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A Note on Gaussian Series Rings

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Abstract. In this paper, we define a new kind of formal power series rings by using Gaussian binomial coefficients and investigate some properties. More precisely, we call such a ring a Gaussian series ring and study McCoy's theorem, Hermite properties and Noetherian properties.

1. Introduction

1.1. Gaussian binomial coefficients

Let $\mathbb{N}_0$ be the set of nonnegative integers, q a prime power, and GF(q) the Galois field with q elements. For $n, k \in \mathbb{N}_0$ with $n \geq k$, the *Gaussian binomial coefficient* (or *q-binomial coefficient*) is defined to be the number of k-dimensional subspaces of an n-dimensional vector space over GF(q), and is denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$. It

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is well known that for $n, k \in \mathbb{N}_0$ with $n \geq k$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0. \end{cases}$$

For an $n \in \mathbb{N}_0$ and a prime power q, the *q-bracket* (or *q-number*) is given by

$$[n]_q = \begin{cases} \frac{q^n-1}{q-1} & \text{if } n \geq 1 \\ 1 & \text{if } n = 0, \end{cases}$$

and the *q-factorial* is defined to be

$$[n]_q! = \begin{cases} [n]_q[n-1]_q \cdots [1]_q & \text{if } n \geq 1 \\ 1 & \text{if } n = 0. \end{cases}$$

Then for all $n, k \in \mathbb{N}_0$ with $n \geq k$ and a prime power q, we obtain

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q[k-1]_q \cdots [1]_q} = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

For $n, k \in \mathbb{N}_0$, $\binom{n}{k}$ denotes the binomial coefficient and $n!$ means the factorial. Note that if we regard the Gaussian binomial coefficient as a function of the real variable q (where n and k are fixed nonnegative integers with $n \geq k$), then easy calculation shows that $\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$ and $\lim_{q \to 1}[n]_q! = n!$; so the Gaussian binomial coefficient (resp., q-factorial) can be viewed as the q-analogue of the binomial coefficient (resp., factorial).

For more on Gaussian binomial coefficients, the readers can refer to [2, Section 9.2].

1.2. Motivation

A study of ring extensions is one of important topics in commutative algebra. In particular, it has been actively studied how to extend some properties of base rings to formal power series rings. In [5], Keigher investigated a special kind of formal power series rings, which is the so-called Hurwitz series ring. Later, the author in [6] found out that Hurwitz in [3] first considered the product of power series by using binomial coefficients, and called it a Hurwitz series in order to celebrate the contribution of Hurwitz.

We now review the definition of Hurwitz series rings. Let R be a commutative ring with identity and $\mathrm{H}(R)$ the set of formal power series over R. Define the addition $+$ and the multiplication $*$ on $\mathrm{H}(R)$ as follows: For $f = \sum_{n=0}^{\infty} a_n X^n, g = \sum_{n=0}^{\infty} b_n X^n \in \mathrm{H}(R)$,

$$f + g := \sum_{n=0}^{\infty} (a_n + b_n) X^n \text{ and } f * g := \sum_{n=0}^{\infty} c_n X^n,$$

where $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$. Then $\mathrm{H}(R)$ becomes a commutative ring with identity under these operations and is called the *Hurwitz series ring* over R. It was shown in [6, Propositions 2.3 and 2.4] that the Hurwitz series ring is isomorphic to the formal power series ring if and only if the base ring contains the field of rational numbers; so Hurwitz series rings generally have different algebraic structures from formal power series rings.

In [1], Benhissi and Koja studied several properties of Hurwitz series rings including McCoy's theorem and Noetherian properties. In [9], Liu investigated L-Hermite property of Hurwitz series rings. In [8], the authors studied chain conditions on composite Hurwitz series rings.

Motivated by the construction of Hurwitz series rings and the relation between the binomial coefficient and the Gaussian binomial coefficient, in this paper, we define a new kind of formal power series rings and call it the Gaussian series ring. (Definition of Gaussian series rings will be given in Section 2.) We also study McCoy's theorem, Hermite properties, and Noetherian properties in Gaussian series rings.

2. Basic Results

In this section, we define the notion of Gaussian series rings and study some properties. Let R be a commutative ring with identity, q a prime power, and $\mathrm{G}_q(R)$ the set of formal power series over R. Define the addition $+$ and the multiplication $\star$ on $\mathrm{G}_q(R)$ as follows: For $f = \sum_{n=0}^{\infty} a_n X^n, g = \sum_{n=0}^{\infty} b_n X^n \in \mathrm{G}_q(R)$,

$$f + g := \sum_{n=0}^{\infty} (a_n + b_n) X^n \text{ and } f \star g := \sum_{n=0}^{\infty} c_n X^n,$$

where $c_n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q a_i b_{n-i}$. With these operations, we can show that $\mathrm{G}_q(R)$ is a commutative ring with identity. The proof is routine; so we omit it.

Since the Gaussian binomial coefficient is the q-analogue of the binomial coefficient, the concept of Gaussian series rings may be regarded as the q-analogue of that of Hurwitz series rings.

Definition 2.1. Let R be a commutative ring with identity and q a prime power. Then $(G_q(R), +, \star)$ is called the *Gaussian series ring* over R with respect to q and an element of $G_q(R)$ is said to be a *Gaussian series.*

Our first result gives a relation between the Gaussian series ring and the formal power series ring.

Proposition 2.2. *Let R be a commutative ring with identity and q a prime power. Then a map $\varphi : R[\![X]\!] \to G_q(R)$ defined by $\varphi\big(\sum_{n=0}^{\infty} a_n X^n\big) = \sum_{n=0}^{\infty} [n]_q! a_n X^n$ is a ring homomorphism. In particular, φ is an isomorphism if and only if $\frac{1}{[n]_q!} \in R$ for all $n \in \mathbb{N}_0$.*

Proof. Let $\sum_{n=0}^{\infty} a_nX^n, \sum_{n=0}^{\infty} b_nX^n \in R[\![X]\!]$. Then we obtain

$$\begin{aligned}
\varphi\left(\sum_{n=0}^{\infty} a_nX^n + \sum_{n=0}^{\infty} b_nX^n\right) &= \varphi\left(\sum_{n=0}^{\infty}(a_n+b_n)X^n\right) \\
&= \sum_{n=0}^{\infty}[n]_q!(a_n+b_n)X^n \\
&= \sum_{n=0}^{\infty}[n]_q!a_nX^n + \sum_{n=0}^{\infty}[n]_q!b_nX^n \\
&= \varphi\left(\sum_{n=0}^{\infty} a_nX^n\right) + \varphi\left(\sum_{n=0}^{\infty} b_nX^n\right).
\end{aligned}$$

Also, we have

$$\begin{aligned}
\varphi\left(\left(\sum_{n=0}^{\infty} a_nX^n\right)\left(\sum_{n=0}^{\infty} b_nX^n\right)\right) &= \varphi\left(\sum_{n=0}^{\infty}\left(\sum_{i+j=n} a_ib_j\right)X^n\right) \\
&= \sum_{n=0}^{\infty}\left([n]_q!\sum_{i+j=n} a_ib_j\right)X^n
\end{aligned}$$

and

$$\begin{aligned}
\varphi\left(\sum_{n=0}^{\infty} a_nX^n\right) \star \varphi\left(\sum_{n=0}^{\infty} b_nX^n\right) &= \left(\sum_{n=0}^{\infty}[n]_q!a_nX^n\right) \star \left(\sum_{n=0}^{\infty}[n]_q!b_nX^n\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{i+j=n}\begin{bmatrix} n \\ i \end{bmatrix}_q [i]_q![j]_q!a_ib_j\right)X^n \\
&= \sum_{n=0}^{\infty}\left([n]_q!\sum_{i+j=n} a_ib_j\right)X^n;
\end{aligned}$$

so $\varphi\Big(\Big(\sum_{n=0}^{\infty} a_nX^n\Big)\Big(\sum_{n=0}^{\infty} b_nX^n\Big)\Big) = \varphi\Big(\sum_{n=0}^{\infty} a_nX^n\Big) \star \varphi\Big(\sum_{n=0}^{\infty} b_nX^n\Big)$. Thus φ is a ring homomorphism.

To show the second statement, we first suppose that φ is an isomorphism. Then for any $n \in \mathbb{N}_0$, there exists an element $\frac{1}{[n]_q!}X^n \in R[\![X]\!]$ such that $\varphi\Big(\frac{1}{[n]_q!}X^n\Big) = X^n$. Thus $\frac{1}{[n]_q!} \in R$. For the converse, we suppose that $\frac{1}{[n]_q!} \in R$ for all $n \in \mathbb{N}_0$. If $\varphi\Big(\sum_{n=0}^{\infty} a_nX^n\Big) = 0$, then $\sum_{n=0}^{\infty}[n]_q!a_nX^n = 0$; so $[n]_q!a_n = 0$ for all $n \in \mathbb{N}_0$. Since $\frac{1}{[n]_q!} \in R$ for all $n \in \mathbb{N}_0$, $a_n = 0$ for all $n \in \mathbb{N}_0$. Hence $\sum_{n=0}^{\infty} a_nX^n = 0$, which shows that φ is one-to-one. Let $\sum_{n=0}^{\infty} b_nX^n \in \mathrm{G}_q(R)$. Then by the assumption, we can find an element $\sum_{n=0}^{\infty} \frac{1}{[n]_q!} b_n X^n \in R[\![X]\!]$ such that $\varphi\left(\sum_{n=0}^{\infty} \frac{1}{[n]_q!} b_n X^n\right) = \sum_{n=0}^{\infty} b_n X^n$. Hence φ is onto. Thus φ is an isomorphism. $\square$

Let R be a commutative ring with identity, $\mathbb{Z}$ the ring of integers, and q a prime power. Then R may be viewed as a $\mathbb{Z}$-module in the usual sense. Recall that R is *torsion-free* if whenever $na = 0$ for $n \in \mathbb{Z}$ and $a \in R$, $n = 0$ or $a = 0$. As the q-analogue of 'torsion-free', we define the concept of 'q-torsion-free. We say that R is *q-torsion-free* if whenever $\begin{bmatrix} n \\ r \end{bmatrix}_q a = 0$ for $a \in R$ and $n, r \in \mathbb{N}_0$ with $n \geq r$, $a = 0$. Clearly, if R is torsion-free, then R is q-torsion-free. The following examples show that R being q-torsion-free does not imply that R is torsion-free.

Example 2.3. Let q be a prime power.

(1) Note that for all $n, r \in \mathbb{N}_0$ with $n \geq r$, $\begin{bmatrix} n \\ k \end{bmatrix}_q \equiv 1 \pmod{q}$. Hence $\mathbb{Z}_q$ is q-torsion-free but not torsion-free.

(2) If $k = 0$ or $k = n$, then $\begin{bmatrix} n \\ k \end{bmatrix}_q = 1$; so we assume that $n \geq 2$ and $k \in \{1, \ldots, n-1\}$. Then $\begin{bmatrix} n \\ k \end{bmatrix}_q \equiv q + 1 \pmod{q^2}$. Now, it is easy to check that if $(q+1)a \equiv 0 \pmod{q^2}$ for $a \in \mathbb{Z}_{q^2}$, then $a \equiv 0 \pmod{q^2}$. Hence $\mathbb{Z}_{q^2}$ is q-torsion-free but not torsion-free.

The next result characterizes when the Gaussian series ring is an integral domain.

Proposition 2.4. *Let R be a commutative ring with identity and q a prime power. Then the following conditions are equivalent.*

(1) *R is a q-torsion-free integral domain.*

(2) *$\mathrm{G}_q(R)$ is an integral domain.*

Proof. (1) $\Rightarrow$ (2) Suppose that $f = \sum_{n=0}^{\infty} a_n X^n$, $g = \sum_{n=0}^{\infty} b_n X^n$ are nonzero elements of $\mathrm{G}_q(R)$ such that $f \star g = 0$, and let m_1, m_2 be the smallest nonnegative integers such that $a_{m_1} \neq 0$ and $b_{m_2} \neq 0$. Then by comparing the coefficients of $X^{m_1+m_2}$ in $f \star g = 0$, we have

$$\begin{bmatrix} m_1 + m_2 \\ m_1 \end{bmatrix}_q a_{m_1} b_{m_2} = 0.$$

Since R is q-torsion-free, $a_{m_1} b_{m_2} = 0$. Since R is an integral domain, $a_{m_1} = 0$ or $b_{m_2} = 0$, a contradiction. Thus $\mathrm{G}_q(R)$ is an integral domain.

(2) $\Rightarrow$ (1) Let $a, b \in R$ such that $ab = 0$. Then $a, b \in \mathrm{G}_q(R)$ such that $a \star b = 0$. Since $\mathrm{G}_q(R)$ is an integral domain, $a = 0$ or $b = 0$. Thus R is an integral domain. Suppose that $\begin{bmatrix} n \\ r \end{bmatrix}_q a = 0$ for $a \in R$ and $n, r \in \mathbb{N}_0$ with $n \geq r$. Then $aX^r \star X^{n-r} =$

$\begin{bmatrix} n \\ r \end{bmatrix}_q aX^n = 0$. Since $\mathrm{G}_q(R)$ is an integral domain and $X^{n-r} \neq 0$, $aX^r = 0$; so $a = 0$. Thus R is q-torsion-free. $\square$

Let R be a commutative ring with identity. Then $\mathrm{Idem}(R)$ denotes the set of idempotent elements of R.

Proposition 2.5. *Let R be a commutative ring with identity and q a prime power. Then $Idem(G_q(R)) = Idem(R)$.*

Proof. Clearly, $\mathrm{Idem}(R) \subseteq \mathrm{Idem}(\mathrm{G}_q(R))$. For the reverse containment, let $f = \sum_{i=0}^{\infty} a_i X^i \in \mathrm{Idem}(\mathrm{G}_q(R))$. We first claim that $a_0 a_n = 0$ for all $n \in \mathbb{N}$. Since $a_0^2 = a_0$ and $2a_0a_1 = a_1$, $2a_0a_1 = a_0a_1$; so $a_0a_1 = 0$. Suppose that $a_0a_1 = \cdots = a_0a_m = 0$ for some positive integer m. Then by comparing the coefficients of X^{m+1} in $f \star f = f$, we obtain

$$\sum_{i=0}^{m+1} \begin{bmatrix} m+1 \\ i \end{bmatrix}_q a_i a_{m+1-i} = a_{m+1}.$$

By multiplying a_0 in both sides, $2a_0^2 a_{m+1} = a_0 a_{m+1}$. Since $a_0^2 = a_0$, $a_0 a_{m+1} = 0$. Hence by the induction, $a_0 a_n = 0$ for all $n \in \mathbb{N}$.

We next show that $f \in R$. Suppose to the contrary that $f \notin R$. Let k be the smallest positive integer such that $a_k \neq 0$, and set $g = \sum_{i=k}^{\infty} a_i X^i$. Then $f = a_0 + g$ and $a_0 \star g = 0$. Since $f \star f = f$ and $a_0^2 = a_0$, $g \star g = g$. Hence by comparing the coefficients of X^k in $g \star g = g$, $a_k = 0$. This contradicts the choice of k. Thus $f \in R$, which means that $f \in \mathrm{Idem}(R)$. $\square$

The final result in this section gives an equivalent condition for a Gaussian series to be a unit.

Proposition 2.6. *Let R be a commutative ring with identity and q a prime power. Then $\sum_{n=0}^{\infty} a_n X^n$ is a unit in $G_q(R)$ if and only if a_0 is a unit in R.*

Proof. ($\Rightarrow$) Suppose that $\sum_{n=0}^{\infty} a_n X^n$ is a unit in $\mathrm{G}_q(R)$. Then there exists an element $\sum_{n=0}^{\infty} b_n X^n \in \mathrm{G}_q(R)$ such that $\left(\sum_{n=0}^{\infty} a_n X^n\right) \star \left(\sum_{n=0}^{\infty} b_n X^n\right) = 1$; so $a_0 b_0 = 1$. Thus a_0 is a unit in R.

($\Leftarrow$) To show that $\sum_{n=0}^{\infty} a_n X^n$ is a unit in $\mathrm{G}_q(R)$, we construct a suitable element $\sum_{n=0}^{\infty} b_n X^n \in \mathrm{G}_q(R)$ such that $\left(\sum_{n=0}^{\infty} a_n X^n\right) \star \left(\sum_{n=0}^{\infty} b_n X^n\right) = 1$. Since a_0 is a unit in R, we can find an element $b_0 \in R$ such that $a_0 b_0 = 1$. For each $n \in \mathbb{N}$, set

$$b_n = \frac{-\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q a_i b_{n-i}}{a_0}.$$

Then it is easy to check that $\left(\sum_{n=0}^{\infty} a_n X^n\right) \star \left(\sum_{n=0}^{\infty} b_n X^n\right) = 1$. Thus $\sum_{n=0}^{\infty} a_n X^n$ is a unit in $\mathrm{G}_q(R)$. $\square$

3. McCoy's Theorem

Let R be a commutative ring with identity and q a prime power. Then *McCoy's theorem for* $\mathrm{G}_q(R)$ holds if for any zero-divisor $f \in \mathrm{G}_q(R)$, there exists a nonzero element $a \in R$ such that $a \star f = 0$. In this section, we study McCoy's theorem for $\mathrm{G}_q(R)$.

Proposition 3.1. *Let R be a commutative ring with identity, q a prime power, and let f, g be nonzero elements of $G_q(R)$ such that $f \star g = 0$. If R is q-torsion-free and the ideal of R generated by the first nonzero coefficient of g contains a nonzero idempotent element c, then $c \star f = 0$.*

Proof. Write $f = \sum_{i=0}^{\infty} a_i X^i$ and $g = \sum_{i=m}^{\infty} b_i X^i$ with $b_m \neq 0$, and let $c \in (b_m)$ be a nonzero idempotent element. Then $c = rb_m$ for some $r \in R$. Since $f \star g = 0$, $f \star (r \star g) = 0$; so $ca_0 = 0$. Suppose that $ca_0 = \cdots = ca_n = 0$ for some nonnegative integer n. By comparing the coefficients of X^{m+n+1} in $f \star (r \star g) = 0$, we obtain

$$\begin{bmatrix} m+n+1 \\ 0 \end{bmatrix}_q a_0(rb_{m+n+1}) + \cdots + \begin{bmatrix} m+n+1 \\ n+1 \end{bmatrix}_q a_{n+1}c = 0.$$

By multiplying c in both sides, we have

$$\begin{bmatrix} m+n+1 \\ n+1 \end{bmatrix}_q a_{n+1}c^2 = 0.$$

Since R is q-torsion-free and c is idempotent, $ca_{n+1} = 0$. Hence by the induction, $ca_k = 0$ for all $k \in \mathbb{N}_0$. Thus $c \star f = 0$. □

Theorem 3.2. *Let R be a commutative ring with identity and q a prime power. If McCoy's theorem for $G_q(R)$ holds, then R is q-torsion-free.*

Proof. Suppose to the contrary that there exist a nonzero element $a \in R$ and nonnegative integers $n > k$ such that $\begin{bmatrix} n \\ k \end{bmatrix}_q a = 0$. Then $X^{n-k} \star aX^k = \begin{bmatrix} n \\ k \end{bmatrix}_q aX^n = 0$; so X^{n-k} is a zero-divisor of $\mathrm{G}_q(R)$. Note that $b * X^{n-k} \neq 0$ for all $b \in R \setminus \{0\}$. This contradicts the assumption that McCoy's theorem for $\mathrm{G}_q(R)$ holds. Thus R is q-torsion-free. □

Corollary 3.3. *Let R be a commutative ring with identity and q a prime power. If $char(R) = \begin{bmatrix} n \\ k \end{bmatrix}_q$ for some positive integers $n > k$, then McCoy's theorem for $G_q(R)$ does not hold.*

Proof. Note that for any nonzero element $a \in R$, $\begin{bmatrix} n \\ k \end{bmatrix}_q a = 0$; so R is not q-torsion-free. Thus by Theorem 3.2, McCoy's theorem for $\mathrm{G}_q(R)$ does not hold. □

Let R be a commutative ring with identity. Then R is said to be *reduced* if it has no nonzero nilpotent elements. The next result shows that if R is reduced, then the converse of Theorem 3.2 holds.

Theorem 3.4. *Let R be a commutative ring with identity, q a prime power, and let $f = \sum_{i=0}^{\infty} a_i X^i, g = \sum_{i=0}^{\infty} b_i X^i \in G_q(R)$. If R is reduced and q-torsion-free, then $f \star g = 0$ if and only if $a_i b_j = 0$ for all $i, j \in \mathbb{N}_0$.*

Proof. The "if" part is obvious. To show the converse, we suppose that $f \star g = 0$. Then $a_0 b_0 = 0$. We first claim that $a_n b_0 = 0$ for all $n \in \mathbb{N}_0$. Suppose that $a_0 b_0 = \cdots = a_m b_0 = 0$ for some nonnegative integer m. Then by calculating the coefficients of X^{m+1} in $f \star g = 0$, we obtain

$$\sum_{i=0}^{m+1} \begin{bmatrix} m+1 \\ i \end{bmatrix}_q a_i b_{m+1-i} = 0.$$

By multiplying b_0 in both sides, $a_{m+1} b_0^2 = 0$; so $(a_{m+1} b_0)^2 = 0$. Since R is reduced, $a_{m+1} b_0 = 0$. Hence by the induction, $a_n b_0 = 0$ for all $n \in \mathbb{N}_0$.

We next prove that $a_i b_j = 0$ for all $i, j \in \mathbb{N}_0$. Suppose that $a_n b_0 = a_n b_1 = \cdots = a_n b_m = 0$ for all $n \in \mathbb{N}_0$ and some nonnegative integer m. If $b_{m+1} = 0$, then $a_n b_{m+1} = 0$ for all $n \in \mathbb{N}_0$; so we assume that $b_{m+1} \neq 0$. Then we obtain

$$\begin{aligned} 0 &= f \star g \\ &= \left(\sum_{i=0}^{\infty} a_i X^i \right) \star \left(\sum_{i=m+1}^{\infty} b_i X^i \right); \end{aligned}$$

so $a_0 b_{m+1} = 0$. If $a_0 b_{m+1} = \cdots = a_k b_{m+1} = 0$ for some nonnegative integer k, then a similar argument as in the proof of the previous claim shows that $a_{k+1} b_{m+1} = 0$. Hence by the induction, $a_n b_{m+1} = 0$ for all $n \in \mathbb{N}_0$. Thus by applying the induction again, $a_i b_j = 0$ for all $i, j \in \mathbb{N}_0$. $\square$

Let R be a commutative ring with identity and let $a \in R$. Then $\operatorname{ann}_R(a)$ denotes the annihilator of a, *i.e.*, $\operatorname{ann}_R(a) := \{r \in R \mid ra = 0\}$. Also, if I is an ideal of R, then $\mathrm{G}_q(I) := \{\sum_{n=0}^{\infty} a_n X^n \mid a_n \in I \text{ for all } n \in \mathbb{N}_0\}$ is an ideal of $\mathrm{G}_q(R)$.

Corollary 3.5. *Let R be a commutative ring with identity, q a prime power, and let $f = \sum_{i=0}^{\infty} a_i X^i \in G_q(R)$. If R is reduced and q-torsion-free, then the following assertions hold.*

(1) $ann_{G_q(R)}(f) = G_q\big(\bigcap_{i=0}^{\infty} ann_R(a_i)\big)$.

(2) *f is a regular element of $G_q(R)$ if and only if $\bigcap_{i=0}^{\infty} ann_R(a_i) = (0)$.*

Proof. (1) If $g = \sum_{i=0}^{\infty} b_i X^i \in \operatorname{ann}_{\mathrm{G}_q(R)}(f)$, then $f \star g = 0$. By Theorem 3.4, $a_i b_j = 0$ for all $i, j \in \mathbb{N}_0$; so $b_j \in \bigcap_{i=0}^{\infty} \operatorname{ann}_R(a_i)$ for all $j \in \mathbb{N}_0$. Thus $g \in \mathrm{G}_q\big(\bigcap_{i=0}^{\infty} \operatorname{ann}_R(a_i)\big)$. To show the reverse inclusion, let $h = \sum_{i=0}^{\infty} c_i X^i \in$

$\mathrm{G}_q\big(\bigcap_{i=0}^{\infty}\mathrm{ann}_R(a_i)\big)$. Then $a_ic_j=0$ for all $i,j\in\mathbb{N}_0$. Thus by Theorem 3.4, $f\star h=0$, which means that $h\in\mathrm{ann}_{\mathrm{G}_q(R)}(f)$.

(2) If $b\in\bigcap_{i=0}^{\infty}\mathrm{ann}_R(a_i)$, then $b\star f=0$ by (1). Since f is a regular element of $\mathrm{G}_q(R)$, $b=0$. Thus $\bigcap_{i=0}^{\infty}\mathrm{ann}_R(a_i)=(0)$. Conversely, if $g=\sum_{i=0}^{\infty}b_iX^i$ is an element of $\mathrm{G}_q(R)$ with $f\star g=0$, then by (1), $b_j\in\bigcap_{i=0}^{\infty}\mathrm{ann}_R(a_i)$ for all $j\in\mathbb{N}_0$. Since $\bigcap_{i=0}^{\infty}\mathrm{ann}_R(a_i)=(0)$, $b_j=0$ for all $j\in\mathbb{N}_0$. Hence $g=0$, and thus f is a regular element of $\mathrm{G}_q(R)$. □

Let R be a commutative ring with identity. Then $\mathrm{Nil}(R)$ denotes the set of nilpotent elements of R, and $\mathrm{Z}(R)$ means the set of zero-divisors of R. Also, a prime ideal of R is said to be *divided* if it is comparable to any principal ideal of R. We are closing this section by considering the case when R is not reduced.

Theorem 3.6. *Let R be a commutative ring with identity and q a prime power. If R is not reduced and $Nil(R)$ is a divided prime ideal of R which is different from $Z(R)$, then any element of $G_q(R)$ with constant term in $Z(R)\setminus Nil(R)$ is annihilated by a nonzero element of $G_q(Nil(R))$.*

Proof. Let $f=\sum_{n=0}^{\infty}a_nX^n\in\mathrm{G}_q(R)$ with $a_0\in\mathrm{Z}(R)\setminus\mathrm{Nil}(R)$. We now construct a nonzero element $g=\sum_{n=0}^{\infty}b_nX^n\in\mathrm{G}_q(\mathrm{Nil}(R))$ such that $f\star g=0$. Since a_0 is a zero-divisor of R, there exists a nonzero element $b_0\in R$ such that $a_0b_0=0$. Since $\mathrm{Nil}(R)$ is a prime ideal of R and $a_0\notin\mathrm{Nil}(R)$, $b_0\in\mathrm{Nil}(R)$. Also, since $\mathrm{Nil}(R)$ is divided and $a_0\notin\mathrm{Nil}(R)$, $\mathrm{Nil}(R)\subsetneq(a_0)$. Note that $-a_1b_0\in\mathrm{Nil}(R)$; so $-a_1b_0=a_0b_1$ for some $b_1\in R$. Since $\mathrm{Nil}(R)$ is a prime ideal of R and $a_0\notin\mathrm{Nil}(R)$, $b_1\in\mathrm{Nil}(R)$. Suppose that $b_0,\dots,b_n\in\mathrm{Nil}(R)$ for some nonnegative integer n. Then $-\left(\begin{bmatrix}n+1\\1\end{bmatrix}_q a_1b_n+\cdots+\begin{bmatrix}n+1\\n+1\end{bmatrix}_q a_{n+1}b_0\right)\in\mathrm{Nil}(R)$. Since $\mathrm{Nil}(R)\subsetneq(a_0)$, we obtain

$$-\left(\begin{bmatrix}n+1\\1\end{bmatrix}_q a_1b_n+\cdots+\begin{bmatrix}n+1\\n+1\end{bmatrix}_q a_{n+1}b_0\right)=a_0b_{n+1}$$

for some $b_{n+1}\in R$. Since $\mathrm{Nil}(R)$ is a prime ideal of R and $a_0\notin\mathrm{Nil}(R)$, $b_{n+1}\in\mathrm{Nil}(R)$. By the induction, we obtain an infinite sequence $(b_n)_{n\geq 0}$ in $\mathrm{Nil}(R)$ such that $\begin{bmatrix}n\\0\end{bmatrix}_q a_0b_n+\cdots+\begin{bmatrix}n\\n\end{bmatrix}_q a_nb_0=0$ for all $n\in\mathbb{N}_0$. Thus by setting $g=\sum_{n=0}^{\infty}b_nX^n\in\mathrm{G}_q(\mathrm{Nil}(R))$, we deduce that $f\star g=0$. □

4. Hermite Rings

Let R be a commutative ring with identity. Recall from [4, page 465] that R is a *right K-Hermite ring* if for any $a,b\in R$, there exist an element $r\in R$ and a 2×2 invertible matrix M over R such that $\begin{pmatrix}a & b\end{pmatrix}M=\begin{pmatrix}r & 0\end{pmatrix}$.

Theorem 4.1. *Let R be a commutative ring with identity and q a prime power. If $G_q(R)$ is a right K-Hermite ring, then R is a right K-Hermite ring.*

Proof. Let $a, b \in R$. Then $a, b \in \mathrm{G}_q(R)$. Since $\mathrm{G}_q(R)$ is a right K-Hermite ring, we can find an element $f \in \mathrm{G}_q(R)$ and an invertible matrix $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ over $\mathrm{G}_q(R)$ such that $\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} f & 0 \end{pmatrix}$; so $\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} g_{11}(0) & g_{12}(0) \\ g_{21}(0) & g_{22}(0) \end{pmatrix} = \begin{pmatrix} f(0) & 0 \end{pmatrix}$. Note that the determinant of $\begin{pmatrix} g_{11}(0) & g_{12}(0) \\ g_{21}(0) & g_{22}(0) \end{pmatrix}$ is the constant term of the determinant of $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$; so by Proposition 2.6, the determinant of $\begin{pmatrix} g_{11}(0) & g_{12}(0) \\ g_{21}(0) & g_{22}(0) \end{pmatrix}$ is a unit in R. Hence $\begin{pmatrix} g_{11}(0) & g_{12}(0) \\ g_{21}(0) & g_{22}(0) \end{pmatrix}$ is an invertible matrix over R. Thus R is a right K-Hermite ring. $\square$

The next example shows that the Gaussian series ring over a right K-Hermite ring need not be a right K-Hermite ring.

Example 4.2. Let $\mathbb{Z}$ be the ring of integers.

(1) Let $a, b \in \mathbb{Z}$. If $b = 0$, then $\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \end{pmatrix}$. Suppose that $b \neq 0$. If $a = 0$, then $\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \end{pmatrix}$; so we assume that $a \neq 0$. Let d be the greatest common divisor of a and b. Then $a = da'$ and $b = db'$ for some $a', b' \in \mathbb{Z}$. Since a' and b' are relative prime, there exist $\alpha, \beta \in \mathbb{Z}$ such that $a'\alpha + b'\beta = 1$; so $\begin{pmatrix} a' & b' \end{pmatrix} \begin{pmatrix} \alpha & -b' \\ \beta & a' \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Hence $\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \alpha & -b' \\ \beta & a' \end{pmatrix} = \begin{pmatrix} d & 0 \end{pmatrix}$. Note that $\begin{pmatrix} \alpha & -b' \\ \beta & a' \end{pmatrix}$ is invertible. Thus $\mathbb{Z}$ is a K-Hermite ring.

(2) Let q be any prime power, and let $f = 2 + X + X^2 + \cdots$, $g = X + X^2 + \cdots \in \mathrm{G}_q(\mathbb{Z})$. If $\mathrm{G}_q(\mathbb{Z})$ is a K-Hermite ring, then there exist an element $h \in \mathrm{G}_q(\mathbb{Z})$ and an invertible matrix $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ over $\mathrm{G}_q(\mathbb{Z})$ such that $\begin{pmatrix} f & g \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} h & 0 \end{pmatrix}$; so $f \star h_{12} + g \star h_{22} = 0$. Write $h_{12} = \sum_{n=0}^{\infty} c_n X^n$ and $h_{22} = \sum_{n=0}^{\infty} d_n X^n$. Then we obtain

$$\begin{aligned} 0 &= f \star h_{12} + g \star h_{22} \\ &= 2c_0 + (c_0 + 2c_1 + d_0)X + \cdots ; \end{aligned}$$

so $c_0 = 0$ and d_0 is a multiple of 2. Hence the constant term of the determinant of $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ cannot be ± 1, which means that $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ is not invertible by Proposition 2.6. This is absurd. Thus $\mathrm{G}_q(\mathbb{Z})$ is not a K-Hermite ring.

Let R be a commutative ring with identity. Recall that R is an *L-Hermite ring* if for any $(a_1, \ldots, a_m) \in R^m$ with $a_1 R + \cdots + a_m R = R$, there exists an $m \times m$ invertible matrix over R with the first row vector $(a_1, \ldots, a_m)$.

Theorem 4.3. *Let R be a commutative ring with identity and q a prime power. Then the following statements are equivalent.*

(1) *R is an L-Hermite ring.*

(2) *$G_q(R)$ is an L-Hermite ring.*

Proof. (1) $\Rightarrow$ (2) Let $(f_1, \ldots, f_m) \in \mathrm{G}_q(R)^m$ be such that $f_1 \star \mathrm{G}_q(R) + \cdots + f_m \star \mathrm{G}_q(R) = \mathrm{G}_q(R)$. Then we can find an element $(g_1, \ldots, g_m) \in \mathrm{G}_q(R)^m$ such that $f_1 \star g_1 + \cdots + f_m \star g_m = 1$; so $f_1(0)g_1(0) + \cdots + f_m(0)g_m(0) = 1$. Therefore $f_1(0)R + \cdots + f_m(0)R = R$. Since R is an L-Hermite ring, there exists an $m \times m$ invertible matrix $P = \begin{pmatrix} f_1(0) & f_2(0) & \cdots & f_m(0) \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mm} \end{pmatrix}$ over R. Let $Q = \begin{pmatrix} f_1 & f_2 & \cdots & f_m \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mm} \end{pmatrix}$. Then Q is an $m \times m$ matrix over $\mathrm{G}_q(R)$. Note that the constant term of the determinant of Q is precisely the same as the determinant of P; so by Proposition 2.6, the determinant of Q is a unit in $\mathrm{G}_q(R)$. Hence Q is invertible. Thus $\mathrm{G}_q(R)$ is an L-Hermite ring.

(2) $\Rightarrow$ (1) Let $(a_1, \ldots, a_m) \in R^m$ be such that $a_1R + \cdots + a_mR = R$. Then $a_1 \star \mathrm{G}_q(R) + \cdots + a_m \star \mathrm{G}_q(R) = \mathrm{G}_q(R)$. Since $\mathrm{G}_q(R)$ is an L-Hermite ring, there exists an $m \times m$ invertible matrix $M = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ f_{21} & f_{22} & \cdots & f_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mm} \end{pmatrix}$ over $\mathrm{G}_q(R)$. Let $N = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ f_{21}(0) & f_{22}(0) & \cdots & f_{2m}(0) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1}(0) & f_{m2}(0) & \cdots & f_{mm}(0) \end{pmatrix}$. Since the determinant of M is a unit in $\mathrm{G}_q(R)$, Proposition 2.6 shows that the determinant of N is a unit in R. Hence N is an $m \times m$ invertible matrix over R. Thus R is an L-Hermite ring. $\square$

Let R be a commutative ring with identity and M a unitary R-module. Then M is *free* if M has a basis; and M is *stably free* if there exist positive integers m and n such that $M \oplus R^m = R^n$. Clearly, free modules are stably free.

Corollary 4.4. *Let R be a commutative ring with identity and q a prime power. Then the following assertions are equivalent.*

(1) *Every stably free R-module is free.*

(2) *Every stably free $G_q(R)$-module is free.*

Proof. Note that if T is a commutative ring with identity, then T is an L-Hermite ring if and only if every stably free T-module is free [7, Chapter I, Corollary 4.5]. Thus the equivalence follows from Theorem 4.3. $\square$

Remark 4.5. Note that by Theorem 4.3, the Gaussian series ring over an L-Hermite ring is L-Hermite; so Example 4.2 indicates that the notion of L-Hermite rings is different from that of right K-Hermite rings.

5. Noetherian Properties

Let R be a commutative ring with identity and q a prime power. Recall that R is a *Noetherian ring* if every ideal of R is finitely generated (or equivalently, R satisfies the ascending chain condition on integral ideals). In this section, we study the Noetherian properties in $\mathrm{G}_q(R)$.

Lemma 5.1. *Let R be a commutative ring with identity, q a prime power, and I an ideal of R. Then $G_q(I) = I \star G_q(R)$ if and only if for any countable subset C of I, there exists a finitely generated ideal F of R such that $C \subseteq F \subseteq I$.*

Proof. ($\Rightarrow$) Let $C = \{c_i \,|\, i \in \mathbb{N}_0\}$ be a countable subset of I, and let $f = \sum_{i=0}^{\infty} c_i X^i \in \mathrm{G}_q(I)$. Since $\mathrm{G}_q(I) = I \star \mathrm{G}_q(R)$, we can find $b_1, \dots, b_n \in I$ and $g_1, \dots, g_n \in \mathrm{G}_q(R)$ such that $f = b_1 \star g_1 + \cdots + b_n \star g_n$. Let $F = (b_1, \dots, b_n)$. Then F is a finitely generated ideal of R such that $C \subseteq F \subseteq I$.

($\Leftarrow$) Clearly, $I \star \mathrm{G}_q(R) \subseteq \mathrm{G}_q(I)$, because $I \subseteq \mathrm{G}_q(I)$. For the reverse containment, let $f = \sum_{i=0}^{\infty} a_i X^i \in \mathrm{G}_q(I)$. Then there exist suitable elements $b_1, \dots, b_n \in I$ such that $\{a_i \,|\, i \in \mathbb{N}_0\} \subseteq (b_1, \dots, b_n)$; so for each $i \in \mathbb{N}_0$, $a_i = \sum_{j=1}^{n} b_j c_{ij}$ for some $c_{ij} \in R$. Hence we obtain

$$f = \sum_{i=0}^{\infty} \left(\sum_{j=1}^{n} b_j c_{ij} \right) X^i = \sum_{j=1}^{n} \left(b_j \star \sum_{i=0}^{\infty} c_{ij} X^i \right) \in I \star \mathrm{G}_q(R).$$

Thus $\mathrm{G}_q(I) = I \star \mathrm{G}_q(R)$. □

As an immediate consequence of Lemma 5.1, we obtain

Proposition 5.2. *Let R be a commutative ring with identity and q a prime power. If I is a finitely generated ideal of R, then $G_q(I) = I \star G_q(R)$.*

Theorem 5.3. *Let R be a commutative ring with identity and q a prime power. Then the following statements are equivalent.*

(1) *R is a Noetherian ring.*

(2) *For each ideal I of R, $G_q(I) = I \star G_q(R)$.*

Proof. (1) $\Rightarrow$ (2) This implication follows from Proposition 5.2, because every ideal of a Noetherian ring is finitely generated.

(2) $\Rightarrow$ (1) Suppose to the contrary that R is not a Noetherian ring. Then there exists a strictly ascending chain of ideals $(I_n)_{n \geq 0}$ of R; so for each $n \geq 1$, we can choose an element $a_n \in I_n \setminus I_{n-1}$. Let $I = \bigcup_{n=0}^{\infty} I_n$. Then I is an ideal of R. Since $\mathrm{G}_q(I) = I \star \mathrm{G}_q(R)$, Lemma 5.1 indicates that there exists a finitely generated ideal

F of R such that $\{a_n \,|\, n \in \mathbb{N}\} \subseteq F \subseteq I$. Since F is finitely generated, $F \subseteq I_k$ for some positive integer k. Hence $a_{k+1} \in I_k$, which is a contradiction. Thus R is a Noetherian ring. $\square$

Let R be a commutative ring with identity and I an ideal of R. Then $\sqrt{I}$ denotes the radical of I. We end this article with the radical property of Gaussian series rings.

Proposition 5.4. *Let R be a commutative ring with identity and q a prime power. If I and J are ideals of R with $\mathrm{G}_q(J) = J \star \mathrm{G}_q(R)$ and $J \subseteq \sqrt{I}$, then there exists a positive integer n such that $J^n \subseteq I$.*

Proof. Deny the conclusion. Then for each $m \geq 1$, there exist $b_{m1}, \dots, b_{mm} \in J$ such that $b_{m1} \cdots b_{mm} \notin I$. Let C be the ideal of R generated by $\{b_{mi} \,|\, m \in \mathbb{N}$ and $1 \leq i \leq m\}$. Then C is a countably generated subideal of J such that $C^m \not\subseteq I$ for all $m \in \mathbb{N}$. Since $\mathrm{G}_q(J) = J \star \mathrm{G}_q(R)$, by Lemma 5.1, there exists a finitely generated ideal F of R such that $C \subseteq F \subseteq J$. Since F is finitely generated and $J \subseteq \sqrt{I}$, $F^k \subseteq I$ for some $k \in \mathbb{N}$. Hence $C^k \subseteq I$, which is absurd. Thus $J^n \subseteq I$ for some $n \geq 1$. $\square$

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