KYUNGPOOK Math. J. 57(2017), 393-400
https://doi.org/10.5666/KMJ.2017.57.3.393
pISSN 1225-6951 eISSN 0454-8124

Extension of Generalized Hurwitz-Lerch Zeta Function and Associated Properties

Junesang Choi*
Department of Mathematics, Dongguk University, Gyeongju 38066, Republic of Korea
e-mail : junesang@mail.dongguk.ac.kr

Rakesh Kumar Parmar
Department of Mathematics, Government College of Engineering and Technology, Bikaner 334004, Rajasthan State, India
e-mail : rakeshparmar27@gmail.com

Ravinder Krishna Raina
M. P. University of Agri. and Technology, Udaipur, Rajasthan, India
Present address: 10/11 Ganpati Vihar, Opposite Sector 5, Udaipur 313002, India
e-mail : rkraina_7@hotmail.com

Abstract. Very recently, Srivastava *et al.* [8] introduced an extension of the Pochhammer symbol and used it to define a generalization of the generalized hypergeometric functions. In this paper, by using the generalized Pochhammer symbol, we extend the generalized Hurwitz-Lerch Zeta function by Goyal and Laddha [6] and investigate some interesting properties which include various integral representations, Mellin transforms, differential formula and generating function. Some interesting special cases of our main results are also considered.

1. Introduction and Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ denote the sets of positive integers, negative integers, real numbers, positive real numbers and complex numbers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}$.

---

* Corresponding Author.
Received July 4, 2014; revised December 15, 2014; accepted April 27, 2015.
2010 Mathematics Subject Classification: Primary 11M06, 11M35, 33B15; Secondary 11B68, 33C65, 33C90.
Key words and phrases: generalized Pochhammer symbol, modified Bessel function, Hurwitz-Lerch Zeta function, generalized Hurwitz-Lerch Zeta function, Mellin Transform, generating function.

The *Hurwitz-Lerch Zeta function* $\Phi(z,s,a)$ is defined by (see, *e.g.*, [4, p. 27, Eq. 1.11(1)]; see also [9, p. 121] and [10, p. 194]):

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.1}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z|<1;\ \Re(s)>1 \quad \text{when} \quad |z|=1\right).$$

It is known (see, *e.g.*, [4, p. 27, Eq. 1.11(3)]; see also [10, p. 194, Eq. 2.5(4)]) that

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-(a-1)t}}{e^t - z}dt \tag{1.2}$$

$$\left(\Re(a)>0;\ \Re(s)>0 \quad \text{when} \quad |z| \leqq 1\ (z \neq 1);\ \Re(s)>1 \quad \text{when} \quad z=1\right).$$

For further properties and the special cases of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$, one may refer to [4, Chapter I], [9], [10] and [13, p. 280, Example 8], respectively.

Various generalizations of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ have been investigated by many authors (see, *e.g.*, [1, 2, 3, 4, 5]). Very recently, Srivastava [7] and Srivastava *et al.* [11, 12] have investigated certain generalizations of Hurwitz-Lerch Zeta function with their applications in a systematic and extensive way.

In particular, Goyal and Laddha [6, p. 100, Eq. (1.5)] generalized the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ as follows:

$$\Phi_\mu^*(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s} \tag{1.3}$$

$$\left(\mu \in \mathbb{C};\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z|<1; \Re(s-\mu)>1 \quad \text{when} \quad |z|=1\right)$$

and equivalently, by means of an integral representation as follows:

$$\Phi_\mu^*(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{(1-ze^{-t})^\mu}dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-(a-1)t}}{(e^t - z)^\mu}dt \tag{1.4}$$

$$\left(\Re(a)>0;\ \Re(s)>0 \quad \text{when} \quad |z| \leqq 1\ (z \neq 1);\ \Re(s)>1 \quad \text{when} \quad z=1\right).$$

Very recently, Srivastava *et al.* [8, p. 487, Eq.(15)] introduced and studied, in a rather systematic manner, the following family of generalized hypergeometric functions:

$${}_rF_s\left[\begin{array}{r} (\alpha_1,p),\alpha_2,\cdots,\alpha_r; \\ \beta_1,\cdots,\beta_s; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1;p)_n(\alpha_2)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n} \frac{z^n}{n!} \tag{1.5}$$

in terms of the generalized Pochhammer symbol $(\lambda;p)_\nu$ [8, p. 485, Eq.(8)]:

$$(\lambda;p)_\nu := \begin{cases} \dfrac{\Gamma_p(\lambda+\nu)}{\Gamma(\lambda)} & (\Re(p)>0;\ \lambda,\ \nu\in\mathbb{C}), \\ (\lambda)_\nu & (p=0;\ \lambda,\ \nu\in\mathbb{C}). \end{cases} \tag{1.6}$$

or, equivalently, by means of an integral representation [8, p. 485, Eq.(9)] as follows:

$$(\lambda;p)_\nu = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda+\nu-1}\exp\left(-t-\frac{p}{t}\right)dt \tag{1.7}$$

$$\big(\Re(p)>0;\ \Re(\lambda+\nu)>0 \quad \text{when} \quad p=0\big).$$

Here, and in what follows, $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$ denotes the Pochhammer symbol (or the shifted factorial) which is defined (in general) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}), \end{cases} \tag{1.8}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [10, p. 2 and p. 5]).

It is easy to see that when $p=0$, (1.5) and (1.6) reduces to the usual Pochhammer symbol $(\lambda)_\nu$ (1.8) and familiar generalized hypergeometric function ${}_pF_q$.

Motivated essentially by the demonstrated potential for applications of these extended generalized hypergeometric functions (1.5), we choose to extend the generalized Hurwitz-Lerch Zeta function (1.3) via the generalized Pochhammer symbol in (1.6) and investigate certain properties of the extended generalized Hurwitz-Lerch Zeta function which include their various integral representations, Mellin transforms, differential formula and consider also a generating function.

2. Extended Generalized Hurwitz-Lerch Zeta Function

In terms of the extended generalized Pochhammer symbol $(\lambda;p)_n$ defined by (1.6), we propose a mild extension of the generalized Hurwitz-Lerch Zeta function defined by (1.3) as follows:

$$\Phi^*_{\mu,p}(z,s,a) := \sum_{n=0}^\infty \frac{(\mu;p)_n}{n!}\frac{z^n}{(n+a)^s} \tag{2.1}$$

$$\big(\mu\in\mathbb{C};\ a\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ s\in\mathbb{C} \quad \text{when} \quad |z|<1;\ \Re(s-\mu)>1 \quad \text{when} \quad |z|=1;\ \Re(p)\geq 0\big).$$

Since the generalized Pochhammer symbol $(\lambda;p)_\nu$ is related to the modified Bessel function of the third kind (or the MacDonald function) $K_\mu(z)$ (see [8]) as follows:

$$(\lambda;p)_\nu = \frac{2p^{\frac{\lambda+\nu}{2}}}{\Gamma\lambda}K_{\lambda+\nu}(2\sqrt{p}) \qquad (\Re(p)>0), \tag{2.2}$$

therefore, the extended generalized Hurwitz-Lerch Zeta function defined by (2.1) can also be written in the following form:

$$\Phi^*_{\mu,p}(z,s,a) = \frac{2p^{\frac{\mu}{2}}}{\Gamma(\mu)} \sum_{n=0}^{\infty} \frac{p^{\frac{n}{2}} K_{\mu+n}(2\sqrt{p})}{(n+a)^s} \frac{z^n}{n!}. \tag{2.3}$$

The following representations can also be deduced from the expression (2.3):
(2.4)

$$\Phi^*_{\frac{1}{2},p}(z,s,a) = e^{-2\sqrt{p}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(p^{\frac{n}{2}} 4\sqrt{p})^{-m}}{m!} \frac{1}{(n+a)^s} \frac{(n+m)!}{(n-m)!} \frac{z^n}{n!} \quad (\Re(p) > 0;\ n \in \mathbb{N}_0)$$

and

$$\Phi^*_{1,p}(z,s,a) = 2\sqrt{p} \sum_{n=0}^{\infty} \frac{p^{\frac{n}{2}} K_{n+1}(2\sqrt{p})}{(n+a)^s} \frac{z^n}{n!} \qquad (\Re(p) > 0). \tag{2.5}$$

Remark 2.1. The special case of (2.1) when $p = 0$ and $(p, \mu) = (0, 1)$ are easily seen to reduce the generalized Hurwitz-Lerch Zeta function (1.3) and the Hurwitz-Lerch Zeta function (1.1), respectively.

3. Integral Representations and Derivative Formula of $\Phi^*_{\mu,p}(z,s,a)$

In this section, we present certain integral representations of the extended generalized Hurwitz-Lerch Zeta function defined by (2.1).

Theorem 3.1. *The following integral representation for* $\Phi^*_{\mu,p}(z,s,a)$ in (2.1) *holds true*:

$$\Phi^*_{\mu,p}(z,s,a) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at}\ {}_1F_0((\mu,p);-;ze^{-t})dt \tag{3.1}$$

$\big(\Re(p) > 0,\ \Re(a) > 0;\ \Re(s) > 0 \quad \text{when} \quad |z| \leqq 1\ (z \neq 1);\ \Re(s) > 1 \quad \text{when} \quad z = 1\big).$

Proof. Using the following Eulerian integral:

$$\frac{1}{(n+a)^s} := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(n+a)t} dt \quad (\min\{\Re(s),\ \Re(a)\} > 0;\ n \in \mathbb{N}_0) \tag{3.2}$$

in (2.1) and interchanging the order of summation and integration, we get

$$\Phi^*_{\mu,p}(z,s,a) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} \left(\sum_{n=0}^{\infty} (\mu;p)_n \frac{(ze^{-t})^n}{n!} \right) dt$$

Finally, using the definition (1.5), we are led to the desired result. □

Remark 3.2. The special case of (3.1) when $p = 0$ yields the integral representation (1.4).

Theorem 3.3. *The following integral representation for* $\Phi^*_{\mu,p}(z, s, a)$ in (2.1) *holds true*:

$$\Phi^*_{\mu,p}(z, s, a) := \frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} e^{-t-\frac{p}{t}} E^{(a)}_{1,1}(s; zt) dt \tag{3.3}$$

$\big(\Re(p) \geqq 0,\ \Re(a) > 0;\ \Re(s) > 0 \quad \text{when} \quad |z| \leqq 1\ (z \neq 1);\ \Re(s) > 1 \quad \text{when} \quad z = 1\big)$,

where $E^{(a)}_{\kappa,\nu}(s; z)$ *is the Mittag-Leffler type function studied by Barnes* [1] (*see also* [5, Section 18.1] and [12, p. 492, Eq.(1.24)]) *and is defined by*

$$E^{(a)}_{\kappa,\nu}(s; z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu + \kappa n)(n + a)^s} \qquad \big(\nu,\, a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(\kappa) > 0; s \in \mathbb{C}\big)\,. \tag{3.4}$$

Proof. Using the integral representation of the generalized Pochhammer symbol $(\mu; p)_n$ defined by (1.7) in (2.1) and using the relation (3.4), we are led to the desired integral representation (3.3). □

Differentiating n times both sides of (2.1) with respect to z, we can easily obtain a derivative formula for the extended generalized Hurwitz-Lerch Zeta function $\Phi^*_{\mu,p}(z, s, a)$ which is contained in the following theorem.

Theorem 3.4 *The following derivative formula for* $\Phi^*_{\mu,p}(z, s, a)$ in (2.1) *holds true*:

$$\frac{d^n}{dz^n} \left\{\Phi^*_{\mu,p}(z, s, a)\right\} = (\mu)_n \Phi^*_{\mu+n,p}(z, s, a + n) \qquad (n \in \mathbb{N}_0)\,. \tag{3.5}$$

4. Mellin Transform and Generating Function of $\Phi^*_{\mu,p}(z, s, a)$

The Mellin transform of a suitable integrable function $f(t)$ with index α is defined, as usual, by

$$\mathcal{M}\{f(\tau) : \tau \to \alpha\} := \int_0^\infty \tau^{\alpha-1} f(\tau)\, d\tau, \tag{4.1}$$

whenever the improper integral in (4.1) exists.

Theorem 4.1. *The following Mellin transform of the* $\Phi^*_{\mu,p}(z, s, a)$ in (2.1) *holds true*:

$$\mathcal{M}\left\{\Phi^*_{\mu,p}(z, s, a) : p \to \alpha\right\} := \frac{\Gamma(\alpha)\Gamma(\mu + \alpha)}{\Gamma(\mu)} \Phi^*_{\mu+\alpha}(z, s, a) \tag{4.2}$$

$$(\Re(\alpha) > 0 \quad \textit{and} \quad \Re(\mu + \alpha) > 0).$$

Proof. Using the definition (4.1) of the Mellin transform, we find from (2.1) that

$$\mathcal{M}\left\{\Phi^*_{\mu,p}(z,s,a) : p \to \alpha\right\} := \int_0^\infty p^{\alpha-1}\left(\sum_{n=0}^\infty \frac{(\mu;p)_n}{n!}\frac{z^n}{(n+a)^s}\right)dp$$
$$= \sum_{n=0}^\infty \frac{z^n}{n!(n+a)^s}\frac{1}{\Gamma(\mu)}\int_0^\infty p^{\alpha-1}\Gamma_p(\mu+n)\,dp.$$

Applying now the result of Chaudhry and Zubair [2, p. 16, Eq. (1.110)] given by

$$\int_0^\infty p^{\alpha-1}\Gamma_p(\gamma+n)dp = \Gamma(\gamma+\alpha+n)\Gamma(\alpha) \qquad (\Re(\alpha)>0), \tag{4.3}$$

we get

$$\mathcal{M}\left\{\Phi^*_{\mu,p}(z,s,a) : p \to \alpha\right\} = \frac{\Gamma(\alpha)}{\Gamma(\mu)}\sum_{n=0}^\infty \frac{\Gamma(\mu+\alpha+n)}{(n+a)^s}\frac{z^n}{n!}.$$

which, after a little simplification and using the definition (2.1), yields the desired representation (4.2). □

Remark 4.2. The case $\alpha = 1$ in (4.2) is seen to yield an interesting relation between the extended generalized Hurwitz-Lerch Zeta function and the generalized Hurwitz-Lerch Zeta function as follows:

$$\int_0^\infty \Phi^*_{\mu,p}(z,s,a)dp = \mu\Phi^*_{\mu+1}(z,s,a). \tag{4.4}$$

Next, we derive the generating function for $\Phi^*_{\mu,p}(z,s,a)$ given by following theorem.

Theorem 4.3. *The following generating function for* $\Phi^*_{\mu,p}(z,s,a)$ in (2.1) *holds true*:

$$\sum_{n=0}^\infty \frac{(s)_n}{n!}\,\Phi^*_{\mu,p}(z,s+n,a)\,t^n = \Phi^*_{\mu,p}(z,s,a-t) \qquad (p \geqq 0;\ |t|<|a|;\ s\neq 1) \tag{4.5}$$

Proof. Using (2.1) in the right-hand side of the assertion (4.5), we have

$$\Phi^*_{\mu,p}(z,s,a-t) = \sum_{k=0}^\infty (\mu;p)_k\frac{z^k}{k!(k+a-t)^s} = \sum_{k=0}^\infty (\mu;p)_k\frac{z^k}{k!(k+a)^s}\left(1-\frac{t}{k+a}\right)^{-s}$$
$$= \sum_{k=0}^\infty (\mu;p)_k\frac{z^k}{k!(k+a)^s}\left\{\sum_{n=0}^\infty \frac{(s)_n}{n!}\frac{t^n}{(k+a)^n}\right\}$$
$$= \sum_{n=0}^\infty \frac{(s)_n}{n!}\left\{\sum_{k=0}^\infty (\mu;p)_k\frac{z^k}{k!(k+a)^{s+n}}\right\}t^n.$$

Now on making use of (2.1), we get the desired generating function (4.5). □

Acknowledgements. The authors should express their deepest gratitude for the reviewers' very useful and detailed comments to improve this paper as in the present concise exposition. This research was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (2010-0011005). Also this work was supported by the Dongguk University Research Fund of 2014.

References

[1] E. W. Barnes, *The asymptotic expansion of integral functions defined by Taylor series*, Philos. Trans. Roy. Soc. London Ser. A, **206**(1906), 249–297.

[2] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, Chapman and Hall, (CRC Press Company), Boca Raton, London, New York and Washington, D. C., 2001.

[3] J. Choi, D. S. Jang and H. M. Srivastava, *A generalization of the Hurwitz-Lerch Zeta function*, Integral Transforms Spec. Funct., **19**(2008), 65–79.

[4] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. **I**, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. **III**, McGraw-Hill Book Company, New York, Toronto and London, 1955.

[6] S. P. Goyal and R. K. Laddha, *On the generalized Zeta function and the generalized Lambert function*, Ganita Sandesh, **11**(1997), 99–108.

[7] H. M. Srivastava, *A new family of the λ-generalized Hurwitz-Lerch Zeta functions with applications*, Appl. Math. Inf. Sci., **8(4)**(2014), 1485–1500.

[8] H. M. Srivastava, A. Çetinkaya and İ Onur Kıyamaz, *A Certain generalized Pochhammer symbol and its applications to hypergeometric functions*, Appl. Math. Comput., **226**(2014), 484–491.

[9] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer, Acedemic Publishers, Dordrecht, Boston and London, 2001.

[10] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science, Publishers, Amsterdam, London and New York, 2012.

[11] H. M. Srivastava, M.-J. Luo and R. K. Raina, *New results involving a class of generalized Hurwitz-Lerch Zeta functions and their applications*, Turkish J. Anal. Number Theory **1(1)**(2013), 26–35.

[12] H. M. Srivastava, R. K. Saxena, T. K. Pogány and R. Saxena, *Integral and computational representations of the extended Hurwitz-Lerch Zeta function*, Integral Transforms Spec. Funct., **22(7)**(2011), 487–506.

[13] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*: *An Introduction to the General Theory of Infinite Processes and of Analytic Functions*; *With an Account of the Principal Transcendental Functions*, Fourth edition, Cambridge University Press, Cambridge, London and New York, 1963.