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Radius of Starlikeness for Analytic Functions with Fixed Second Coefficient

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Abstract. Sharp radius constants for certain classes of normalized analytic functions with fixed second coefficient, to be in the classes of starlike functions of positive order, parabolic starlike functions, and Sokół-Stankiewicz starlike functions are obtained. Our results extend several earlier works.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, which are normalized by the conditions $f(0) = 0$, and $f'(0) = 1$ and let $\mathcal{S}$ denote its subclass consisting of univalent functions. The well-known Bieberbach theorem states that the second coefficient in the Maclaurin series of functions in $\mathcal{S}$ is bounded by two. This estimate for the second coefficient plays an important role in the study

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of the class $\mathcal{S}$, and for that reason, there has been considerable continued interest in the investigation of the class $\mathcal{S}_b \subset \mathcal{S}$ of functions $f(z) = z + a_2 z^2 + \cdots$, $a_2 = 2b$ for a fixed b with $|b| \leq 1$. The investigation on $\mathcal{S}_b$ was initiated as early as 1920 by Gronwall [7], where growth and distortion estimates were obtained for functions in $\mathcal{S}_b$. Recently, Ali et al. [5] extended the theory of second-order differential subordination to the class of analytic functions with fixed second coefficient. Pursuant to that work, Nagpal and Ravichandran [15] obtained sufficient conditions for starlikeness and close-to-convexity. Differential superordinations were considered by Mendiratta et al. [13, 14], while Lee et al. [9] investigated other applications of differential subordination for functions with fixed second coefficient. Livingston problems for close-to-convex functions with fixed second coefficient were studied by Mendiratta and Ravichandran [12]. A survey on functions with fixed initial coefficient can be found in [2]. For $0 \leq \alpha < 1$, the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of *starlike functions of order* α and *convex functions of order* α consist of functions $f \in \mathcal{S}$ satisfying respectively $\operatorname{Re}(zf'(z)/f(z)) > \alpha$, and $\operatorname{Re}(1 + zf''(z)/f'(z)) > \alpha$; the classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$ are the familiar classes of starlike and convex functions respectively. The second coefficient of functions in these classes satisfies respectively the inequalities $|a_2| \leq 2(1-\alpha)$ and $|a_2| \leq 1-\alpha$. For notational convenience, let us denote by $\mathcal{A}_b$, the class of normalized analytic functions of the form $f(z) = z + bz^2 + \cdots$. For $|b| \leq 1$ and $0 \leq \alpha < 1$, let $\mathcal{S}^*_b(\alpha) := \mathcal{S}^*(\alpha) \bigcap \mathcal{A}_{2b(1-\alpha)}$ and $\mathcal{K}_b(\alpha) := \mathcal{K}(\alpha) \bigcap \mathcal{A}_{b(1-\alpha)}$. Functions in these classes are respectively called starlike and convex functions of order α with fixed second coefficient. Let $\mathcal{S}^*_b := \mathcal{S}^*_b(0)$ and $\mathcal{K}_b := \mathcal{K}_b(0)$. The class $\mathcal{S}^*_L$ of Sokół-Stankiewicz starlike functions [22] consists of functions $f \in \mathcal{A}$ for which $zf'(z)/f(z)$ lies in the region bounded by the right half-plane of the lemniscate of Bernoulli: $|w^2 - 1| = 1$. A function $f \in \mathcal{S}$ is *uniformly convex* if and only if $\operatorname{Re}(1+zf''(z)/f'(z)) > |zf''(z)/f'(z)|$. The corresponding class of starlike functions connected with the Alexander relation is the class of parabolic starlike functions, introduced by Rønning [19], given by

$$\mathcal{S}^*_P := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \right\}.$$

For a survey of uniformly starlike/convex functions, see [1]. For $\beta > 1$, the class $\mathcal{M}(\beta)$ consists of functions $f \in \mathcal{A}$ satisfying $\operatorname{Re}(zf'(z)/f(z)) < \beta$. This class contains non-univalent functions and was investigated in [17, 24] (see also [4]). Clearly, $\mathcal{S}^*_L \subset \mathcal{S}^*$, $\mathcal{S}^*_P \subset \mathcal{S}^*(1/2)$ while $\mathcal{M}(\beta) \not\subset \mathcal{S}^*$.

The classes of starlike, convex and several other functions are related to the class $\mathcal{P}(\alpha)$, of analytic functions $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ satisfying $\operatorname{Re}(p(z)) > \alpha$ $(0 \leq \alpha < 1)$, $\mathcal{P} := \mathcal{P}(0)$. It is well known [16, p. 170] that $|b_n| \leq 2(1-\alpha)$ for $p \in \mathcal{P}(\alpha)$. We shall denote by $\mathcal{P}_b(\alpha)$ the subclass of $\mathcal{P}(\alpha)$ consisting of functions of the form $p(z) = 1 + 2b(1-\alpha)z + \cdots$, $|b| \leq 1$, and let $\mathcal{P}_b := \mathcal{P}_b(0)$.

Given two sub-families S_1 and S_2 of $\mathcal{A}$, the S_1-radius of S_2 is defined to be the largest number ρ such that $r^{-1}f(rz) \in S_1$ for all $0 < r \leq \rho$ and for all $f \in S_2$. Several works on radius problems can be found in [18, 21, 23]. In a recent paper, Ali et al. [4] obtained sharp radius estimates for functions $f \in \mathcal{A}$ satisfying certain conditions on the ratio f/g for a given $g \in \mathcal{A}$. The radii results presented here are nice extensions of Ali et al. [4] and the works of [2, 18, 21, 23] for functions with fixed second coefficient, and include the radii results for the classes of starlike functions of positive order, parabolic starlike functions, and the Sokół-Stankiewicz starlike functions.

2. Preliminaries

The results that are required in the present investigation are enlisted below:

Lemma 2.1.([11, Theorem 2]) *Let* $|b| \le 1$ *and* $0 \le \alpha < 1$*. If* $p \in \mathcal{P}_b(\alpha)$*, then, for* $|z| = r < 1$,

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{2(1-\alpha)r}{1-r^2}\frac{|b|r^2+2r+|b|}{(1-2\alpha)r^2+2(1-\alpha)|b|r+1}.$$

Lemma 2.2.([10, Lemma 1]) *Let* $|b| \le 1$ *and* $0 \le \alpha < 1$*. If* $p \in \mathcal{P}_b(\alpha)$*, then, for* $|z| = r < 1$, $|p(z) - C_b| \le D_b$*, where*

$$C_b = \frac{(1+|b|r)^2 + (1-2\alpha)(|b|+r)^2r^2}{(1+2|b|r+r^2)(1-r^2)}, \quad D_b = \frac{2(1-\alpha)(|b|+r)(1+|b|r)r}{(1+2|b|r+r^2)(1-r^2)}.$$

Lemma 2.3.([10, Theorem 1]) *Let* $|b| \le 1$ *and* $0 \le \alpha < 1$*. Suppose* $p \in \mathcal{P}_b(\alpha)$*. Then, for* $|z| = r < 1$,

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right) \ge \begin{cases} \frac{-2(1-\alpha)(|b|+2r+|b|r^2)r}{(1+2\alpha|b|r+(2\alpha-1)r^2)(1+2|b|r+r^2)}, & R' \le R_b; \\ (2\sqrt{\alpha C_1} - C_1 - \alpha)/(1-\alpha), & R' \ge R_b, \end{cases}$$

where $R_b = C_b - D_b, R' = \sqrt{\alpha C_1}$, C_b *and* D_b *are as given in Lemma .*

Lemma 2.4.([5, Theorem 5.1]) *If* $f(z) = z + a_2z^2 + \cdots \in \mathcal{K}$*, then* $f \in \mathcal{S}^*(\alpha)$*, where* α *is the smallest positive root of the equation* $2\alpha^3 - |a_2|\alpha^2 - 4\alpha + 2 = 0$*, in the interval* $[1/2, 2/3]$.

Lemma 2.5.([3, Lemma 2.2]) *For* $0 < a < \sqrt{2}$*, let* r_a *be given by*

$$r_a = \begin{cases} (\sqrt{1-a^2} - (1-a^2))^{1/2}, & 0 < a \le 2\sqrt{2}/3; \\ \sqrt{2} - a, & 2\sqrt{2}/3 \le a < \sqrt{2}, \end{cases}$$

and for $a > 0$*, let* R_a *be given by*

$$R_a = \begin{cases} \sqrt{2} - a, & 0 < a \le 1/\sqrt{2}; \\ a, & 1/\sqrt{2} \le a. \end{cases}$$

Then $\{w : |w-a| < r_a\} \subset \{w : |w^2 - 1| < 1\} \subset \{w : |w-a| < R_a\}$.

Lemma 2.6.([20, Section 3]) *Let $a > 1/2$. If the number R_a is given by*

$$R_a = \begin{cases} a - 1/2, & 1/2 < a \le 3/2; \\ \sqrt{2a-2}, & a \ge 3/2, \end{cases}$$

then $\{w \in \mathbb{C} : |w - a| < R_a\} \subset \{w \in \mathbb{C} : |w - a| < \operatorname{Re} w\}$.

3. Radius constants

Let

$$f(z) = z + a_2 z^2 + \cdots \tag{3.1}$$

and if $\operatorname{Re}(f(z)/z) > 0$, then $f(z)/z \in \mathcal{P}$ and hence $|a_2| \le 2$. So such functions can be given the series expansion: $f(z) = z + 2bz^2 + \cdots$, where $|b| \le 1$.

Definition 3.1. *For $|b| \le 1$, let $\mathcal{F}_b^1$ be the class of functions $f \in \mathcal{A}_{2b}$ such that* $\operatorname{Re}(f(z)/z) > 0$.

We now give below the radius constants pertaining to the class $\mathcal{F}_b^1$:

Theorem 3.2. *The sharp radius constants for the class $\mathcal{F}_b^1$ are enlisted below:*

1. *The $\mathcal{S}_L^*$−radius is the smallest positive root $r_0 \in (0,1)$ of*

$$(\sqrt{2} - 1)r^4 + 2\sqrt{2}|b|r^3 + 4r^2 + 2|b|(2 - \sqrt{2})r - \sqrt{2} + 1 = 0, \tag{3.2}$$

2. *The $\mathcal{M}(\beta)$−radius is the smallest positive root $r_1 \in (0,1)$ of*

$$(\beta - 1)r^4 + 2|b|\beta r^3 + 4r^2 + 2|b|(2 - \beta)r - \beta + 1 = 0. \tag{3.3}$$

3. *The $\mathcal{S}^*(\alpha)$−radius is the smallest positive root $r_2 \in (0,1)$ of*

$$(1 - \alpha)r^4 + 2|b|(2 - \alpha)r^3 + 4r^2 + 2|b|\alpha r + \alpha - 1 = 0. \tag{3.4}$$

4. *The $\mathcal{S}_P^*$−radius is the smallest positive root $r_3 \in (0,1)$ of*

$$r^4 + 6|b|r^3 + 8r^2 + 2|b|r - 1 = 0. \tag{3.5}$$

Proof. Clearly, the function $p(z) = f(z)/z = 1 + 2bz + \cdots \in \mathcal{P}_b$ and

$$\frac{zp'(z)}{p(z)} = \frac{zf'(z)}{f(z)} - 1.$$

Now by taking $\alpha = 0$ in Lemma 2.1, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{zp'(z)}{p(z)}\right| \le \frac{2r(|b|r^2 + 2r + |b|)}{(1 - r^2)(r^2 + 2|b|r + 1)}. \tag{3.6}$$

(1) From Lemma 2.5, we see that

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1$$

whenever the following inequality holds:

$$\frac{2r(|b|r^2 + 2r + |b|)}{(1-r^2)(r^2 + 2|b|r + 1)} \leq \sqrt{2} - 1,$$

which upon simplification, becomes

$$1 - \sqrt{2} + 2|b|\left(2 - \sqrt{2}\right) r + 4r^2 + 2\sqrt{2}|b|r^3 + \left(\sqrt{2} - 1\right) r^4 \leq 0.$$

Therefore, the $\mathcal{S}_L^*$-radius for the class $\mathcal{F}_b^1$, is the smallest positive root $r_0 \in (0,1)$ of (3.2).

To prove the sharpness, consider the function f_0 defined by

$$f_0(z) = \frac{z(1 + 2bz + z^2)}{1 - z^2} \tag{3.7}$$

together with $w(z) := z(z+b)/(1+bz)$. Then we see that

$$\frac{f_0(z)}{z} = \frac{1 + w(z)}{1 - w(z)},$$

where w is an analytic function satisfying the conditions of Schwarz's lemma in the unit disk $\mathbb{D}$, which leads to $\mathrm{Re}(f_0(z)/z) > 0$ in $\mathbb{D}$ and hence $f_0 \in \mathcal{F}_b^1$. Thus, for $z = r_0$, the root of (3.2), we have

$$\frac{zf_0'(z)}{f_0(z)} = \frac{1 + 4br_0 + 4r_0^2 - r_0^4}{(1 - r_0^2)(1 + 2br_0 + r_0^2)} = \sqrt{2},$$

it follows that

$$\left|\left(\frac{zf_0'(z)}{f_0(z)}\right)^2 - 1\right| = 1 \quad (z = r_0),$$

which establishes sharpness of the result.

(2) The inequality (3.6) shows that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq 1 + \frac{2r(|b|r^2 + 2r + |b|)}{(1-r^2)(r^2 + 2|b|r + 1)} \leq \beta,$$

if the following inequality

$$(\beta - 1)r^4 + 2|b|\beta r^3 + 4r^2 + 2(2-\beta)|b|r + 1 - \beta \leq 0$$

holds. Therefore, the $\mathcal{M}(\beta)$-radius of the class $\mathcal{F}_b^1$ is the smallest positive root $r_1 \in (0,1)$ of (3.3). The result is sharp due to the function given in (3.7) as, for $z = r_1$, the root of (3.3), we see that

$$\frac{zf_0'(z)}{f_0(z)} = \frac{1+4br_1+4r_1^2-r_1^4}{(1-r_1^2)\,(1+2br_1+r_1^2)} = \beta.$$

(3) In view of (3.6), it follows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq 1 - \frac{2r(|b|r^2+2r+|b|)}{(1-r^2)(r^2+2|b|r+1)} \geq \alpha,$$

whenever the following inequality

$$(1-\alpha)r^4 + 2|b|(2-\alpha)r^3 + 4r^2 + 2|b|\alpha r + \alpha - 1 \leq 0$$

holds. Thus, the $\mathcal{S}^*(\alpha)-$radius of the class $\mathcal{F}_b^1$ is the smallest positive root $r_2 \in (0,1)$ of (3.4).

The function f_0 defined by

$$f_0(z) = \frac{z(1-z^2)}{1-2bz+z^2} \tag{3.8}$$

is in the class $\mathcal{F}_b^1$ because for the function f_0 defined in (3.8), we have $f_0(z)/z = (1-w(z))/(1+w(z))$, where $w(z) = z(z-b)/(1-bz)$ is an analytic function satisfying the conditions of Schwarz's lemma in the unit disk $\mathbb{D}$, and hence $\operatorname{Re}(f_0(z)/z) > 0$ in $\mathbb{D}$. The result is sharp for the function given in (3.8) as, for $z = -r_2$, the root of (3.4), we have

$$\operatorname{Re}\left(\frac{zf_0'(z)}{f_0(z)}\right) = \frac{zf_0'(z)}{f_0(z)} = \frac{1-r_2^2\left(4-4br_2+r_2^2\right)}{(1-r_2^2)\,(1-2br_2+r_2^2)} = \alpha,$$

which demonstrates sharpness.

(4) Lemma 2.6 shows that the disk (3.6) lies inside the parabolic region $\Omega = \{w : |w-1| < \operatorname{Re} w\}$ provided that

$$\frac{2r(|b|r^2+2r+|b|)}{(1-r^2)(r^2+2|b|r+1)} \leq \frac{1}{2},$$

or equivalently, if the inequality $r^4 + 6|b|r^3 + 8r^2 + 2|b|r - 1 \leq 0$ holds. Thus, the $\mathcal{S}_P^*-$radius of the class $\mathcal{F}_b^1$ is the smallest positive root $r_3 \in (0,1)$ of (3.5).

The function defined in (3.8), for $z = -r_3$ satisfies

$$\frac{zf_0'(z)}{f_0(z)} = \frac{1-r_3^2\left(4-4br_3+r_3^2\right)}{(1-r_3^2)\,(1-2br_3+r_3^2)} = \frac{1}{2},$$

which demonstrates sharpness. The following figures illustrate sharpness of the result. □

Remark 3.3. For $\alpha = 0$, part (3) of Theorem 3.1 reduces to the result [8, Theorem 2] of Goel.

Let

$$g(z) = z + g_2 z^2 + \cdots \tag{3.9}$$

and assume that $g(z)/z \in \mathcal{P}$. Let f be given by (3.1) and $\mathrm{Re}(f(z)/g(z)) > 0$. Then we have $|a_2| \le |g_2| + 2 \le 4$. Our next theorem focuses on the class of functions involving these functions f and g with fixed second coefficients, whose series expansions are given respectively by $f(z) = z + 4bz^2 + \cdots$ and $g(z) = z + 2cz^2 + \cdots$, where $|b| \le 1$ and $|c| \le 1$.

Definition 3.4. *For $|b| \le 1$ and $|c| \le 1$, let*

$$\mathcal{F}^2_{b,c} := \left\{ f \in \mathcal{A}_{4b} : \mathrm{Re}\left(\frac{f(z)}{g(z)}\right) > 0 \text{ and } \mathrm{Re}\left(\frac{g(z)}{z}\right) > 0, \text{ where } g \in \mathcal{A}_{2c} \right\}.$$

Here below, we furnish the radius constants for the class $\mathcal{F}^2_{b,c}$:

Theorem 3.5. *Assume that $\gamma := |2b - c|$. Then the sharp radius constants for the class $\mathcal{F}^2_{b,c}$ are enlisted below:*

(1) *The $\mathcal{S}^*_L$−radius is the smallest positive root $r_0 \in (0,1)$ of*

$$\begin{aligned}(\sqrt{2}-1)r^6 &+ (|c|+\gamma)2\sqrt{2}r^5 + (7+\sqrt{2}+4(1+\sqrt{2})|c|\gamma)r^4 + 12(|c|+\gamma)r^3 \\ &+ (9-\sqrt{2}+4(3-\sqrt{2})|c|\gamma)r^2 + 2(2-\sqrt{2})(|c|+\gamma)r - \sqrt{2} + 1 = 0.\end{aligned} \tag{3.10}$$

(2) *The $\mathcal{M}(\beta)$−radius is the smallest positive root $r_1 \in (0,1)$ of*

$$\begin{aligned}(\beta-1)r^6 &+ 2\beta(|c|+\gamma)r^5 + (7+\beta+4(1+\beta)|c|\gamma)r^4 + 12(|c|+\gamma)r^3 \\ &+ (9-\beta+4(3-\beta)|c|\gamma)r^2 + 2(2-\beta)(|c|+\gamma)r - \beta + 1 = 0.\end{aligned} \tag{3.11}$$

(3) *The $\mathcal{S}^*(\alpha)$−radius is the smallest positive root $r_2 \in (0,1)$ of*

$$\begin{aligned}(1-\alpha)r^6 &+ 2(2-\alpha)(|c|+\gamma)r^5 + (9-\alpha+4(3-\alpha)|c|\gamma)r^4 + 12(|c|+\gamma)r^3 \\ &+ (7+\alpha+4(1+\alpha)|c|\gamma)r^2 + 2(|c|+\gamma)\alpha r + \alpha - 1 = 0.\end{aligned} \tag{3.12}$$

(4) *The $\mathcal{S}^*_P$−radius is the smallest positive root $r_3 \in (0,1)$ of*

$$r^6 + 6(|c|+\gamma)r^5 + (17+20\gamma|c|)r^4 + 24(|c|+\gamma)r^3 + (15+12\gamma|c|)r^2 + 2(|c|+\gamma)r - 1 = 0. \tag{3.13}$$

Proof. Let the functions p and h be defined by $p(z) = g(z)/z$, and $h(z) = f(z)/g(z)$. Then

$$p(z) = 1 + 2cz + \cdots \quad \text{and} \quad h(z) = 1 + 2(2b - c)z + \cdots$$

or $p \in \mathcal{P}_c$ and $h \in \mathcal{P}_{2b-c}$. Since $f(z) = zp(z)h(z)$, from Lemma 2.1 with $\alpha = 0$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\frac{zp'(z)}{p(z)}\right| + \left|\frac{zh'(z)}{h(z)}\right| \leq \frac{2r}{1-r^2}\left(\frac{|c|r^2 + 2r + |c|}{r^2 + 2|c|r + 1} + \frac{\gamma r^2 + 2r + \gamma}{r^2 + 2\gamma r + 1}\right). \tag{3.14}$$

(1) By Lemma 2.5, the function f satisfies $\left|(zf'(z)/f(z))^2 - 1\right| < 1$, for $|z| < r$, if the following inequality holds

$$\frac{2r}{1-r^2}\left(\frac{|c|r^2 + 2r + |c|}{r^2 + 2|c|r + 1} + \frac{\gamma r^2 + 2r + \gamma}{r^2 + 2\gamma r + 1}\right) \leq \sqrt{2} - 1$$

or equivalently, if the following inequality holds:

$$\begin{aligned}(\sqrt{2} - 1)r^6 + (|c| + \gamma)2\sqrt{2}r^5 + (7 + \sqrt{2} + 4(1 + \sqrt{2})|c|\gamma)r^4 + 12(|c| + \gamma)r^3 \\ + (9 - \sqrt{2} + 4(3 - \sqrt{2})|c|\gamma)r^2 + 2(2 - \sqrt{2})(|c| + \gamma)r - \sqrt{2} + 1 \leq 0.\end{aligned}$$

Therefore, the $\mathcal{S}^*_L$-radius of the class $\mathcal{F}^2_{b,c}$ is the smallest positive root $r_0 \in (0, 1)$ of (3.10).

Consider the functions defined by

$$f_0(z) = \frac{z\left(1 + (4b - 2c)z + z^2\right)\left(1 + 2cz + z^2\right)}{(1 - z^2)^2} \quad \text{and} \quad g_0(z) = \frac{z\left(1 + 2cz + z^2\right)}{(1 - z^2)}. \tag{3.15}$$

The function f_0 with the choice of g_0, defined above, is in the class $\mathcal{F}^2_{b,c}$ because

$$\frac{f_0(z)}{g_0(z)} = \frac{1 + w_1(z)}{1 - w_1(z)} \quad \text{and} \quad \frac{g_0(z)}{z} = \frac{1 + w_2(z)}{1 - w_2(z)},$$

where $w_1(z) = z(z + 2b - c)/(1 + (2b - c)z)$ with $|2b - c| \leq 1$ and $w_2(z) = z(z + c)/(1 + cz)$ are analytic functions satisfying the conditions of Schwarz's lemma in the unit disk $\mathbb{D}$, and hence $\operatorname{Re}(g_0(z)/z) > 0$ and $\operatorname{Re}(f_0(z)/g_0(z)) > 0$ in $\mathbb{D}$. Since

$$\frac{zf_0'(z)}{f_0(z)} = 1 + \frac{2}{1 - r_0} + \frac{2}{1 + r_0} - \frac{2(1 + cr_0)}{1 + 2cr_0 + r_0^2} - \frac{2 + 4br_0 - 2cr_0}{1 + r_0(4b - 2c + r_0)} = \sqrt{2}, \quad (z = r_0), \tag{3.16}$$

we have

$$\left|\left(\frac{zf_0(z)}{f_0(z)}\right)^2 - 1\right| = 1.$$

Thus, the result is sharp.

(2) The inequality (3.14) shows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \le 1+\frac{2r}{1-r^2}\left(\frac{|c|r^2+2r+|c|}{r^2+2|c|r+1}+\frac{\gamma r^2+2r+\gamma}{r^2+2\gamma r+1}\right) \le \beta,$$

if the following inequality holds:

$$(\beta-1)r^6+2\beta(|c|+\gamma)r^5+(7+\beta+4(1+\beta)|c|\gamma)r^4+12(|c|+\gamma)r^3 \\ +(9-\beta+4(3-\beta)|c|\gamma)r^2+2(2-\beta)(|c|+\gamma)r-\beta+1\le 0.$$

Hence the $\mathcal{M}(\beta)$-radius of the class $\mathcal{F}^2_{b,c}$ is the smallest positive root $r_1 \in (0,1)$ of (3.11). The result is sharp due to the functions given in (3.15) as it can be seen for $z = r_1$

$$\frac{zf_0'(z)}{f_0(z)} = 1+\frac{2}{1-r_1}+\frac{2}{1+r_1}-\frac{2(1+cr_1)}{1+2cr_1+r_1^2}-\frac{2+4br_1-2cr_1}{1+r_1(4b-2c+r_1)}=\beta. \tag{3.17}$$

(3) In view of (3.14), it follows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge 1-\frac{2r}{1-r^2}\left(\frac{|c|r^2+2r+|c|}{r^2+2|c|r+1}+\frac{\gamma r^2+2r+\gamma}{r^2+2\gamma r+1}\right) \ge \alpha,$$

if the following inequality holds:

$$(1-\alpha)r^6+2(2-\alpha)(|c|+\gamma)r^5+(9-\alpha+4(3-\alpha)|c|\gamma)r^4+12(|c|+\gamma)r^3 \\ +(7+\alpha+4(1+\alpha)|c|\gamma)r^2+2(|c|+\gamma)\alpha r+\alpha-1\le 0.$$

Thus, the $\mathcal{S}^*(\alpha)-$radius of the class $\mathcal{F}^2_{b,c}$ is the smallest positive root $r_2 \in (0,1)$ of (3.12).

Consider the functions defined by

$$f_0(z)=\frac{z\left(1-z^2\right)^2}{(1-(4b-2c)z+z^2)\left(1-2cz+z^2\right)} \quad\text{and}\quad g_0(z)=\frac{z\left(1-z^2\right)}{(1-2cz+z^2)}. \tag{3.18}$$

The function f_0, with the choice of g_0 defined in (3.18), is in the class $\mathcal{F}^2_b$ because

$$\frac{f_0(z)}{g_0(z)}=\frac{1-w_1(z)}{1+w_1(z)} \quad\text{and}\quad \frac{g_0(z)}{z}=\frac{1-w_2(z)}{1+w_2(z)},$$

where $w_1(z) = z(z-(2b-c))/(1-(2b-c)z)$ with $|2b-c| \le 1$ and $w_2(z) = z(z-c)/(1-cz)$ are analytic functions satisfying the conditions of Schwarz's lemma in the unit disk $\mathbb{D}$, and hence $\operatorname{Re}(g_0(z)/z) > 0$ and $\operatorname{Re}(f_0(z)/g_0(z)) > 0$ in $\mathbb{D}$. The functions defined in (3.18) satisfy

$$\frac{zf_0'(z)}{f_0(z)}=1-\frac{2}{1+r_2}-\frac{2}{1-r_2}+\frac{2+2cr_2}{1+2cr_2+r_2^2}+\frac{2(1+2br_2-cr_2)}{1+r_2(4b-2c+r_2)}=\alpha \quad (z=-r_2),$$

which demonstrates the sharpness.

(4) By Lemma 2.6, the disk (3.14) lies inside the parabolic region $\Omega = \{w : |w-1| < \operatorname{Re} w\}$ provided

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{2r}{1-r^2}\left(\frac{|c|r^2+2r+|c|}{r^2+2|c|r+1} + \frac{\gamma r^2+2r+\gamma}{r^2+2\gamma r+1}\right) \leq \frac{1}{2}.$$

or equivalently, if $r^6+6(|c|+\gamma)r^5+(17+20\gamma|c|)r^4+24(|c|+\gamma)r^3+(15+12\gamma|c|)r^2+2(|c|+\gamma)r-1 \leq 0$. Thus, the $\mathcal{S}^*_P-$radius of the class $\mathcal{F}^2_b$ is the smallest positive root $r_3 \in (0,1)$ of (3.13). The functions defined in (3.18) satisfy, for $z=-r_3$,

$$\frac{zf_0'(z)}{f_0(z)} = 1 - \frac{2}{1+r_3} - \frac{2}{1-r_3} + \frac{2(1+cr_3)}{1+2cr_3+r_3^2} + \frac{2(1+2br_3-cr_3)}{1+r_3(4b-2c+r_3)} = \frac{1}{2},$$

which demonstrates sharpness. □

Remark 3.6. Setting $b=1=c$, in Theorem 3.5, we obtain the result [4, Theorem 2.1] of Ali et al.

Let the functions f and g be given by (3.1) and (3.9) respectively. Assume that f and g are satisfying $\operatorname{Re}(f(z)/g(z)) > 0$ and $\operatorname{Re}(g(z)/z) > 1/2$ in $\mathbb{D}$. Then we have $|a_2| \leq |g_2|+2 \leq 3$. In the following theorem we shall discuss some radius problems for functions with fixed second coefficients whose series expansion are given respectively by $f(z) = z+3bz^2+\cdots$ and $g(z) = z+cz^2+\cdots$ with $|b| \leq 1$ and $|c| \leq 1$.

Definition 3.7. *For $|b| \leq 1$ and $|c| \leq 1$, let*

$$\mathcal{F}^3_{b,c} := \left\{f \in \mathcal{A}_{3b} : \operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0,\ \textit{and}\ \operatorname{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2},\ \textit{where}\ g \in \mathcal{A}_c\right\}.$$

Now the radius constants for the class $\mathcal{F}^3_{b,c}$ are established in the following result.

Theorem 3.8. *Assume that $\gamma_1 = |3b-c|$. For the class $\mathcal{F}^3_{b,c}$,*

1. *the $\mathcal{S}^*_L-$radius is the smallest positive root $r_0 \in (0,1)$ of*

$$\begin{aligned}&\sqrt{2}|c|r^5+(1+\sqrt{2})(1+|c|\gamma_1)r^4+(6|c|+\sqrt{2}(1+\sqrt{2})\gamma_1)r^3\\ &+(6+(3-\sqrt{2})|c|\gamma_1)r^2+\sqrt{2}(\sqrt{2}-1)(|c|+\gamma_1)r-\sqrt{2}+1=0\end{aligned} \tag{3.19}$$

   and it is sharp.

2. *the $\mathcal{M}(\beta)-$radius is the smallest positive root $r_1 \in (0,1)$ of*

$$\begin{aligned}&|c|\beta r^5+(1+\beta)(1+|c|\gamma_1)r^4+(6|c|+(2+\beta)\gamma_1)r^3\\ &+(6+(3-\beta)|c|\gamma_1)r^2+(2-\beta)(|c|+\gamma_1)r-\beta+1=0\end{aligned} \tag{3.20}$$

   and it is sharp.

3. *the* $\mathcal{S}^*(\alpha)-$*radius is the smallest positive root* $r_2 \in (0,1)$ *of*

$$\begin{aligned}&-|c|\alpha r^7 + (|c|(1-\alpha)(\gamma_1+2|c|) - 1 - \alpha)r^6 + (|c|(2-\alpha)(3+2|c|\gamma_1) - \alpha\gamma_1)r^5\\ &\quad + (5+8|c|^2 - \alpha + 2(3-\alpha)|c|\gamma_1)r^4 + ((12+\alpha)|c| + 2(2+|c|^2\alpha)\gamma_1)r^3\\ &\quad + (5-2|c|^2+\alpha+2|c|^2\alpha + (1+3\alpha)|c|\gamma_1)r^2 + (2|c|+3|c|\alpha+\alpha\gamma_1)r\\ &\quad + \alpha - 1 = 0. \end{aligned} \tag{3.21}$$

4. *the* $\mathcal{S}^*_P-$*radius is the smallest positive root* $r_3 \in (0,1)$ *of*

$$\begin{aligned}&|c|r^7 + (1+4|c|^2+3|c|\gamma_1)r^6 + (17|c|+3\gamma_1+8|c|^2\gamma_1)r^5 + (13+20|c|^2+16|c|\gamma_1)r^4\\ &\quad + (31|c|+8\gamma_1+4|c|^2\gamma_1)r^3 + (11+5|c|\gamma_1)r^2 + (\gamma_1-|c|)r - 1 = 0.\end{aligned} \tag{3.22}$$

Proof. Define the functions p and h by $p(z) = g(z)/z$ and $h(z) = f(z)/g(z)$

$$p(z) = 1 + cz + \cdots \quad \text{and} \quad h(z) = \frac{f(z)}{g(z)} = 1 + (3b-c)z + \cdots \tag{3.23}$$

or $p \in \mathcal{P}_{c/2}(1/2)$ and $h \in \mathcal{P}_{(3b-c)/2}$. Lemma 2.1 with $\alpha = 0$ and $\alpha = 1/2$ respectively lead to

$$\left|\frac{zh'(z)}{h(z)}\right| \le \frac{r}{1-r^2}\frac{\gamma_1 r^2 + 4r + \gamma_1}{r^2+\gamma_1 r + 1} \quad \text{and} \quad \left|\frac{zp'(z)}{p(z)}\right| \le \frac{r}{1-r^2}\frac{|c|r^2+2r+|c|}{|c|r+1}. \tag{3.24}$$

From (3.23), $f(z)/z = p(z)h(z)$, and so the inequalities in (3.24) yields

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &\le \left|\frac{zh'(z)}{h(z)}\right| + \left|\frac{zp'(z)}{p(z)}\right|\\ &\le \frac{r}{1-r^2}\left(\frac{\gamma_1 r^2+4r+\gamma_1}{r^2+\gamma_1 r+1} + \frac{|c|r^2+2r+|c|}{|c|r+1}\right).\end{aligned} \tag{3.25}$$

(1) By Lemma 2.5, the function f satisfies $\left|(zf'(z)/f(z))^2 - 1\right| < 1$, for $|z| < r$, if the following inequality holds

$$\frac{r}{1-r^2}\left(\frac{\gamma_1 r^2+4r+\gamma_1}{r^2+\gamma_1 r+1} + \frac{|c|r^2+2r+|c|}{|c|r+1}\right) \le \sqrt{2} - 1$$

or equivalently, if the following inequality holds:

$$\begin{aligned}&\sqrt{2}|c|r^5 + (1+\sqrt{2})(1+|c|\gamma_1)r^4 + (6|c| + \sqrt{2}(1+\sqrt{2})\gamma_1)r^3\\ &\quad + (6+(3-\sqrt{2})|c|\gamma_1)r^2 + \sqrt{2}(\sqrt{2}-1)(|c|+\gamma_1)r - \sqrt{2} + 1 \le 0.\end{aligned}$$

Therefore, the $\mathcal{S}_L^*$-radius of the class $\mathcal{F}_{b,c}^3$ is the smallest positive root $r_0 \in (0,1)$ of (3.19). Consider the functions defined by

$$f_0(z) = \frac{z(1+(3b-c)z+z^2)(1+cz)}{(1-z^2)^2} \quad \text{and} \quad g_0(z) = \frac{z(1+cz)}{(1-z^2)}. \tag{3.26}$$

The function f_0 with the choice g_0, defined in (3.26), is in the class $\mathcal{F}_{b,c}^3$ because

$$\frac{f_0(z)}{g_0(z)} = \frac{1+w_1(z)}{1-w_1(z)} \quad \text{and} \quad \frac{g_0(z)}{z} = \frac{1+w_2(z)}{1-w_2(z)},$$

where $w_1(z) = z(z+(3b-c)/2)/(1+((3b-c)z/2)$ with $|3b-c| \leq 2$ and $w_2(z) = z(z+c/2)/(1+cz/2)$ are analytic functions satisfying the conditions of Schwarz's lemma in the unit disk $\mathbb{D}$, and hence $\operatorname{Re}(g_0(z)/z) > 1/2$ and $\operatorname{Re}(f_0(z)/g_0(z)) > 0$ in $\mathbb{D}$. Since

$$\frac{zf_0'(z)}{f_0(z)} = \frac{2}{1-r_0} + \frac{2}{1+r_0} - \frac{1}{1+cr_0} - \frac{2+3br_0-cr_0}{1+r_0(3b-c+r_0)} = \sqrt{2},$$

for $z = r_0$, the root of (3.19), we have $|(zf_0(z)/f_0(z))^2 - 1| = 1$. Thus, the result is sharp.

(2) The inequality (3.25) shows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq \frac{r}{1-r^2}\left(\frac{\gamma_1 r^2+4r+\gamma_1}{r^2+\gamma_1 r+1} + \frac{|c|r^2+2r+|c|}{|c|r+1}\right) + 1 \leq \beta,$$

if the following inequality holds:

$$\begin{aligned}&\beta|c|r^5 + (1+\beta)(1+|c|\gamma_1)r^4 + (6|c| + (2+\beta)\gamma_1)r^3\\ &\quad + (6+(3-\beta)|c|\gamma_1)r^2 + (2-\beta)(|c|+\gamma_1)r - \beta + 1 \leq 0.\end{aligned}$$

Therefore the $\mathcal{M}(\beta)-$radius of the class $\mathcal{F}_{b,c}^3$ is the smallest positive root $r_1 \in (0,1)$ of (3.20). The result is sharp for the functions given in (3.26) as it can be seen that, for $z = r_1$, the root of (3.20), we have

$$\frac{zf_0'(z)}{f_0(z)} = \frac{2}{1-r_1} + \frac{2}{1+r_1} - \frac{1}{1+cr_1} - \frac{2+3br_1-cr_1}{1+r_1(3b-c+r_1)} = \beta.$$

(3) Since $f(z)/z = p(z)h(z)$, it follows from Lemma 2.1 and Lemma 2.3 that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq 1 - \frac{(\gamma_1 r^2+4r+\gamma_1)r}{(r^2+\gamma_1 r+1)(1-r^2)} + \frac{(|c|+2r+|c|r^2)r}{(1+2|c|r+r^2)(1+|c|r)} \geq \alpha, \tag{3.27}$$

if the following inequality holds:

$$\begin{aligned}&-|c|\alpha r^7 + (|c|(1-\alpha)(\gamma_1+2|c|) - 1 - \alpha)r^6 + (|c|(2-\alpha)(3+2|c|\gamma_1) - \alpha\gamma_1)r^5\\ &\quad + (5+8|c|^2 - \alpha + 2(3-\alpha)|c|\gamma_1)r^4 + ((12+\alpha)|c| + 2(2+|c|^2\alpha)\gamma_1)r^3\\ &\quad + (5-2|c|^2+\alpha+2|c|^2\alpha + (1+3\alpha)|c|\gamma_1)r^2 + (2|c|+3|c|\alpha+\alpha\gamma_1)r + \alpha - 1 \leq 0.\end{aligned}$$

Thus, the $\mathcal{S}^*(\alpha)-$radius of the class $\mathcal{F}^3_{b,c}$ is the smallest positive root $r_2 \in (0,1)$ of (3.21).

(4) From (3.25) and (3.27), it is clear that $|(zf'(z)/f(z))-1| < \operatorname{Re}(zf'(z)/f(z))$ provided

$$\begin{aligned}1 &- \frac{(\gamma_1 r^2+4r+\gamma_1)r}{(r^2+\gamma_1 r+1)(1-r^2)} + \frac{(|c|+2r+|c|r^2)r}{(1+2|c|r+r^2)(1+|c|r)}\\ &\geq \frac{r}{1-r^2}\left(\frac{\gamma_1 r^2+4r+\gamma_1}{r^2+\gamma_1 r+1} + \frac{|c|r^2+2r+|c|}{|c|r+1}\right)\end{aligned}$$

or equivalently, if the following inequality holds:

$$\begin{aligned}&|c|r^7+(1+4|c|^2+3|c|\gamma_1)r^6+(17|c|+3\gamma_1+8|c|^2\gamma_1)r^5+(13+20|c|^2+16|c|\gamma_1)r^4\\ &\quad+(31|c|+8\gamma_1+4|c|^2\gamma_1)r^3+(11+5|c|\gamma_1)r^2+(\gamma_1-|c|)r-1\leq 0.\end{aligned}$$

Thus the $\mathcal{S}^*_P-$radius of the class $\mathcal{F}^3_{b,c}$ is the smallest positive root r_3 in $(0,1)$ of (3.22). ☐

Remark 3.9. Putting $b = 1 = c$ in Theorem 3.8, we obtain the result [4, Theorem 2.2] of Ali et al.

Now consider the functions f and g, given by (3.1) and (3.9) respectively. Suppose that f and g satisfy the conditions $|f(z)/g(z)-1| < 1$ and $\operatorname{Re}(g(z)/z) > 0$ in $\mathbb{D}$. Then it follows that $|a_2| \leq |g_2|+2 \leq 3$. Thus such functions with fixed second coefficient, satisfying the above conditions can have the series expansion namely $f(z) = z+3bz^2+\cdots$ and $g(z) = z+2cz^2+\cdots$ with $|b| \leq 1$ and $|c| \leq 1$.

Definition 3.10. *For $|b| \leq 1$ and $|c| \leq 1$, let*

$$\mathcal{F}^4_{b,c} := \left\{ f\in\mathcal{A}_{3b} : \left|\frac{f(z)}{g(z)}-1\right| < 1, \textit{and } \operatorname{Re}\left(\frac{g(z)}{z}\right) > 0,\ \textit{where } g\in\mathcal{A}_{2c}\right\}.$$

Now in the following result, we provide the radius constants for the class $\mathcal{F}^3_{b,c}$.

Theorem 3.11. *Assume that $\delta := |2c-3b|$. For the class $\mathcal{F}^4_{b,c}$,*

1. *the $\mathcal{S}^*_L-$radius is the smallest positive root $r_0 \in (0,1)$ of*

$$\begin{aligned}&\sqrt{2}\delta r^5+(1+\sqrt{2})(1+2\delta|c|))r^4+2(3\delta+\sqrt{2}(1+\sqrt{2})|c|)r^3\\ &\quad+2(3+(3-\sqrt{2})\delta|c|)r^2+(2-\sqrt{2})(\delta+2|c|)r-\sqrt{2}+1=0.\end{aligned}\tag{3.28}$$

2. *the $\mathcal{M}(\beta)-$radius is the smallest positive root $r_1 \in (0,1)$ of*

$$\begin{aligned}&\beta\delta r^5+(1+\beta)(1+2\delta|c|)r^4+2(3\delta+2|c|+|c|\beta)r^3\\ &\quad+2(3+(3-\beta)\delta|c|)r^2+(2-\beta)(\delta+2|c|)r-\beta+1=0.\end{aligned}\tag{3.29}$$

3. *the* $f \in \mathcal{S}^*(\alpha)-$*radius is the smallest positive root* $r_2 \in (0,1)$ *of*

$$\begin{aligned}&(2-\alpha)\delta r^5 + (1+2\delta|c|)(3-\alpha)r^4 + 2(3\delta + 4|c| - |c|\alpha)r^3 \\ &\quad + (\delta + 2|c|)\alpha r^2 + 2(3 + (1+\alpha)\delta|c|)r + \alpha - 1 = 0.\end{aligned} \tag{3.30}$$

4. *the* $\mathcal{S}^*_P-$*radius is the smallest positive root* $r_3 \in (0,1)$ *of*

$$3\delta r^5 + 5(1+2\delta|c|)r^4 + 2(6\delta+7|c|)r^3 + 6(2+\delta|c|)r^2 + (\delta+2|c|)r - 1 = 0. \tag{3.31}$$

Proof. It is easy to see that $|f(z)/g(z) - 1| < 1$ if and only if $\operatorname{Re}(g(z)/f(z)) > 1/2$. Define the functions p and h by $p(z) = g(z)/z$, and $h(z) = g(z)/f(z)$. Then

$$p(z) = 1 + 2cz + \cdots \quad \text{and} \quad h(z) = \frac{g(z)}{f(z)} = 1 + (2c - 3b)z + \cdots$$

or $p \in \mathcal{P}_c$ and $h \in \mathcal{P}_{(2c-3b)/2}(1/2)$. Lemma 2.1 with $\alpha = 0$ and $\alpha = 1/2$ respectively lead to

$$\left|\frac{zh'(z)}{h(z)}\right| \le \frac{2r(|c|r^2 + 2r + |c|)}{(1-r^2)(r^2 + 2|c|r + 1)} \quad \text{and} \quad \left|\frac{zp'(z)}{p(z)}\right| \le \frac{r(\delta r^2 + 2r + \delta)}{(1-r^2)(\delta r + 1)} \tag{3.32}$$

respectively, where $\delta := |2c - 3b|$. Since $zp(z) = f(z)h(z)$, from (3.32), we have

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &\le \left|\frac{zp'(z)}{p(z)}\right| + \left|\frac{zh'(z)}{h(z)}\right| \\ &\le \frac{r}{1-r^2}\left(\frac{2(|c|r^2 + 2r + |c|)}{r^2 + 2qr + 1} + \frac{(\delta r^2 + 2r + \delta)}{\delta r + 1}\right).\end{aligned} \tag{3.33}$$

(1) By Lemma 2.5, the function f satisfies $\left|(zf'(z)/f(z))^2 - 1\right| < 1$, if the following inequality holds:

$$\frac{r}{1-r^2}\left(\frac{2(|c|r^2 + 2r + |c|)}{r^2 + 2qr + 1} + \frac{(\delta r^2 + 2r + \delta)}{\delta r + 1}\right) \le \sqrt{2} - 1,$$

or equivalently, if

$$\begin{aligned}&\sqrt{2}\delta r^5 + (1+\sqrt{2})(1 + 2\delta|c|))r^4 + 2(3\delta + \sqrt{2}(1+\sqrt{2})|c|)r^3 \\ &\quad + 2(3 + (3-\sqrt{2})\delta|c|)r^2 + (2-\sqrt{2})(\delta + 2|c|)r - \sqrt{2} + 1 \le 0.\end{aligned}$$

Therefore the $\mathcal{S}^*_L-$radius of the class $\mathcal{F}^4_{b,c}$ is the smallest positive root $r_0 \in (0,1)$ of (3.28).

(2) Using (3.33), we get

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \le 1 + \frac{r}{1-r^2}\left(\frac{2(|c|r^2 + 2r + |c|)}{r^2 + 2qr + 1} + \frac{(\delta r^2 + 2r + \delta)}{\delta r + 1}\right) \le \beta$$

if the following inequality holds:

$$\beta\delta r^5+(1+\beta)(1+2\delta|c|)r^4+2(3\delta+2|c|+|c|\beta)r^3 \\ +2(3+(3-\beta)\delta|c|)r^2+(2-\beta)(\delta+2|c|)r-\beta+1\leq 0.$$

Therefore, the $\mathcal{M}(\beta)-$radius of the class $\mathcal{F}^4_{b,c}$ is the smallest positive root $r_1\in(0,1)$ of (3.29).

(3) Inequality in (3.33) implies that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)\geq 1-\frac{r}{1-r^2}\left(\frac{2(|c|r^2+2r+|c|)}{r^2+2qr+1}+\frac{(\delta r^2+2r+\delta)}{\delta r+1}\right)\geq\alpha$$

if the following inequality holds:

$$(2-\alpha)\delta r^5+(1+2\delta|c|)(3-\alpha)r^4+2(3\delta+4|c|-|c|\alpha)r^3 \\ +(\delta+2|c|)\alpha r^2+2(3+(1+\alpha)\delta|c|)r+\alpha-1\leq 0.$$

Thus, the $\mathcal{S}^*(\alpha)-$radius of the class $\mathcal{F}^4_{b,c}$ is the smallest positive root in $r_2\in(0,1)$ of (3.30).

(4) Lemma 2.6 shows that the disk (3.33) lies inside $\Omega=\{w:|w-1|<\operatorname{Re}w\}$, the parabolic region, provided

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\frac{r}{1-r^2}\left(\frac{2(|c|r^2+2r+|c|)}{r^2+2qr+1}+\frac{(\delta r^2+2r+\delta)}{\delta r+1}\right)\leq\frac{1}{2}$$

if the following inequality holds:

$$3\delta r^5+5(1+2\delta|c|)r^4+2(6\delta+7|c|)r^3+6(2+\delta|c|)r^2+(\delta+2|c|)r-1\leq 0.$$

Therefore, the $\mathcal{S}^*_P-$radius of the class $\mathcal{F}^4_{b,c}$ is the smallest positive root $r_3\in(0,1)$ of (3.31). □

Remark 3.12. *Note that, in addition, we obtain the following sharp results of [4, Theorem 2.3] of Ali et al. as special case to parts (3) and (4) of Theorem 3.11 when* $b=1=c$.

For the class $\mathcal{F}^4_{1,1}$,

(1) *the* $\mathcal{S}^*_L-$*radius,* $r_0=\frac{2(2-\sqrt{2})}{\sqrt{2}(\sqrt{17-4\sqrt{2}}+3)}$,

(2) *the* $\mathcal{M}(\beta)-$*radius,* $r_1=\frac{2(\beta-1)}{3+\sqrt{9+4\beta(\beta-1)}}$,

(3) *the sharp* $f\in\mathcal{S}^*(\alpha)-$*radius,* $r_2=\frac{2(1-\alpha)}{3+\sqrt{9+4\beta(1-\alpha)(2-\alpha)}}$,

(4) *the sharp* $\mathcal{S}^*_P-$*radius,* $r_3=\frac{2\sqrt{3}-3}{3}$.

Consider the functions f and g given by (3.1) and (3.9) respectively. Further assume that f and g satisfy the condition $|f(z)/g(z)-1|<1$ and g is a convex function in the unit disk $\mathbb{D}$. Then we have $|a_2|\leq|g_2|+1\leq 2$. In the next theorem, we consider such functions with fixed second coefficient, whose series expansion are given by $f(z)=z+2bz^2+\cdots$ and $g(z)=z+cz^2+\cdots$ with $|b|\leq 1$ and $|c|\leq 1$.

Definition 3.13. *For $|b|\leq 1$ and $|c|\leq 1$, let*

$$\mathcal{F}^5_{b,c}:=\left\{f\in\mathcal{A}_{2b}:\operatorname{Re}\left(\frac{f(z)}{g(z)}\right)>0,\ \text{where } g\in\mathcal{A}_c\cap\mathcal{K}=\mathcal{K}_c\right\}.$$

We now obtain the radius constants for the class $\mathcal{F}^3_{b,c}$ in the following result.

Theorem 3.14. *Assume that $\delta_1:=|c-2b|$. For the class $\mathcal{F}^5_{b,c}$,*

1. *the $\mathcal{S}^*(\lambda)-$radius is the smallest root $r_0\in(0,1)$ of*

$$\begin{aligned}&(\delta_1+\beta_0\delta_1-\delta_1\lambda)r^5+(2+\beta_0+3|c|\delta_1+|c|\beta_0\delta_1-\lambda-2|c|\delta_1\lambda)r^4\\&\quad+(5|c|+|c|\beta_0+3\delta_1-\beta_0\delta_1-2|c|\lambda)r^3+(3-\beta_0+(1-\beta_0+2\lambda)\delta_1|c|)r^2\\&\quad+(2|c|\lambda+\delta_1\lambda-|c|-|c|\beta_0)r+\lambda-1=0,\end{aligned}\tag{3.34}$$

where $\beta_0=2\alpha_0-1$ and $\alpha_0\in(0,1)$ is the smallest positive root of the equation

$$2\alpha^3-q\alpha^2-4\alpha+2=0$$

in the interval $[1/2,2/3]$.

2. *the $\mathcal{S}^*_P-$radius is the smallest root $r_1\in(0,1)$ of*

$$\begin{aligned}&(\delta_1+2\beta_0\delta_1)r^5+(3+2\beta_0+4|c|\delta_1-2|c|\beta_0\delta_1)r^4\\&\quad+(8|c|-2|c|\beta_0+6\delta_1-2\beta_0\delta_1+2|c|\beta_0\delta_1-2|c|^2\beta_0\delta_1)r^3\\&\quad+(6-2\beta_0+2|c|\beta_0-2|c|^2\beta_0+4|c|\delta_1-2|c|\beta_0\delta_1)r^2\\&\quad+(-2|c|\beta_0+\delta_1)r-1=0.\end{aligned}\tag{3.35}$$

3. *the $\mathcal{S}^*_L-$radius is the smallest root $r_2\in(0,1)$ of*

$$\begin{aligned}&(\delta_1+\sqrt{2}\delta_1-\beta_0\delta_1)r^5+(2+\sqrt{2}-\beta_0+3q\delta_1+2\sqrt{2}|c|\delta_1-|c|\beta_0\delta_1)r^4\\&\quad+(5|c|+2\sqrt{2}|c|-|c|\beta_0+3\delta_1+2|c|^2\delta_1-\beta_0\delta_1-|c|\beta_0\delta_1-|c|^2\beta_0\delta_1)r^3\\&\quad+(3+2|c|^2-\beta_0-|c|\beta_0-|c|^2\beta_0+5|c|\delta_1-2\sqrt{2}|c|\delta_1-|c|\beta_0\delta_1)r^2\\&\quad+(3|c|-2\sqrt{2}|c|-|c|\beta_0+2\delta_1-\sqrt{2}\delta_1)r-\sqrt{2}+1=0.\end{aligned}\tag{3.36}$$

4. *the* $\mathcal{M}(\beta)-$*radius is the smallest root* $r_3 \in (0,1)$ *of*

$$\begin{aligned}(3.37)\quad &(\delta_1+\beta\delta_1-\beta_0\delta_1)r^5+(2+\beta-\beta_0+3|c|\delta_1+2\beta|c|\delta_1-|c|\beta_0\delta_1)r^4\\ &+(5|c|+2\beta|c|-|c|\beta_0+3\delta_1+2|c|^2\delta_1-\beta_0\delta_1-|c|\beta_0\delta_1-|c|^2\beta_0\delta_1)r^3\\ &+(3+2|c|^2-\beta_0-|c|\beta_0-|c|^2\beta_0+5|c|\delta_1-2b|c|\delta_1-|c|\beta_0\delta_1)r^2\\ &+(|c|-2\beta|c|-|c|\beta_0+2\delta_1-\beta\delta_1)r-\beta+1=0.\end{aligned}$$

Proof. Define the functions h and p by $h(z)=g(z)/f(z)$, and $p(z)=zg'(z)/g(z)$. Then

$$h(z)=1+(c-2b)z+\cdots \quad\text{and}\quad p(z)=1+cz+\cdots.$$

Since $|f(z)/g(z)-1|<1$ if and only if $\mathrm{Re}(g(z)/f(z))>1/2$, we have $h\in\mathcal{P}_{(c-2b)/2}(1/2)$. An application of Lemma 2.1 to the function $h(z)$, gives

$$\left|\frac{zh'(z)}{h(z)}\right|\le\frac{(\delta_1r^2+2r+\delta_1)r}{(\delta_1r+1)(1-r^2)}, \tag{3.38}$$

where $\delta_1:=|c-2b|$. Since $g(z)=z+cz^2+\cdots\in\mathcal{K}_c$, it follows from Lemma that

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right)>\alpha_0,$$

where α_0 is the smallest positive root of the equation $2\alpha^3-|c|\alpha^2-4\alpha+2=0$ in the interval $[1/2,2/3]$. Thus $\mathrm{Re}(p(z))>\alpha_0$.
(1) An application of Lemma with $\alpha=\alpha_0$, gives

$$|p(z)-C_c|\le D_c, \tag{3.39}$$

where

$$C_c=\frac{(1+|c|r)^2-\beta_0(|c|+r)^2r^2}{(1+2|c|r+r^2)(1-r^2)},\quad D_c=\frac{(1-\beta_0)(|c|+r)(1+|c|r)r}{(1+2|c|r+r^2)(1-r^2)}\ \text{ and }\ \beta_0=2\alpha_0-1.$$

Since $h(z)=g(z)/f(z)$ and $p(z)=zg'(z)/g(z)$, we have

$$\left|\frac{zf'(z)}{f(z)}-C_c\right|\le|p(z)-C_c|+\left|\frac{zh'(z)}{h(z)}\right|. \tag{3.40}$$

From (3.39), (3.38) and (3.40), we have

$$\left|\frac{zf'(z)}{f(z)}-C_c\right|\le D_c+\frac{(\delta_1r^2+2r+\delta_1)r}{(\delta_1r+1)(1-r^2)}. \tag{3.41}$$

Clearly $f\in\mathcal{S}^*(\lambda)$, provided that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)\ge C_c-D_c-\frac{(\delta_1r^2+2r+\delta_1)r}{(\delta_1r+1)(1-r^2)})\ge\lambda$$

or equivalently, if the following inequality holds:

$$\begin{aligned}(\delta_1+\beta_0\delta_1-\delta_1\lambda)r^5&+(2+\beta_0+3|c|\delta_1+|c|\beta_0\delta_1-\lambda-2|c|\delta_1\lambda)r^4\\&+(5|c|+|c|\beta_0+3\delta_1-\beta_0\delta_1-2|c|\lambda)r^3+(3-\beta_0+(\delta_1-\beta_0\delta_1+2\delta_1\lambda)|c|)r^2\\&+(-|c|-|c|\beta_0+2|c|\lambda+\delta_1\lambda)r-1+\lambda\le 0.\end{aligned}$$

Thus, the $\mathcal{S}^*(\lambda)-$radius of the class $\mathcal{F}^5_{b,c}$ is the smallest positive root $r_0\in(0,1)$ of (3.34).

(2) In view of Lemma 2.6, the disk given in (3.41) lies inside the parabolic region given by $\Omega:=\{w:|w-1|<\operatorname{Re}w\}$ provided

$$D_c+\frac{(\delta_1r^2+2r+\delta_1)r}{(\delta_1r+1)(1-r^2)})\le C_c-1/2$$

or equivalently, if the following inequality holds:

$$\begin{aligned}(\delta_1+2\beta_0\delta_1)r^5&+(3+2\beta_0+4|c|\delta_1-2|c|\beta_0\delta_1)r^4\\&+(8|c|-2|c|\beta_0+6\delta_1-2\beta_0\delta_1+2|c|\beta_0\delta_1-2|c|^2\beta_0\delta_1)r^3\\&+(6-2\beta_0+2|c|\beta_0-2|c|^2\beta_0+4|c|\delta_1-2|c|\beta_0\delta_1)r^2+(\delta_1-2|c|\beta_0)r-1\le 0.\end{aligned}$$

Hence the $\mathcal{M}(\beta)-$radius of the class $\mathcal{F}^5_{b,c}$ is the smallest positive root $r_1\in(0,1)$ of (3.35).

(3) From Lemma 2.5, the function f satisfies $\left|(zf'(z)/f(z))^2-1\right|<1$, in $|z|<r$, if the following inequality holds:

$$D_c+\frac{(\delta_1r^2+2r+\delta_1)r}{(\delta_1r+1)(1-r^2)})\le\sqrt{2}-C_c,$$

or equivalently, if the following inequality holds:

$$\begin{aligned}(\delta_1+\sqrt{2}\delta_1-\beta_0\delta_1)r^5&+(2+\sqrt{2}-\beta_0+3q\delta_1+2\sqrt{2}|c|\delta_1-|c|\beta_0\delta_1)r^4\\&+(5|c|+2\sqrt{2}|c|-|c|\beta_0+3\delta_1+2|c|^2\delta_1-\beta_0\delta_1-|c|\beta_0\delta_1-|c|^2\beta_0\delta_1)r^3\\&+(3+2|c|^2-\beta_0-|c|\beta_0-|c|^2\beta_0+5|c|\delta_1-2\sqrt{2}|c|\delta_1-|c|\beta_0\delta_1)r^2\\&+(3|c|-2\sqrt{2}|c|-|c|\beta_0+2\delta_1-\sqrt{2}\delta_1)r-\sqrt{2}+1\le 0.\end{aligned}$$

Therefore the $\mathcal{S}^*_L-$radius of the class $\mathcal{F}^5_{b,c}$ is the smallest positive root $r_2\in(0,1)$ of (3.36).

(4) From (3.41), we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)\le C_c+D_c+\frac{(\delta_1r^2+2r+\delta_1)r}{(\delta_1r+1)(1-r^2)}\le\beta.$$

if the following inequality holds:

$$\begin{aligned}&(\delta_1+\beta\delta_1-\beta_0\delta_1)r^5+(2+\beta-\beta_0+3|c|\delta_1+2\beta|c|\delta_1-|c|\beta_0\delta_1)r^4\\&\quad+(5|c|+2\beta|c|-|c|\beta_0+3\delta_1+2|c|^2\delta_1-\beta_0\delta_1-|c|\beta_0\delta_1-|c|^2\beta_0\delta_1)r^3\\&\quad+(3+2|c|^2-\beta_0-|c|\beta_0-|c|^2\beta_0+5|c|\delta_1-2b|c|\delta_1-|c|\beta_0\delta_1)r^2\\&\quad+(|c|-2\beta|c|-|c|\beta_0+2\delta_1-\beta\delta_1)r-\beta+1=0.\end{aligned}$$

Therefore, the $\mathcal{M}(\beta)-$radius of the class $\mathcal{F}^5_{b,c}$ is the smallest positive root $r_3\in(0,1)$ of (3.37). □

Remark 3.15. Note that for $b=1=c$, Theorem 3.14 reduces to the result [4, Theorem 2.5] of Ali *et al.*

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