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# Note on Almost Generalized Pseudo-Ricci Symmetric Manifolds

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Abstract. The purpose of the present paper is to study an almost generalized pseudo-Ricci symmetric manifold. The existence of such manifold is ensured by an example. Furthermore, having found, faulty example in [13], the present paper also attempts to construct a non-trivial example of an almost pseudo Ricci symmetric manifold.

## 1. Introduction

In the spirit of Chaki and Kawaguchi [9], a non-flat $n$-dimensional Riemannian manifold $(M^n, g)(n > 3)$ is defined to be an almost pseudo-Ricci symmetric manifold, if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and satisfies the equation

$$(\nabla_X S)(Y, U) = [A(X) + B(X)]S(Y, U) + A(Y)S(X, U) + A(U)S(X, Y) \tag{1.1}$$

where $A$ and $B$ are two non-zero 1-forms defined by $A(X) = g(X, \theta)$ and $B(X) = g(X, \varrho)$ $\forall$ $X$, $\nabla$ being the operator of the covariant differentiation. The local expression of the above equation is

$$R_{ik,l} = (A_l + B_l)R_{ik} + A_k R_{il} + A_i R_{lk}, \tag{1.2}$$

where $A_l$ and $B_l$ are two non-zero co-vectors and comma followed by indices denotes the covariant differentiation with respect to the metric tensor $g$. An $n$-dimensional manifold of this kind is abbreviated by $A(PRS)_n$.

Keeping in tune with Dubey[11], the author in [6] has introduced the notion of an *almost generalized pseudo-Ricci symmetric manifold* which is abbreviated by $A(GPRS)_n$-manifold and defined as follows

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A non-flat $n$-dimensional Riemannian manifold $(M^n, g)(n > 3)$, is termed as an almost generalized pseudo-Ricci symmetric manifold, if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and admits the identity

$$\begin{aligned}(\nabla_X S)(Y,U) &= [A(X)+B(X)]S(Y,U)+A(Y)S(X,U)+A(U)S(X,Y)\\ &\quad +[C(X)+D(X)]g(Y,U)+C(Y)g(X,U)+C(U)g(X,Y)\end{aligned} \tag{1.3}$$

where $A$, $B$, $C$ and $D$ are non-zero 1-forms defined by $A(X) = g(X, \theta)$, $B(X) = g(X, \varrho)$, $C(X) = g(X, \pi)$ and $D(X) = g(X, \delta)\ \forall\ X$. The beauty of such $A(GPRS)_n$-space is that it has the flavour of

(1) Ricci symmetric space in the sense of Cartan (for $A = B = C = D = 0$),

(2) Ricci recurrent space by E. M. Patterson [4] (for $B \neq 0$ and $A = C = D = 0$),

(3) generalized Ricci recurrent space by De, Guha and Kamilya [14] (for $B \neq 0$, $D \neq 0$ and $A = C = 0$),

(4) pseudo-Ricci symmetric space by Chaki [8] (for $A = B \neq 0$ and $C = D = 0$),

(5) generalized pseudo-Ricci symmetric space, by Baishya [5] (for $A = B \neq 0$ and $C = D \neq 0$) and

(6) almost pseudo-Ricci symmetric manifold by Chaki and Kawaguchi [9] (for $A = B \neq 0$ and $C = D = 0$).

We structured the present paper as follows: Section 2 is dealt with some basic properties of an almost generalized pseudo-Ricci symmetric manifold. In section 3, we have constructed a non-trivial example of an almost pseudo-Ricci symmetric manifold which is not an almost generalized pseudo-Ricci symmetric manifold. Finally, it is investigated that there exists a Riemannian manifold $(\mathbb{R}^4, g)$ which is an almost generalized pseudo-Ricci symmetric for some choice of the 1-forms.

## 2. $A(GPRS)_n$-manifold

In this section, we assume a non-flat $n$-dimensional Riemannian manifold $(M^n, g)(n > 3)$ to be an almost generalized pseudo-Ricci symmetric manifold. Next, we consider that the 1-forms $A$ and $B$ are co-directional with that of $C$ & $D$ respectively, that is $C(X) = \phi A(X)$ & $D() = \phi B(X)\ \forall\ X$, where $\phi$ being a non-zero constant function, then the relation (1.3) turnes into

$$(\nabla_X Z)(Y,U) = [A(X)+B(X)]Z(Y,U)] + A(Y)\ Z(X,U)\ + A(U)Z(X,U)$$

where $Z(X, Y) = S(X, Y) + \phi\ g(X, Y)$ is well known $Z$-tensor introduced in [2, 3]. This leads to the following

**Theorem 2.1.** *Every $A(GPRS)_n$-manifold is an almost pseudo $Z$-symmetric manifold provided that the 1-forms $A$ & $B$ are co-directional with that of $C$ & $D$ respectively.*

**Definition 2.1.** A non-flat Riemannian manifold $(M^n, g)(n > 3)$ is said to be a *quasi-Einstein manifold* [10] if its Ricci tensor $S$ of type (0, 2) is not identically zero and satisfies the condition

$$S(X,Y) = \lambda g(X,Y) + \mu\psi(X)\psi(Y),$$

where $\lambda$, $\mu \in \mathbb{R}$ and $\psi$ is a non-zero 1-form such that $g(X,U) = \psi(X)$, for all vector fields $X$.

Now, contracting $Y$ over $U$ in (1.1) we obtain

$$\mathrm{dr}(X) = r[A(X) + B(X)] + 2\bar{A}(X) + (n+2)C(X) + nD(X) \tag{2.1}$$

where $\bar{A}(X) = S(X,,\theta)$. Again, from (1.1), one can easily bring out

$$\begin{aligned}(\nabla_X S)(Y,U) - (\nabla_U S)(X,Y) &= B(X)S(Y,U) - B(U)S(X,Y)\\ &\quad +D(X)g(Y,U) - D(U)g(X,Y)\end{aligned} \tag{2.2}$$

after further contraction which leaves

$$dr(X) = 2rB(X) - 2\bar{B}(X) + 2(n-1)D(X), \tag{2.3}$$

where $\bar{B}(X) = S(X,\varrho)$.

It is known ([7], p, 41) that a conformally flat $(M^n, g)$ possesses the relation

$$(\nabla_X S)(Y,U) - (\nabla_U S)(X,Y) = \frac{1}{2(n-1)}[g(Y,U)dr(X) - g(X,Y)dr(U)]. \tag{2.4}$$

By virtue of (2.2), (2.3) and (2.4) we find

$$\begin{aligned}&(n-1)[B(X)S(Y,U) - B(U)S(X,Y)]\\ =\ &[rB(X) - \bar{B}(X)]g(Y,U) - [rB(U) - \bar{B}(U)]g(X,Y).\end{aligned} \tag{2.5}$$

which yields

$$B(X)\bar{B}(U) = B(U)\bar{B}(X) \tag{2.6}$$

for $Y = \varrho$. Assuming the Ricci tensor of the manifold as codazzi type (in the sense of [12]) and then making use of (2.3), we obtain from (2.6) that

$$B(X)D(U) = B(U)D(X)\ \forall\ X \text{ and } U. \tag{2.7}$$

This motivate us to state

**Proposition 2.1.** *In a conformally flat $A(GPRS)_4$-manifold with codazzi type of Ricci tensor, the 1-forms $B$ and $D$ are co-directional.*

Again, for constant scalar curvature tensor (or codazzi type of Ricci tensor) by virtue of (2.3), (2.5), (2.7), we can easily find out

$$S(Y,U) = -\frac{D(\varrho)}{B(\varrho)}g(Y,U) + \frac{1}{B(\varrho)}[rB(Y) + nD(Y)]B(U), \tag{2.8}$$

where $\frac{D(U)}{B(U)} = k$, $\forall\ U$. If the 1-forms $B$ and $D$ are co-directional, then (2.8) takes the following form

$$S(Y,U) == \alpha g(Y,U) + \beta B(Y)B(U). \tag{2.9}$$

This leads to the followings

**Corollary 2.1.** *A conformally flat $A(GPRS)_n$-manifold with codazzi type of Ricci tensor, is a quasi-Einstein manifold.*

**Corollary 2.2.** *A conformally flat almost generalized pseudo-Ricci symmetric manifold with constant scalar curvature is a space of quasi constant curvature [1].*

## 3. Existence of $A(PRS)_n$-manifold

In the example given in ([13], Example 5, p. 515–516) authors have found the value of the covariant derivatives corresponding to the vanishing component of the Ricci tensor $R_{14}$, $R_{24}$ & $R_{34}$ (namely, $R_{14},_1$ , $R_{24},_2$ & $R_{34},_3$) to be zero. But those value are calculated to be $R_{14},_1 = R_{24},_2 = R_{34},_3 = \frac{8}{9(x^4)^{5/3}}$ which are non-zero. Consequently for their [13] choice of the 1-forms

$$A_i(x) = \begin{cases} -\dfrac{3}{x^4}, & \text{for } i = 4, \\ 0, & \text{otherwise,} \end{cases} \qquad B_i(x) = \begin{cases} -\dfrac{1}{x^4}, & \text{for } i = 1, \\ 0, & \text{otherwise,} \end{cases}$$

the relations

$$\begin{aligned} R_{14},_1 &= (A_1 + B_1)R_{14} + A_1R_{14} + A_4R_{11}, \\ R_{24},_2 &= (A_2 + B_2)R_{24} + A_2R_{24} + A_4R_{22}, \\ R_{34},_3 &= (A_3 + B_3)R_{34} + A_3R_{34} + A_4R_{33}, \end{aligned}$$

do not stand. Hence, $(\mathbb{R}^4, g)$ under-considered metric ([13], page 516) can not be an almost pseudo-Ricci symmetric manifold.

**Example 3.1.** Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian space endowed with the Riemannian metric $g$ given by

$$ds^2 = g_{ij}dx^idx^j = e^{-x^1}[(dx^1)^2 + (dx^2)^2 + 2\ dx^3dx^4], \tag{3.1}$$

where $i, j = 1, 2, 3, 4$.

The non-zero components of Riemannian curvature tensor, Ricci tensors and scalar curvature (up to symmetry and skew-symmer) are

$$R_{2324} = -\frac{1}{4}e^{-x^1} = -R_{3434},$$
$$R_{22} = -\frac{1}{2} = R_{34},$$
$$r = -\frac{3}{2}e^{x^1}.$$

Covariant derivatives of Ricci tensors is expressed as

$$R_{12},_2 = R_{13},_4 = R_{14},_3 = -\frac{1}{4},$$
$$R_{22},_1 = R_{34},_1 = -\frac{1}{2}.$$

For the following choice of the1-forms

$$A_i = \begin{cases} \frac{1}{2}, & \text{for } i = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad B_i = \begin{cases} \frac{1}{2}, & \text{for } i = 1, \\ 0, & \text{otherwise,} \end{cases}$$

one can easily verify the followings

$$\begin{aligned}
R_{12},_k &= (A_k + B_k)\, R_{12} + A_1 R_{k2} + A_2 R_{1k}, \\
R_{13},_k &= (A_k + B_k)\, R_{13} + A_1 R_{k3} + A_3 R_{1k}, \\
R_{14},_k &= (A_k + B_k)\, R_{14} + A_1 R_{k4} + A_4 R_{1k}, \\
R_{23},_k &= (A_k + B_k)\, R_{23} + A_2 R_{k3} + A_3 R_{2k}, \\
R_{24},_k &= (A_k + B_k)\, R_{24} + A_2 R_{k4} + A_4 R_{2k}, \\
R_{34},_k &= (A_k + B_k)\, R_{34} + A_3 R_{k4} + A_4 R_{3k}, \\
R_{11},_k &= (A_k + B_k)\, R_{11} + A_1 R_{k1} + A_1 R_{1k}, \\
R_{22},_k &= (A_k + B_k)\, R_{22} + A_2 R_{k2} + A_2 R_{2k}, \\
R_{33},_k &= (A_k + B_k)\, R_{33} + A_3 R_{k3} + A_3 R_{3k}, \\
R_{44},_k &= (A_k + B_k)\, R_{44} + A_4 R_{k4} + A_4 R_{4k},
\end{aligned}$$

where $k = 1, 2, 3, 4$. In consequence of the above, one can say that

**Theorem 3.1.** *There exists a manifold $(\mathbb{R}^4, g)$ which is an almost pseudo-Ricci symmetric manifold with the above mentioned choice of the 1-forms.*

It is obvious that the manifold bearing the metric given by (3.1) can not be Ricci symmetric, Ricci recurrent, generalized Ricci recurrent as well as almost generalized pseudo-Ricci symmetric manifold.

## 4. Existence of $A(GPRS)_n$-manifold

**Example 4.1.** Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian space endowed with the Riemann metric $g$ given by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{4/3}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 \ ] + (dx^4)^2, \tag{4.1}$$

where $i, j = 1, 2, 3, 4$. The non-zero components of Ricci tensors are

$$R_{11} = R_{22} = R_{33} = -\frac{2}{3(x^4)^{2/3}}, \quad R_{44} = \frac{2}{3(x^4)^2}.$$

Covariant derivative $R_{ik,l}$ of Ricci tensors is expressed by

$$R_{11,4} = \frac{4}{3(x^4)^{5/3}} = R_{22,4} = R_{33,4}\,, \quad R_{44,4} = -\frac{4}{3(x^4)^3}$$
$$R_{14,1} = R_{24,2} = R_{34,3} = \frac{8}{9(x^4)^{5/3}}.$$

For following choice of the 1-forms

$$A_i = \begin{cases} \frac{1}{x^4}, & \text{for } i = 4, \\ 0, & \text{otherwise,} \end{cases} \qquad B_i = \begin{cases} -\frac{19}{3x^4}, & \text{for } i = 4, \\ 0, & \text{otherwise,} \end{cases}$$
$$C_i = \begin{cases} \frac{14}{9(x^4)^3}, & \text{for } i = 4, \\ 0, & \text{otherwise,} \end{cases} \quad D_i = \begin{cases} -\frac{34}{9(x^4)^3}, & \text{for } i = 4, \\ 0, & \text{otherwise,} \end{cases}$$

One can verify the followings

$$\begin{aligned}
R_{12,k} &= (A_k + B_k)\, R_{12} + A_1 R_{k2} + A_2 R_{1k} + (C_k + D_k)\, g_{12} + C_1 g_{k2} + C_2 g_{1k},\\
R_{13,k} &= (A_k + B_k)\, R_{13} + A_1 R_{k3} + A_3 R_{1k} + (C_k + D_k)\, g_{13} + C_1 g_{k3} + C_3 g_{1k},\\
R_{14,k} &= (A_k + B_k)\, R_{14} + A_1 R_{k4} + A_4 R_{1k} + (C_k + D_k)\, g_{14} + C_1 g_{k4} + C_4 g_{1k},\\
R_{23,k} &= (A_k + B_k)\, R_{23} + A_2 R_{k3} + A_3 R_{2k} + (C_k + D_k)\, g_{23} + C_2 g_{k3} + C_3 g_{2k},\\
R_{24,k} &= (A_k + B_k)\, R_{24} + A_2 R_{k4} + A_4 R_{2k} + (C_k + D_k)\, g_{24} + C_2 g_{k4} + C_4 g_{2k},\\
R_{34,k} &= (A_k + B_k)\, R_{34} + A_3 R_{k4} + A_4 R_{3k} + (C_k + D_k)\, g_{34} + C_3 g_{k4} + C_4 g_{3k},\\
R_{11,k} &= (A_k + B_k)\, R_{11} + A_1 R_{k1} + A_1 R_{1k} + (C_k + D_k)\, g_{11} + C_1 g_{k1} + C_1 g_{1k},\\
R_{22,k} &= (A_k + B_k)\, R_{22} + A_2 R_{k2} + A_2 R_{2k} + (C_k + D_k)\, g_{22} + C_2 g_{k2} + C_2 g_{2k},\\
R_{33,k} &= (A_k + B_k)\, R_{33} + A_3 R_{k3} + A_3 R_{3k} + (C_k + D_k)\, g_{33} + C_3 g_{k3} + C_3 g_{3k},\\
R_{44,k} &= (A_k + B_k)\, R_{44} + A_4 R_{k4} + A_4 R_{4k} + (C_k + D_k)\, g_{44} + C_4 g_{k4} + C_4 g_{4k},
\end{aligned}$$

where $k = 1, 2, 3, 4$. In consequence of the above, one can say that

**Theorem 4.1.** *There exists a manifold $(\mathbb{R}^4, g)$ which is an almost generalized pseudo-Ricci symmetric for the above mentioned choice of the1-forms.*

It is obvious that the manifold bearing the metric given by (4.1) can not be Ricci symmetric, Ricci recurrent, generalized Ricci recurrent as well as pseudo-Ricci symmetric.

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