\sqrt{8}r)$ and $\beta = \sqrt{2}\cot(\sqrt{2}r)$ where $r \in (0, \frac{\pi}{\sqrt{8}})$, it makes an contradiction. Hence the model space of (M_A) does not satisfy our condition (C-2).

Moreover, for $(\mathcal{T}_A)$ (and $(\mathcal{H}_A)$, resp.) we obtain the contradiction, since $\alpha = 2\coth(2r)$ (and $\alpha = 2$, resp.) and $\beta = \coth(r)$ (and $\beta = 1$, resp.) where $r \in (0, \infty)$.

Summing up these observations, we assert the corollary in Section 1. □

Acknowledgements. This work was supported by grant Proj. Nos. NRF-2016-R1A6A3A-11931947 and NRF-2017-R1A2B-4005317 from National Research Foundation of Korea.

References

[1] J. Berndt, *Riemannian geometry of complex two-plane Grassmannians*, Rend. Sem. Mat. Univ. Politec. Torino, **55**(1997), 19–83.

[2] J. Berndt, H. Lee and Y. J. Suh, *Contact hypersurfaces in noncompact complex Grassmannians of rank two*, Internat. J. Math., **24(11)**(2013), 1350089, 11 pp.

[3] J. Berndt and Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians*, Monatsh. Math., **127**(1999), 1–14.

[4] J. Berndt and Y. J. Suh, *Real hypersurfaces with isometric Reeb flows in complex two-plane Grassmannians*, Monatsh. Math., **137**(2002), 87–98.

[5] J. Berndt and Y. J. Suh, *Hypersurfaces in noncompact complex Grassmannians of rank two*, Internat. J. Math., **23(10)**(2012), 1250103, 35 pp.

[6] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Birkhäuser, Boston, 2002.

[7] I. Jeong, H. Lee and Y. J. Suh, *Levi-Civita and generalized Tanaka-Webster covariant derivatives for real hypersurfaces in complex two-plane Grassmannians*, Ann. Mat. Pura Appl., **194(3)**(2015), 919–930.

[8] I. Jeong, C. Machado, J.D. Pérez and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with $\mathfrak{D}^{\perp}$-parallel structure Jacobi operator*, Internat. J. Math., **22(5)**(2011), 655–673.

[9] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. I*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996.

[10] H. Lee and Y. J. Suh, *Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector*, Bull. Korean Math. Soc., **47(3)**(2010), 551–561.

[11] H. Lee, Y. J. Suh and C. Woo, *Real hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting structure Jacobi operators*, Meditterr. J. Math., **13**(2016), 3389–3407.

[12] J. D. Pérez and Y. J. Suh, *The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians*, J. Korean Math. Soc., **44**(2007), 211–235.

[13] Y. J. Suh, *Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians*, Adv. in Appl. Math., **50**(2013), 645–659.

[14] Y. J. Suh, *Real hypersurfaces in complex hyperbolic two-plane Grassmannians with Reeb vector field*, Adv. in Appl. Math., **55**(2014), 131–145.

[15] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan J. Math. (N.S.), **2(1)**(1976), 131–190.

[16] S. Tanno, *Variational problems on contact Riemannian manifolds*, Trans. Amer. Math. Soc., **314(1)**(1989), 349–379.

[17] S. M. Webster, *Pseudo-Hermitian structures on a real hypersurface*, J. Differential Geom., **13**(1978), 25–41.