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A Characterization of Dedekind Domains and ZPI-rings

Esmaeil Rostami
Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran
e-mail : e_rostami@uk.ac.ir

Abstract. It is well known that an integral domain D is a Dedekind domain if and only if D is a Noetherian almost Dedekind domain. In this paper, we show that an integral domain D is a Dedekind domain if and only if D is an almost Dedekind domain such that Max(D) is a Noetherian topological space as a subspace of Spec(D) with respect to the Zariski topology. We also give a new characterization of ZPI-rings.

1. Introduction

Let R denote throughout a commutative ring with 1. Recall that a ring R is called a ZPI-ring if every proper ideal of R is a product of prime ideals. Also, an integral domain D is called a Dedekind domain if every proper ideal of D is a product of prime ideals. It is well known that if D is a Dedekind domain, then $D_{\mathfrak{m}}$ is a Noetherian valuation domain for each maximal ideal $\mathfrak{m}$ of D and the converse is true if D is Noetherian. An integral domain D is called almost Dedekind if $D_{\mathfrak{m}}$ is a Noetherian valuation domain for each maximal ideal $\mathfrak{m}$ of D. Thus, a Dedekind domain is an almost Dedekind domain and a Noetherian almost Dedekind domain is a Dedekind domain. In [6], Loper discussed methods for constructing nonNoetherian almost Dedekind domains.

Almost Dedekind domains and ZPI-rings have many of the important properties of Dedekind domains that make them two useful and attractive classes of rings. We give below some results which demonstrates some of the properties of Dedekind domains, almost Dedekind domains and ZPI-rings.

Theorem 1.1.([5, Theorem 6.20]) *If D is a Noetherian integral domain, then the following statements are equivalent:*

(1) *D is a Dedekind domain.*

(2) *D is integrally closed and every nonzero proper prime ideal of D is maximal.*

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(3) *Every nonzero ideal of D generated by two elements is invertible.*

(4) *If $AB = AC$, where A, B, C are ideals of D and $A \neq 0$, then $B = C$.*

(5) *For every maximal ideal $\mathfrak{m}$ of D, the ring of quotients $D_{\mathfrak{m}}$ is a valuation ring.*

Theorem 1.2.([5, Theorem 9.4]) *If D is an integral domain which is not a field, then the following statements are equivalent:*

(1) *D is an almost Dedekind domain.*

(2) *D has Krull dimension one and each primary ideal of D is a power of its radical.*

Theorem 1.3.([5, Theorem 9.10]) *The following statements are equivalent for a ring R.*

(1) *R is a ZPI-ring.*

(2) *R is a Noetherian ring such that for each maximal ideal $\mathfrak{m}$ of R, there are no ideals of R strictly between $\mathfrak{m}$ and $\mathfrak{m}^2$.*

(3) *R is a direct sum of a finite number of Dedekind domains and special primary rings.*

For more information about Dedekind domains, almost Dedekind domains and ZPI-rings, refer to [4] and [5].

In this paper by using Max(D) as a subspace of Spec(D) with respect to the Zariski topology, we show that an integral domain D is a Dedekind domain if and only if D is an almost Dedekind domain such that Max(D) is a Noetherian topological space. We also give a new characterization of ZPI-rings.

2. Main Results

Recall that a ring R is said to be indecomposable, if R cannot be written as $R = R_1 \times R_2$, where R_1 and R_2 are both nonzero rings. It is well known, and not difficult to prove, that a ring R is indecomposable if and only if 1 is the only nonzero idempotent of R.

Lemma 2.1. *The following statements are equivalent for an ideal I in R.*

(1) *The ideal I can be written as $I = J_1 \cap J_2 \cap ... \cap J_n$, where J_1, J_2,...,J_n are ideals in R such that each of the R/J_i is indecomposable.*

(2) *The ideal I can be written as $I = K_1K_2...K_m = K_1 \cap K_2 \cap ... \cap K_m$, where K_1, K_2,...,K_m are pairwise comaximal ideals in R such that each of the R/K_i is indecomposable.*

(3) *R/I has only finitely many idempotents.*

Proof. (1) $\Rightarrow$ (2) Let I can be written as $I = J_1 \cap J_2 \cap ... \cap J_n$, where J_1, J_2,...,J_n are ideals in R such that each of the R/J_i is indecomposable. Let two of the J_i's , say J_1 and J_2, are contained in a maximal ideal $\mathfrak{m}$ of R. If $e + (J_1 \cap J_2)$ is an idempotent in $R/(J_1 \cap J_2)$, then $e + J_1$ and $e + J_2$ are idempotents in R/J_1 and R/J_2, respectively. Now since R/J_1 and R/J_2 are indecomposable and $J_1 \subseteq \mathfrak{m}$ and $J_2 \subseteq \mathfrak{m}$, we have $e + (J_1 \cap J_2) = 0 + (J_1 \cap J_2)$ or $e + (J_1 \cap J_2) = 1 + (J_1 \cap J_2)$. Hence, $R/(J_1 \cap J_2)$ is also indecomposable. Set $I_1' = (J_1 \cap J_2)$. Hence, $I = I_1' \cap I_3 \cap ... \cap J_n$ such that $R/I_1', R/J_3, ..., R/J_n$ are indecomposable. Repeating this argument, we see that the ideal I can be written as $I = K_1 \cap K_2 \cap ... \cap K_m$, where K_1, K_2,...,K_m are pairwise comaximal ideals of R such that each of the R/K_i is indecomposable. Now since K_1, K_2,...,K_m are pairwise comaximal, we have $K_1 K_2 ... K_m = K_1 \cap K_2 \cap ... \cap K_m$.
(2) $\Rightarrow$ (3) Let the ideal I can be written as $I = K_1 K_2 ... K_m = K_1 \cap K_2 \cap ... \cap K_m$, where K_1, K_2,...,K_m are pairwise comaximal ideals of R such that each of the R/K_i is indecomposable. By the Chinese Remainder Theorem, $R/I \cong \bigoplus_{i=1}^{m} R/K_i$. Now since each of the R/K_i has no nontrivial idempotents, R/I has only finitely many idempotents.
(3) $\Rightarrow$ (1) If R/I is indecomposable, there is nothing to prove. Suppose that R/I is not indecomposable. Then there exists a nontrivial idempotent $r + I$ in R/I. Thus, $\{0 + I\} = I/I = (\langle I, r\rangle/I) \cap (\langle I, r-1\rangle/I) = (\langle I, r\rangle/I)(\langle I, r-1\rangle/I)$. Hence, $I = \langle I, r\rangle \cap \langle I, r-1\rangle$ and $R/I \cong R/\langle I, r\rangle \oplus R/\langle I, r-1\rangle$. Now if $R/\langle I, r\rangle$ and $R/\langle I, r-1\rangle$ are indecomposable, the proof is complete. Otherwise, $R/\langle I, r\rangle$ or $R/\langle I, r-1\rangle$ can be written as a direct sum of nonzero rings as above. Since R/I has finitely many idempotents, this process terminates after finite steps. This completes the proof. □

For a ring R, let Spec(R) and Max(R) denote the set of all prime ideals and all maximal ideals of R, respectively. The Zariski topology on Spec(R) is the topology obtained by taking the collection of sets of the form $\mathcal{D}(I) = \{P \in \mathrm{Spec}(R) \mid I \nsubseteq P\}$ (resp. $\mathcal{V}(I) = \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$), for every ideal I of R, as the open (resp. closed) sets. When considering as a subspace of Spec(R), Max(R) is called $Max-Spectrum$ of R. So, its closed and open subsets are $\mathbf{D}(I) = \mathcal{D}(I) \cap \mathrm{Max}(R) = \{\mathfrak{m} \in \mathrm{Max}(R) \mid I \nsubseteq \mathfrak{m}\}$ and $\mathbf{V}(I) = \mathcal{V}(I) \cap \mathrm{Max}(R) = \{\mathfrak{m} \in \mathrm{Max}(R) \mid I \subseteq \mathfrak{m}\}$, respectively.

A topological space X is called *Noetherian* if every nonempty set of closed subsets of X, ordered by inclusion, has a minimal element. An ideal I of R is called a *J-radical ideal* if it is an intersection of maximal ideals. Clearly, J-radical ideals of R correspond to closed subsets of Max(R), and Max-Spectrum of R is Noetherian if and only if R satisfies the ascending chain condition for J-radical ideals (See [7] for more details).

Theorem 2.2. *Let R be a ring such that* Max(R) *is a Noetherian topological space as a subspace of* Spec(R) *with respect to the Zariski topology. Then every proper ideal I of R can be written as $I = J_1 \cap J_2 \cap ... \cap J_n$, where J_1, J_2,...,J_n are ideals of R such that each of the R/J_i is indecomposable.*

Proof. Let I be a proper ideal of R. Since Max(R) is Noetherian, Max(R/I) is also

Noetherian. Thus, it is sufficient to show that the result is true for the zero ideal. Suppose, on the contrary, that the zero ideal cannot be written as $I = J_1 \cap J_2 \cap ... \cap J_n$, where J_1, J_2,...,J_n are ideals of R such that each of the R/J_i is indecomposable. By Lemma 2.1, R has infinitely many distinct idempotents, say α_1, α_2,...,α_n,..., and so $f_1 = \alpha_1$, $f_2 = \alpha_1\alpha_2$,..., $f_n = \alpha_1\alpha_2...\alpha_n$,..., are infinitely many distinct idempotents in R with $f_i f_{i+1} = f_{i+1}$. Set $e_i = f_i - f_{i+1}$ for each $i \in \mathbb{N}$. It is easily seen that $\{e_i\}_{i=1}^{\infty}$ is an infinite set of nonzero orthogonal idempotents. For each $i \in \mathbb{N}$, as $1 - e_i \neq 1$, there exists $\mathfrak{m}_i \in \mathrm{Max}(R)$ such that $1 - e_i \in \mathfrak{m}_i$.

Set $J_n = \cap_{i\in\mathbb{N}\setminus\{1,2,...,n\}}\mathfrak{m}_i$ for each $n \in \mathbb{N}$. Thus $J_1 \subseteq J_2 \subseteq J_3 \subseteq ...$, is an ascending chain of J-radical ideals of R. By hypothesis, there exists $k \in \mathbb{N}$ such that $J_k = J_{k+1}$. Therefore, $J_{k+1} = \cap_{i\in\mathbb{N}\setminus\{1,2,...,k+1\}}\mathfrak{m}_i \subseteq J_k = \cap_{i\in\mathbb{N}\setminus\{1,2,...,k\}}\mathfrak{m}_i \subseteq \mathfrak{m}_{k+1}$. Now for all $i \neq k+1$, $e_{k+1} = e_{k+1} - 0 = (1 - e_i)e_{k+1} \in \mathfrak{m}_i$. Thus, $e_{k+1} \in J_{k+1} \subseteq \mathfrak{m}_{k+1}$. Therefore, $e_{k+1} \in \mathfrak{m}_{k+1}$ and $1 - e_{k+1} \in \mathfrak{m}_{k+1}$, a contradiction. □

We will need the following well known fact about indecomposable rings which is a consequence of [1, Proposition 27.1].

Lemma 2.3. *Let I be a proper ideal of R. Then R/I is indecomposable if and only if $R/\sqrt{I}$ is indecomposable.*

An ideal I of a ring R is called *semi-primary* if $\sqrt{I}$ is a prime ideal. In [3] and [2], Gilmer considered rings whose semi-primary ideals are primary. Before proceeding, we state some useful results.

Theorem 2.4.([2, Theorem 2]) *Let R be a ring whose semi-primary ideals are primary. If Q is a P-primary ideal of R where P is a nonmaximal prime ideal of R, then $Q = P$.*

By Lemma 2.3, if I is a semi-primary ideal, then R/I is indecomposable. The next result is a consequence of Theorem 2.4.

Corollary 2.5. *Let R be a ring with the property that if R/I is indecomposable for an ideal I of R, then I is primary. Then if Q is a P-primary ideal of R where P is a nonmaximal prime ideal of R, then $Q = P$.*

Lemma 2.6. *Let R be a one-dimensional ring such that for any chain $P \subset \mathfrak{m}$ of prime ideals of R and $p \in P$, there exists $m \in \mathfrak{m}$ such that $p = pm$. Then if P is a nonmaximal prime ideal of R and I is an ideal of R with $\sqrt{I} = P$, then $I = P$.*

Proof. Let P be a nonmaximal prime ideal of R, and let I be an ideal of R with $\sqrt{I} = P$. Consider the ring $\overline{R} = R/I$, and write "bar" for the quotient map. Since $\sqrt{I} = P$, $\overline{P}$ is the unique nonmaximal prime ideal of $\overline{R}$. Let $\overline{p} \in \overline{P}$. If $\overline{\mathfrak{m}}$ is a maximal ideal of $\overline{R}$, then $\overline{P} \subset \overline{\mathfrak{m}}$. By hypothesis, there exists $\overline{m} \in \overline{\mathfrak{m}}$ such that $\overline{p} = \overline{pm}$. Now let $\overline{x}$ be an element of $\overline{R}$ such that $\overline{x} \notin \overline{\mathfrak{m}}$. Then $(\overline{x} - \overline{xm})\overline{p} = \overline{0}$. Since $\overline{x} - \overline{xm} \notin \overline{\mathfrak{m}}$, the annihilator of $\overline{p}$ must be $\overline{R}$. Hence, $\overline{p} = \overline{pm} = \overline{0}$. Therefore, $I = P$. □

Corollary 2.7.([3, Corollay 2.2]) *Let R be a ring whose semi-primary ideals are primary. Then R has dimension less than two.*

Remark 2.8. It is easily seen that if an ideal I of a ring R can be generated by a set of idempotents, then every element of I is a multiple of an idempotent of I.

Lemma 2.9. *Let I be a ideal of R such that R/I is indecomposable, and let $I' = \langle\{e \in I \mid e^2 = e\}\rangle$. Then R/I' is also indecomposable.*

*Proof.*Let $x^2+I' = x+I'$ for some $x \in R$. Thus, $x^2-x \in I' \subseteq I$ and so $x^2+I = x+I$. Since R/I is indecomposable and $x + I$ is an idempotent element in R/I, we have $x \in I$ or $x-1 \in I$. Suppose that $x \in I$. Now since $x^2 - x \in I'$, by Remark 2.8, there exists $e^2 = e \in I'$ such that $x^2 - x = re$ for some $r \in R$. Thus, $x^2 - x = (x^2 - x)e$. Hence, $(1 - e)x^2 = (1 - e)x$. Thus, $((1 - e)x)^2 = (1 - e)^2x^2 = (1 - e)x^2 = (1 - e)x$. This shows that $(1 - e)x$ is an idempotent element in I, hence $(1 - e)x \in I'$. Now since $e \in I'$, we have $x \in I'$. A similar argument works when $x - 1 \in I$. Therefore, R/I' has no nontrivial idempotents, and so R/I' is indecomposable. □

Lemma 2.10. *Let R be a zero-dimensional ring. If $\{P_\alpha\}_{\alpha\in\Lambda}$ is a family of prime ideals of R such that $R/\cap_{\alpha\in\Lambda} P_\alpha$ is indecomposable, then $\mid \Lambda \mid = 1$.*

Proof. Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be a family of prime ideals of R such that $R/\cap_{\alpha\in\Lambda} P_\alpha$ is indecomposable. Since $\cap_{\alpha\in\Lambda}P_\alpha$ is a radical ideal, $R/\cap_{\alpha\in\Lambda} P_\alpha$ is a zero-dimensional reduced ring, and so $R/\cap_{\alpha\in\Lambda} P_\alpha$ is a $Von\ Neumann$ regular ring. Suppose $r + \cap_{\alpha\in\Lambda}P_\alpha$ is a nonzero element of $R/\cap_{\alpha\in\Lambda} P_\alpha$, if $r + \cap_{\alpha\in\Lambda}P_\alpha$ is nonunit, then there exists $s+\cap_{\alpha\in\Lambda}P_\alpha$ in $R/\cap_{\alpha\in\Lambda} P_\alpha$ such that $r+\cap_{\alpha\in\Lambda}P_\alpha = rsr+\cap_{\alpha\in\Lambda}P_\alpha$. It is easily seen that $sr + \cap_{\alpha\in\Lambda}P_\alpha$ is a nontrivial idempotent of $R/\cap_{\alpha\in\Lambda} P_\alpha$. So $R/\cap_{\alpha\in\Lambda} P_\alpha$ is not indecomposable, a contradiction. Thus, every nonzero element of $R/\cap_{\alpha\in\Lambda} P_\alpha$ is unit, and so $R/\cap_{\alpha\in\Lambda} P_\alpha$ is a field. Thus, $\mid \Lambda \mid = 1$. □

Proposition 2.11. *The following statements are equivalent for a ring R.*

1. *For an ideal I of R if R/I is indecomposable, then I is primary.*
2. *R is a zero-dimensional ring or R is a one-dimensional ring such that every nonmaximal prime ideal of R can be generated by its idempotents.*

Proof. (1) $\Rightarrow$ (2) By Corollary 2.7 and the fact that R/I is indecomposable for every semi-primary ideal I, R has dimension less than two. Let R be a one-dimensional ring, and let P be a nonmaximal prime ideal of R. By Lemma 2.9, R/P' is indecomposable, where $P' = \langle\{e \in P \mid e^2 = e\}\rangle$. By hypothesis, P' is primary. Since P is a minimal prime ideal over P', P' is P-primary ideal. Thus $P' = P$ by Corollary 2.5. Therefore, every nonmaximal prime ideal of R can be generated by its idempotents.
(2) $\Rightarrow$ (1) Let R be a zero-dimensional ring, and let R/I be indecomposable for an ideal I of R. By Lemma 2.3, $R/\sqrt{I} = R/\cap_{I\subseteq P\in\mathrm{Spec}(R)} P$ is indecomposable. Thus, by Lemma 2.10, $\sqrt{I} = \cap_{I\subseteq P\in\mathrm{Spec}(R)}P$ must be a maximal ideal of R. Hence, I is primary in this case.

Now let R be one-dimensional, and let R/I be indecomposable for an ideal I of R. By Lemma 2.3, $R/\sqrt{I} = R/\cap_{I\subseteq P\in\mathrm{Spec}(R)} P$ is indecomposable. If $R/\cap_{I\subseteq P\in\mathrm{Spec}(R)} P$ is a zero-dimension ring, as above, $\sqrt{I} = \cap_{I\subseteq P\in\mathrm{Spec}(R)}P$ is a maximal ideal of R, and so I is primary.

Now let $R/\cap_{I\subseteq P\in \mathrm{Spec}(R)} P$ be a one-dimension ring. We now consider the cases $\cap_{I\subseteq P\in \mathrm{Spec}(R)}P$ is a prime ideal and $\cap_{I\subseteq P\in \mathrm{Spec}(R)}P$ is not a prime ideal.
Case 1. If $\cap_{I\subseteq P\in \mathrm{Spec}(R)}P$ is a prime ideal of R, then $\cap_{I\subseteq P\in \mathrm{Spec}(R)}P$ is a nonmaximal prime ideal of R. If $p\in \cap_{I\subseteq P\in \mathrm{Spec}(R)}P$, by hypothesis and Remark 2.8, there exists an idempotent $e\in \cap_{I\subseteq P\in \mathrm{Spec}(R)}P$ such that $p=re$ for some $r\in R$. Hence, $p=pe^n$ for each $n\in\mathbb{N}$. Since $\sqrt{I}=\cap_{I\subseteq P\in \mathrm{Spec}(R)}P$, thus a power of e is in I. Hence, $p\in I$, and so $\cap_{I\subseteq P\in \mathrm{Spec}(R)}P=I$. Therefore, I is primary.
Case 2. If $\cap_{I\subseteq P\in \mathrm{Spec}(R)}P$ is not a prime ideal, there exists a nonmaximal prime ideal P_1 of a ring R containing $\cap_{I\subseteq P\in \mathrm{Spec}(R)}P$. By hypothesis, there exists a nontrivial idempotent $e\in P_1\setminus \cap_{I\subseteq P\in \mathrm{Spec}(R)}P$. Thus the ring $R/\cap_{I\subseteq P\in \mathrm{Spec}(R)} P$ has a nontrivial idempotent, and so $R/\cap_{I\subseteq P\in \mathrm{Spec}(R)} P$ is not indecomposable, a contradiction. □

Now we can state the main results of this paper.

Theorem 2.12. *The following statements are equivalent for a ring R.*

(1) *R is a ZPI-ring.*

(2) *R satisfies the following conditions:*

  (a) *The dimension of R is at most one and every nonmaximal prime ideal of R can be generated by its idempotents.*

  (b) *Each primary ideal of R is a power of its radical.*

  (c) $\mathrm{Max}(R)$ *is a Noetherian topological space as a subspace of* $\mathrm{Spec}(R)$ *with respect to the Zariski topology.*

Proof. By Theorem 1.3 and the definition of ZPI-rings, it is sufficient to prove (2) $\Rightarrow$ (1). By Theorem 2.2 and Lemma 2.1, every proper ideal I of R can be written as $I=K_1K_2...K_m=K_1\cap K_2\cap...\cap K_m$, where K_1, K_2,...,K_m are pairwise comaximal ideals in R such that each of the R/K_i is indecomposable. Proposition 2.11 implies that each of the K_i is primary, and so every proper ideal of R can be written as a product of primary ideals. By hypothesis, each primary ideal of R is a power of its radical. Thus, every ideal of R can be written as a product of prime ideals of R. Therefore, R is a ZPI-ring. □

Theorem 2.13. *Let D be an integral domain which is not a field, then the following statements are equivalent:*

(1) *D is a Dedekind domain.*

(2) *D is an almost Dedekind domain such that* $\mathrm{Max}(D)$ *is a Noetherian topological space as a subspace of* $\mathrm{Spec}(D)$ *with respect to the Zariski topology.*

Proof. By Theorem 1.1 and 1.2, it is sufficient to prove (2) $\Rightarrow$ (1). Suppose that D is an almost Dedekind domain such that $\mathrm{Max}(D)$ is a Noetherian topological space as a subspace of $\mathrm{Spec}(D)$ with respect to the Zariski topology. Thus, by

Theorem 2.2, every proper ideal I of D can be written as $I = J_1 \cap J_2 \cap ... \cap J_n$, where J_1, J_2,...,J_n are ideals of D such that each of the D/J_i is indecomposable. By Proposition 2.11, each of the J_i is primary, and so every proper ideal of D can be written as an intersection of primary ideals. Since D is an almost Dedekind domain, each nonzero primary ideal of D is a power of its radical. Thus, every nonzero proper ideal I of D can be written as $I = J_1 \cap J_2 \cap ... \cap J_n$, such that each of the J_i is a power of a maximal ideal, say $J_i = \mathfrak{m}_i^{r_i}$ for some $r_i \in \mathbb{N}$. Hence, $I = J_1 \cap J_2 \cap ... \cap J_n = \mathfrak{m}_1^{r_1} \cap \mathfrak{m}_2^{r_2} \cap ... \cap \mathfrak{m}_n^{r_n} = \mathfrak{m}_1^{r_1}\mathfrak{m}_2^{r_2}...\mathfrak{m}_n^{r_n}$. Therefore, D is a Dedekind domain. □

Theorem 2.14. *Let D be an integral domain which is not a field, then the following statements are equivalent:*

(1) *D is an almost Dedekind domain and each nonzero element of D is contained in only finitely many maximal ideals of D.*

(2) *D is an almost Dedekind domain such that* $\mathrm{Max}(D)$ *is a Noetherian topological space as a subspace of* $\mathrm{Spec}(D)$ *with respect to the Zariski topology.*

Proof. By Theorem 2.13 and [4, Theorem 37.2]. □

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