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On the Fine Spectrum of the Lower Triangular Matrix $B(r, s)$ over the Hahn Sequence Space

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Abstract. In this article we have determined the spectrum and fine spectrum of the lower triangular matrix $B(r, s)$ on the Hahn sequence space h. We have also determined the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $B(r, s)$ on the sequence space h.

1. Introduction

By w, we denote the space of all real or complex valued sequences. Throughout the paper c, c_0, bv, cs, bs, ℓ_1, ℓ_∞ represent the spaces of all convergent, null, bounded variation, convergent series, bounded series, absolutely summable and bounded sequences respectively. Also bv_0 denotes the sequence space $bv \cap c_0$.

Fine spectra of various matrix operators on different sequence spaces have been examined by several authors. Fine spectrum of the operator $\Delta_{a,b}$ on the sequence space c was determined by Akhmedov and El-Shabrawy [1]. The fine spectra of the Cesàro operator C_1 over the sequence space bv_p, $(1 \leq p < \infty)$ was determined by Akhmedov and Başar [2]. Altay and Başar [3, 4] determined the fine spectrum of the difference operator Δ and the generalized difference operator $B(r, s)$ on the sequence spaces c_0 and c. The spectrum and fine spectrum of the Zweier Matrix on the sequence spaces ℓ_1 and bv were studied by Altay and Karakuş [5]. Altun [6, 7] determined the fine spectra of triangular Toeplitz operators and tridiagonal symmetric matrices over some sequence spaces. Furkan, Bilgiç and Kayaduman [14] have determined the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces ℓ_1 and bv. Fine spectra of operator $B(r, s, t)$ over the sequence spaces ℓ_1 and bv and generalized difference operator $B(r, s)$ over the sequence spaces ℓ_p and bv_p, $(1 \leq p < \infty)$ were studied by Bilgiç and Furkan [11, 12]. Furkan, Bilgiç and Altay [15] have studied the fine spectrum of operator $B(r, s, t)$ over the sequence

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spaces c_0 and c. Fine spectrum of the operator $B(r,s,t)$ over the sequence spaces ℓ_p and bv_p, $(1 \leq p < \infty)$ were studied by Furkan, Bilgiç and Başar [16]. The spectrum of the operator $D(r,0,0,s)$ over the sequence space bv_0 was investigated by Tripathy and Paul [30]. Tripathy and Paul [29, 31] also determined the spectrum of the operators $D(r,0,0,s)$ and $D(r,0,s,0,t)$ over the sequence spaces ℓ_p and bv_p, $(1 \leq p < \infty)$. Fine spectrum of the generalized difference operator Δ_v on the sequence space ℓ_1 was investigated by Srivastava and Kumar [26]. Panigrahi and Srivastava [23, 24] studied the spectrum and fine spectrum of the second order difference operator Δ^2_{uv} on the sequence space c_0 and generalized second order forward difference operator Δ^2_{uvw} on the sequence space ℓ_1. Fine spectra of upper triangular double-band matrix $U(r,s)$ over the sequence spaces c_0 and c were studied by Karakaya and Altun [20]. Karaisa and Başar [19] have determined the spectrum and fine spectrum of the upper traiangular matrix $A(r,s,t)$ over the sequence space ℓ_p, $(0 < p < \infty)$. Dündar and Başar [13] have studied the fine spectrum of the linear operator Δ^+ defined by an upper triangle double band matrix acting on the sequence space c_0 with respect to the Goldberg's classification. Başar, Durna and Yildirim [9] subdivided the spectra for some generalized difference operators over certain sequence spaces. Başar [10] also determined the spectrum and fine spectrum of some particular limitation matrices over some sequence spaces. Tripathy and Das [27, 28] have studied the fine spectrum of the matrix operators $B(r,0,s)$ and $U(r,s)$ over the sequence space cs. The fine spectrum of the forward difference operator on the Hahn sequence space h was determined by Yeşilkayagil and Kirişci [33].

The Hahn sequence space is defined as

$$h = \left\{ x = (x_n) \in w : \sum_{k=1}^{\infty} k|\Delta x_k| < \infty \quad \text{and} \quad \lim_{k\to\infty} x_k = 0 \right\},$$

where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. This space was defined and studied to some general properties by Hahn [18]. The norm $||x||_h = \sum_{k=1}^{\infty} k|\Delta x_k| + \sup_k |x_k|$ on the space h was defined by Hahn [18]. Rao ([25], Proposition 2.1) defined a new norm on h given by $||x||_h = \sum_{k=1}^{\infty} k|\Delta x_k|$. Many other authors also investigated various properties of the Hahn sequence space.

In this paper, we shall determine the spectrum and fine spectrum of the lower triangular matrix $B(r,s)$ on the Hahn sequence space h . Also we determine the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $B(r,s)$ on the sequence space h.

2. Preliminaries and Background

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$,we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^*

of X defined by $(T^*f)(x) = f(Tx)$, for all $f \in X^*$ and $x \in X$. Let $X \neq \{\theta\}$ be a complex normed linear space, where θ is the zero element and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T, we associate the operator

$$T_\lambda = T - \lambda I,$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1},$$

and call it the *resolvent* operator of T.

A *regular value* λ of T is a complex number such that

(R1) T_λ^{-1} exists,

(R2) T_λ^{-1} is bounded

(R3) T_λ^{-1} is defined on a set which is dense in X i.e. $\overline{R(T_\lambda)} = X$.

The *resolvent set* of T, denoted by $\rho(T, X)$, is the set of all regular values λ of T. Its complement $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point(discrete) spectrum* $\sigma_p(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T, X)$ is called an eigenvalue of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists and satisfies $(R3)$, but not $(R2)$, that is, T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists (and may be bounded or not), but does not satisfy $(R3)$, that is, the domain of T_λ^{-1} is not dense in X.

From Goldberg [17], if X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and T^{-1} :

(I) $R(T) = X$,

(II) $R(T) \neq \overline{R(T)} = X$

(III) $\overline{R(T)} \neq X$

and

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created which may be shown as in Table 1.

	I	II	III
1	$\rho(T,X)$		$\sigma_r(T,X)$
2	$\sigma_c(T,X)$	$\sigma_c(T,X)$	$\sigma_r(T,X)$
3	$\sigma_p(T,X)$	$\sigma_p(T,X)$	$\sigma_p(T,X)$

Table 1: Subdivisions of spectrum of a linear operator

These are labeled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2 and III_3 . If λ is a complex number such that $T_\lambda \in I_1$ or $T_\lambda \in I_2$,then λ is in the resolvent set $\rho(T,X)$ of T . The further classification gives rise to the fine spectrum of T . If an operator is in state II_2 , then $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$ and T_λ^{-1} exists but is discontinuous and we write $\lambda \in II_2\sigma(T,X)$. The state II_1 is impossible as if T_λ is injective, then from Kreyszig $\big[$[22], Problem 6, p.290$\big]$ T_λ^{-1} is bounded and hence continuous if and only if $R(T_\lambda)$ is closed.

Again, following Appell et al. [8], we define the three more subdivisions of the spectrum called as the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $||x_k|| = 1$ and $||Tx_k|| \to 0$ as $k \to \infty$.

The *approximate point spectrum* of T , denoted by $\sigma_{ap}(T,X)$, is defined as the set

$$\sigma_{ap}(T,X) = \{\lambda \in \mathbb{C} : \textit{there exists a Weyl sequence for} \quad T - \lambda I\} \tag{2.1}$$

The *defect spectrum* of T, denoted by $\sigma_\delta(T,X)$, is defined as the set

$$\sigma_\delta(T,X) = \{\lambda \in \mathbb{C} : T - \lambda I \quad \textit{is not surjective}\} \tag{2.2}$$

The two subspectra given by the relations (2.1) and (2.2) form a (not necessarily disjoint) subdivisions

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_\delta(T,X) \tag{2.3}$$

of the spectrum. There is another subspectrum

$$\sigma_{co}(T,X) = \{\lambda \in \mathbb{C} : \overline{R(T - \lambda I)} \neq X\}$$

which is often called the *compression spectrum* of T. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X) \tag{2.4}$$

Clearly, $\sigma_p(T,X) \subseteq \sigma_{ap}(T,X)$ and $\sigma_{co}(T,X) \subseteq \sigma_\delta(T,X)$. Moreover, it is easy to verify that

$$\sigma_r(T,X) = \sigma_{co}(T,X) \setminus \sigma_p(T,X) \quad \text{and} \tag{2.5}$$

$$\sigma_c(T,X) = \sigma(T,X) \setminus \left[\sigma_p(T,X) \cup \sigma_{co}(T,X)\right] \tag{2.6}$$

By the definitions given above, we can illustrate the subdivisions of spectrum of a bounded linear operator in Table 2.

		1	2	3
		T_λ^{-1} exists and is bounded	T_λ^{-1} exists and is not bounded	T_λ^{-1} does not exist
I	$R(T_\lambda) = X$	$\lambda \in \rho(T,X)$	$\cdots$	$\lambda \in \sigma_p(T,X)$ $\lambda \in \sigma_{ap}(T,X)$
II	$\overline{R(T_\lambda)} = X$	$\lambda \in \rho(T,X)$	$\lambda \in \sigma_c(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$	$\lambda \in \sigma_p(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$
III	$\overline{R(T_\lambda)} \neq X$	$\lambda \in \sigma_r(T,X)$ $\lambda \in \sigma_\delta(T,X)$ $\lambda \in \sigma_{co}(T,X)$	$\lambda \in \sigma_r(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$ $\lambda \in \sigma_{co}(T,X)$	$\lambda \in \sigma_p(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$ $\lambda \in \sigma_{co}(T,X)$

Table 2: Subdivisions of spectrum of a linear operator

Proposition 2.1.(Appell et al. [8], Proposition 1.3, p. 28) *Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:*

(a) $\sigma(T^*,X^*) = \sigma(T,X)$.

(b) $\sigma_c(T^*,X^*) \subseteq \sigma_{ap}(T,X)$.

(c) $\sigma_{ap}(T^*,X^*) = \sigma_\delta(T,X)$.

(d) $\sigma_\delta(T^*,X^*) = \sigma_{ap}(T,X)$.

(e) $\sigma_p(T^*,X^*) = \sigma_{co}(T,X)$.

(f) $\sigma_{co}(T^*,X^*) \supseteq \sigma_p(T,X)$.

(g) $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_p(T^*,X^*) = \sigma_p(T,X) \cup \sigma_{ap}(T^*,X^*)$.

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum. Part (g) of Proposition 2.1 implies, in particular, that $\sigma(T,X) = \sigma_{ap}(T,X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (Appell et al. [8]).

Let E and F be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from E into F, and we denote it by $A : E \to F$, if for every sequence $x = (x_k) \in E$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in F, where

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k, \quad n \in \mathbb{N}. \tag{2.7}$$

By $(E : F)$, we denote the class of all matrices such that $A : E \to F$. Thus, $A \in (E : F)$ if and only if the series on the right hand side of (2.7) converges for each $n \in \mathbb{N}$ and every $x \in E$ and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in F$ for all $x \in E$.

The matrix $B(r, s)$ is an infinite lower triangular matrix of the form

$$B(r,s) = \begin{pmatrix} r & 0 & 0 & 0 & \cdots \\ s & r & 0 & 0 & \cdots \\ 0 & s & r & 0 & \cdots \\ 0 & 0 & s & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $s \neq 0$.

The following results will be used in order to establish the results of this article.

Lemma 2.1.(Kirişci [21], Theorem 3.5) *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(h)$ from h to itself if and only if:*

(i) $\sum\limits_{n=1}^{\infty} n|(a_{nk} - a_{n+1,k})|$ *converges, for each k,*

(ii) $\sup\limits_{k} \frac{1}{k} \sum\limits_{n=1}^{\infty} n \left| \sum\limits_{v=1}^{k} (a_{nv} - a_{n+1,v}) \right| < \infty$,

(iii) $\lim\limits_{n\to\infty} a_{nk} = 0$, *for each k.*

Lemma 2.2.(Goldberg [17], Page 59) *T has a dense range if and only if T^* is one to one.*

Lemma 2.3.(Goldberg [17], Page 60) *T has a bounded inverse if and only if T^* is onto.*

3. Spectrum and Fine Spectrum of the Operator $B(r, s)$ over the Sequence Space h

Theorem 3.1. *$B(r, s) : h \to h$ is a bounded linear operator and*

$$\| B(r,s) \|_{(h:h)} \leq |r| + |s|.$$

Proof. From Lemma 2.1, $B(r,s) : h \to h$ is a bounded linear operator on h if

(i) $\sum_{n=1}^{\infty} n|(a_{nk} - a_{n+1,k})|$ converges, for each k,

(ii) $\sup_k \frac{1}{k} \sum_{n=1}^{\infty} n \left| \sum_{v=1}^{k} (a_{nv} - a_{n+1,v}) \right| < \infty$,

(iii) $\lim_{n\to\infty} a_{nk} = 0$, for each k,

where

$$B(r,s) = (a_{nk}) = \begin{pmatrix} r & 0 & 0 & 0 & \cdots \\ s & r & 0 & 0 & \cdots \\ 0 & s & r & 0 & \cdots \\ 0 & 0 & s & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

For each k, it is clear that $\lim_{n\to\infty} a_{nk} = 0$. Also for each k, $\sum_{n=1}^{\infty} n|(a_{nk} - a_{n+1,k})|$ is finite and so is convergent. It is easy to show that, for each k

$$\frac{1}{k} \sum_{n=1}^{\infty} n \left| \sum_{v=1}^{k} (a_{nv} - a_{n+1,v}) \right| \leq |r| + \left(1 + \frac{2}{k}\right)|s|$$

and so

$$\sup_k \frac{1}{k} \sum_{n=1}^{\infty} n \left| \sum_{v=1}^{k} (a_{nv} - a_{n+1,v}) \right| \leq |r| + 3|s| < \infty.$$

Now,

$$\begin{aligned} ||B(r,s)(x)||_h &= \sum_{k=1}^{\infty} k\,|(sx_k + rx_{k+1}) - (sx_{k+1} + rx_{k+2})| \\ &= \sum_{k=1}^{\infty} k|s(x_k - x_{k+1}) + r(x_{k+1} - x_{k+2})| \\ &\leq |s| \sum_{k=1}^{\infty} k|(x_k - x_{k+1})| + |r| \sum_{k=1}^{\infty} k|(x_{k+1} - x_{k+2})| \\ &\leq (|s| + |r|) \parallel x \parallel_h \end{aligned}$$

and hence, $\parallel B(r,s) \parallel_{(h:h)} \leq |r| + |s|$. Hence the result. □

Theorem 3.2. *The spectrum of the operator $B(r,s)$ over h is given by*

$$\sigma(B(r,s), h) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}.$$

Proof. We prove this theorem by showing that $(B(r,s)-\alpha I)^{-1}$ exists and is in $(h:h)$ for $|\alpha-r|>|s|$, and then show that the operator $B(r,s)-\alpha I$ is not invertible for $|\alpha-r|\le|s|$.

Let α be such that $|\alpha-r|>|s|$. Since $s\neq 0$ we have $\alpha\neq r$ and so $B(r,s)-\alpha I$ is a triangle, therefore $(B(r,s)-\alpha I)^{-1}$ exists. Let $y=(y_n)\in h$. On solving $(B(r,s)-\alpha I)x=y$ for x in terms of y we get

$$\begin{aligned}(B(r,s)-\alpha I)^{-1} &= (b_{nk})\\ &= \begin{pmatrix} \frac{1}{r-\alpha} & 0 & 0 & 0 & \cdots\\ -\frac{s}{(r-\alpha)^2} & \frac{1}{r-\alpha} & 0 & 0 & \cdots\\ \frac{s^2}{(r-\alpha)^3} & -\frac{s}{(r-\alpha)^2} & \frac{1}{r-\alpha} & 0 & \cdots\\ -\frac{s^3}{(r-\alpha)^4} & \frac{s^2}{(r-\alpha)^3} & -\frac{s}{(r-\alpha)^2} & \frac{1}{r-\alpha} & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\end{aligned}$$

From Lemma 2.1, $(B(r,s)-\alpha I)^{-1}$ will be a bounded linear operator on h if

(i) $\sum_{n=1}^{\infty} n|(b_{nk}-b_{n+1,k})|$ converges, for each k,

(ii) $\sup_k \frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(b_{nv}-b_{n+1,v})\right|<\infty$,

(iii) $\lim_{n\to\infty} b_{nk}=0$, for each k.

For each k, we get

$$b_{nk}=\frac{(-s)^{n-k}}{(r-\alpha)^{n-k+1}}=\frac{1}{r-\alpha}\left(\frac{-s}{r-\alpha}\right)^{n-k}$$

Since $|\alpha-r|>|s|$, so for each k, $\lim_{n\to\infty} b_{nk}=0$. For each k, it is easy to show that

$$\sum_{n=1}^{\infty} n|(b_{nk}-b_{n+1,k})|\le(2k-1)\frac{1}{|r-\alpha|}+(2k+1)\frac{|s|}{|r-\alpha|^2}+(2k+3)\frac{|s|^2}{|r-\alpha|^3}+\cdots$$

Now for a fixed k, considering $2k-1=a$, from above we get

$$\begin{aligned}\sum_{n=1}^{\infty} n|(b_{nk}-b_{n+1,k})| &\le a\frac{1}{|r-\alpha|}+(a+2)\frac{|s|}{|r-\alpha|^2}+(a+4)\frac{|s|^2}{|r-\alpha|^3}+\cdots\\ &= \frac{a}{|r-\alpha|}\left(1+\frac{|s|}{|r-\alpha|}+\frac{|s|^2}{|r-\alpha|^2}+\cdots\right)+\\ &\quad \frac{2}{|r-\alpha|}\left(\frac{|s|}{|r-\alpha|}+\frac{2|s|^2}{|r-\alpha|^2}+\frac{3|s|^3}{|r-\alpha|^3}+\cdots\right)\end{aligned}$$

Since $|\alpha - r| > |s|$, therefore the two series

$$1 + \frac{|s|}{|r-\alpha|} + \frac{|s|^2}{|r-\alpha|^2} + \cdots \quad \text{and} \quad \frac{|s|}{|r-\alpha|} + \frac{2|s|^2}{|r-\alpha|^2} + \frac{3|s|^3}{|r-\alpha|^3} + \cdots$$

are convergent and converge to $\frac{1}{1-\frac{|s|}{|r-\alpha|}}$ and $\frac{\frac{|s|}{|r-\alpha|}}{\left(1-\frac{|s|}{|r-\alpha|}\right)^2}$ respectively. Therefore, $\sum_{n=1}^{\infty} n|(b_{nk} - b_{n+1,k})|$ converges, for each k. Also, for each k, it is easy to show that

$$\begin{aligned} \frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(b_{nv} - b_{n+1,v})\right| &\leq \frac{1}{|r-\alpha|} + \left(1+\frac{2}{k}\right)\frac{|s|}{|r-\alpha|^2} + \left(1+\frac{4}{k}\right)\frac{|s|^2}{|r-\alpha|^3} + \cdots \\ &\leq \frac{1}{|r-\alpha|} + 3\frac{|s|}{|r-\alpha|^2} + 5\frac{|s|^2}{|r-\alpha|^3} + \cdots \end{aligned}$$

Since $|\alpha - r| > |s|$, so by D'Alembert's ratio test it is easy to show that the series $\frac{1}{|r-\alpha|} + 3\frac{|s|}{|r-\alpha|^2} + 5\frac{|s|^2}{|r-\alpha|^3} + \cdots$ is convergent and therefore we have,

$$\sup_k \frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(b_{nv} - b_{n+1,v})\right| < \infty.$$

So, by Lemma 2.1, $(B(r,s) - \alpha I)^{-1}$ is in $(h : h)$. This shows that $\sigma(B(r,s),h) \subseteq \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$.

Now, let $\alpha \in \mathbb{C}$ be such that $|\alpha - r| \leq |s|$. If $\alpha \neq r$, then $B(r,s) - \alpha I$ is a triangle and hence, $(B(r,s) - \alpha I)^{-1}$ exists. Let $y = (1,0,0,0,...)$. Then $y \in h$. Now, $(B(r,s) - \alpha I)^{-1}y = x$ gives

$$x_n = \frac{(-s)^{n-1}}{(r-\alpha)^n}.$$

Since $|\alpha - r| \leq |s|$, so the sequence (x_n) does not converge to 0 and so, $x = (x_n) \notin h$. Therefore, $(B(r,s) - \alpha I)^{-1}$ is not in $(h : h)$ and so $\alpha \in \sigma(B(r,s),h)$. If $\alpha = r$, then the operator $B(r,s) - \alpha I$ is represented by the matrix

$$B(r,s) - \alpha I = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ s & 0 & 0 & 0 & \cdots \\ 0 & s & 0 & 0 & \cdots \\ 0 & 0 & s & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Since, the range of $B(r,s) - \alpha I$ is not dense, so $\alpha \in \sigma(B(r,s),h)$. Hence,

$$\{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\} \subseteq \sigma(B(r,s),h).$$

This completes the proof. □

Theorem 3.3. *The point spectrum of the operator $B(r,s)$ over h is given by*

$$\sigma_p(B(r,s),h)=\emptyset.$$

Proof. Let α be an eigenvalue of the operator $B(r,s)$. Then there exists $x\neq\theta=(0,0,0,...)$ in h such that $B(r,s)x=\alpha x$. Then, we have

$$\left.\begin{aligned} rx_1&=\alpha x_1\\ sx_1+rx_2&=\alpha x_2\\ sx_2+rx_3&=\alpha x_3\\ &\vdots\\ sx_n+rx_{n+1}&=\alpha x_{n+1}\end{aligned}\right\},$$

where $n\geq 1$. If x_k is the first non-zero entry of the sequence (x_n), then $\alpha=r$. Then from the relation $sx_k+rx_{k+1}=\alpha x_{k+1}$, we have $sx_k=0$. But $s\neq 0$ and hence, $x_k=0$, a contradiction. Hence, $\sigma_p(B(r,s),h)=\emptyset$. □

If $T:h\to h$ is a bounded linear operator represented by a matrix A, then it is known that the adjoint operator $T^*:h^*\to h^*$ is defined by the transpose A^t of the matrix A. It should be noted that the dual space h^* of h is isometrically isomorphic to the Banach space $\sigma_\infty=\left\{x=(x_k)\in w:\sup_n\frac{1}{n}\left|\sum_{k=1}^{n}x_k\right|<\infty\right\}$.

Theorem 3.4. *The point spectrum of the operator $B(r,s)^*$ over h^* is given by*

$$\sigma_p(B(r,s)^*,h^*\cong\sigma_\infty)=\{\alpha\in\mathbb{C}:|\alpha-r|<|s|\}.$$

Proof. Let α be an eigenvalue of the operator $B(r,s)^*$. Then there exists $x\neq\theta=(0,0,0,...)$ in σ_∞ such that $B(r,s)^*x=\alpha x$. Then, we have

$$\left.\begin{aligned} B(r,s)^t x&=\alpha x\\ \Rightarrow rx_1+sx_2&=\alpha x_1\\ rx_2+sx_3&=\alpha x_2\\ &\vdots\\ rx_n+sx_{n+1}&=\alpha x_n\end{aligned}\right\},$$

where $n\geq 1$. Solving, we get

$$x_n=\left(\frac{\alpha-r}{s}\right)^{n-1}x_1,\qquad n\geq 1.$$

and so, $\sup_n \frac{1}{n}\left|\sum_{k=1}^{n} x_k\right| < \infty$ if and only if $|\alpha - r| < |s|$. Hence, $\sigma_p(B(r,s)^*, h^* \cong \sigma_\infty) = \{\alpha \in \mathbb{C} : |\alpha - r| < |s|\}$. □

Theorem 3.5. *The residual spectrum of the operator $B(r,s)$ over h is given by*

$$\sigma_r(B(r,s), h) = \{\alpha \in \mathbb{C} : |\alpha - r| < |s|\}.$$

Proof. From part (e) of Propostion 2.1 and relation (2.5), we get

$$\sigma_r(B(r,s), h) = \sigma_p(B(r,s)^*, h^*) \setminus \sigma_p(B(r,s), h).$$

So we get the required result by using Theorem 3.3 and Theorem 3.4. □

Theorem 3.6. *The continuous spectrum of the operator $B(r,s)$ over h is given by*

$$\sigma_c(B(r,s), h) = \{\alpha \in \mathbb{C} : |\alpha - r| = |s|\}.$$

Proof. Since, $\sigma(B(r,s), h)$ is the disjoint union of $\sigma_p(B(r,s), h)$, $\sigma_r(B(r,s), h)$ and $\sigma_c(B(r,s), h)$, therefore, by Theorem 3.2, Theorem 3.3 and Theorem 3.5, we get $\sigma_c(B(r,s), h) = \{\alpha \in \mathbb{C} : |\alpha - r| = |s|\}$. □

Theorem 3.7. *If $\alpha = r$, then $\alpha \in III_1\sigma(B(r,s), h)$.*

Proof. If $\alpha = r$, the range of $B(r,s) - \alpha I$ is not dense. So, from Table 2 and Theorem 3.3, we have $\alpha \in \sigma_r(B(r,s), h)$. From Table 2,

$$\sigma_r(B(r,s), h) = III_1\sigma(B(r,s), h) \cup III_2\sigma(B(r,s), h).$$

Therefore, $\alpha \in III_1\sigma(B(r,s), h)$ or $\alpha \in III_2\sigma(B(r,s), h)$. Also for $\alpha = r$,

$$B(r,s) - \alpha I = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ s & 0 & 0 & 0 & \cdots \\ 0 & s & 0 & 0 & \cdots \\ 0 & 0 & s & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is easy to show that the operator $(B(r,s) - \alpha I)^* : \sigma_\infty \to \sigma_\infty$ is onto. So by Lemma 2.3 we get the operator $B(r,s) - \alpha I$ has a bounded inverse. This completes the theorem. □

Theorem 3.8. *If $\alpha \neq r$ and $\alpha \in \sigma_r(B(r,s), h)$, then $\alpha \in III_2\sigma(B(r,s), h)$.*

Proof. Let $\alpha \neq r$. Since, $\alpha \in \sigma_r(B(r,s), h)$, therefore, from Table 2,

$$\alpha \in III_1\sigma(B(r,s), h) \quad \text{or} \quad \alpha \in III_2\sigma(B(r,s), h).$$

Now, $\alpha \in \sigma_r(B(r,s), h)$ implies that $|\alpha - r| < |s|$. Therefore, for each k, the sequence $(b_{nk})_n$ in Theorem 3.2 does not converge to 0 as $n \to \infty$ and hence, the operator $B(r,s) - \alpha I$ has no bounded inverse. Therefore, $\alpha \in III_2\sigma(B(r,s), h)$. □

Theorem 3.9. *If* $\alpha \in \sigma_c(B(r,s),h)$ *, then* $\alpha \in II_2\sigma(B(r,s),h)$.

Proof. If $\alpha \in \sigma_c(B(r,s),h)$, then $|\alpha - r| = |s|$. Therefore, for each k, the sequence $(b_{nk})_n$ in Theorem 3.2 does not converge to 0 as $n \to \infty$ and hence, the operator $B(r,s) - \alpha I$ has no bounded inverse. So, α satisfies Goldberg's condition 2.

Now we shall show that the operator $B(r,s) - \alpha I$ is not onto. Let $y = (y_n) = (1,0,0,0,...)$. Clearly, $(y_n) \in h$. Let $x = (x_n)$ be a sequence such that $(B(r,s) - \alpha I)x = y$. On solving, we get

$$x_n = \frac{(-s)^{n-1}}{(r-\alpha)^n}.$$

Since, $|\alpha - r| = |s|$ so the sequence $\{x_n\}$ does not converge to 0 as $n \to \infty$ and so, $x = (x_n) \notin h$. Hence the operator $B(r,s) - \alpha I$ is not onto. So, α satisfies Goldberg's condition II. This completes the proof. □

Theorem 3.10. *The approximate point spectrum of the operator* $B(r,s)$ *over* h *is given by*

$$\sigma_{ap}(B(r,s),h) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\} \setminus \{r\}.$$

Proof. From Table 2,

$$\sigma_{ap}(B(r,s),h) = \sigma(B(r,s),h) \setminus III_1\sigma(B(r,s),h).$$

By Theorem 3.7, $III_1\sigma(B(r,s),h) = \{r\}$ .This completes the proof. □

Theorem 3.11. *The compression spectrum of the operator* $B(r,s)$ *over* h *is given by*

$$\sigma_{co}(B(r,s),h) = \{\alpha \in \mathbb{C} : |\alpha - r| < |s|\}.$$

Proof. From part (e) of Proposition 2.1, we get

$$\sigma_p(B(r,s)^*,h^*) = \sigma_{co}(B(r,s),h).$$

Using Theorem 3.4, we get the required result. □

Theorem 3.12. *The defect spectrum of the operator* $B(r,s)$ *over* h *is given by*

$$\sigma_\delta(B(r,s),h) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}.$$

Proof. From Table 2, we have

$$\sigma_\delta(B(r,s),h) = \sigma(B(r,s),h) \setminus I_3\sigma(B(r,s),h).$$

Also,

$$\sigma_p(B(r,s),h) = I_3\sigma(B(r,s),h) \cup II_3\sigma(B(r,s),h) \cup III_3\sigma(B(r,s),h).$$

By Theorem 3.3, we have $\sigma_p(B(r,s),h) = \emptyset$ and so $I_3\sigma(B(r,s),h) = \emptyset$. Hence, $\sigma_\delta(B(r,s),h) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$. □

Theorem 3.13. *The following statements hold:*

(i) $\sigma_{ap}(B(r,s)^*, h^* \cong \sigma_\infty) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$.

(ii) $\sigma_\delta(B(r,s)^*, h^* \cong \sigma_\infty) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\} \setminus \{r\}$.

Proof. From parts (c) and (d) of Proposition 2.1, we get

$$\sigma_{ap}(B(r,s)^*, h^* \cong \sigma_\infty) = \sigma_\delta(B(r,s),h)$$

and

$$\sigma_\delta(B(r,s)^*, h^* \cong \sigma_\infty) = \sigma_{ap}(B(r,s),h).$$

Using Theorem 3.10 and Theorem 3.12, we get the required results. □

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