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ALTERNATING RESOLVENT ALGORITHMS FOR FINDING A COMMON ZERO OF TWO ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In this paper we introduce a new iterative method by the combination of the prox-Tikhonov regularization and the alternating resolvents for finding a common zero of two accretive operators in Banach spaces. And we will give some applications and numerical examples. The results of this paper improve and extend the corresponding results announced by many others.

1. Introduction

Let E be a real Banach space. For an operator $A:\ E \to 2^E$, we denote its domain, range and graph as $D(A)$, $R(A)$ and $G(A)$, respectively. The inverse A^{-1} of an operator A is defined by $x \in A^{-1}y$ if and only if $y \in Ax$.

An operator A is said to be accretive if for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geq 0$$

for all $u \in Ax$ and $v \in Ay$. An accretive operator A is said to be maximal if there is no proper accretive extension of A, and m-accretive if $R(I + \lambda A) = E$ for all $\lambda > 0$, where I is the identity operator on E. If A is m-accretive, then it is maximal accretive, but the reverse is not true. For an accretive operator A, we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda^A : R(I + \lambda A) \longrightarrow D(A)$ by

$$J_\lambda^A = (I + \lambda A)^{-1}.$$

It is called the resolvent of A.

We know that many problems for the nonlinear analysis and optimization can be formulated as the form:

$$0 \in Ax,$$

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where A is an m-accretive operator.

Martinet [18] first introduced the proximal point algorithm which is a well-known method for solving the equation $0 \in Ax$, where A is a maximal monotone operator in a Hilbert space H. For starting point $x_0 = x \in E$, the proximal point algorithm is the sequence $\{x_n\}$ as following:

$$x_{n+1} = J^A_{r_n}(x_n) \tag{1.1}$$

for all $n \in \mathbb{N}$, where $\{r_n\}$ is a sequence of positive real numbers and $\mathbb{N}$ is the set of all natural numbers.

And also, Rockafellar [24] has given a more practical method which is an inexact variant of the method:

$$x_n + e_n \ni x_{n+1} + c_n A x_{n+1} \tag{1.2}$$

for all $n \in \mathbb{N}$, where $\{e_n\}$ is considered as an error sequence and $\{c_n\}$ is a sequence of positive regularization parameters. We know that the algorithm (1.2) can be written as

$$x_{n+1} = J^A_{r_n}(x_n + e_n) \tag{1.3}$$

for all $n \in \mathbb{N}$. This method is called an inexact proximal point algorithm. Rockafellar [24] proved that if $e_n \to 0$ quickly enough such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then $x_n \rightharpoonup z \in H$ with $0 \in Az$.

Further, Rockafellar [24] presented the open question of whether the sequence in (1.1) converges strongly or not. In 1991, Güler [10] gave an example that Rockafellar's proximal point algorithm does not converge strongly, and Bauschke, Matoušková and Reich [4] also showed that the proximal algorithm only converges weakly but not in norm.

When A is maximal monotone in a Hilbert space H, Lehdili and Moudafi [16] obtained the convergence of the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = J^{A_n}_{c_n}(x_n), \tag{1.4}$$

where $A_n = \mu_n I + A$ is a Tikhonov regularization of A. Next, Xu [31] and Song and Yang [26] used the technique of nonexpansive mappings to prove the convergence theorems for $\{x_n\}$ defined by the perturbed version of the algorithm (1.3) in the form

$$x_{n+1} = J^A_{r_n}(t_n u + (1 - t_n)x_n + e_n). \tag{1.5}$$

The results of Xu [31] and Song and Yang [26] have been extended by Kim and Tuyen [15], Kim and Buong [14], Tuyen [28, 29], Sahu and Yao [25] and many others.

Now, we consider the following problem: Find an element

$$x^* \in S := A^{-1}0 \cap B^{-1}0, \tag{1.6}$$

where $A: \ D(A) \to 2^E$ and $B: \ D(B) \to 2^E$ are two accretive operators.

In 1930, von Neumann [19] proved that for any two closed subsets C_1 and C_2 of a real Hilbert space H, the sequence of alternating projections

$$H \ni x_0 \mapsto x_1 = P_{C_1}x_0 \mapsto x_2 = P_{C_2}x_1 \mapsto x_3 = P_{C_1}x_2 \mapsto \cdots , \tag{1.7}$$

converges strongly to a point in $C_1 \cap C_2$ that is the nearest to the starting x_0. In 1965, Bregman [5] showed that for two arbitrary closed convex sets C_1 and C_2 with nonempty intersection, the sequence $\{x_n\}$ generated by the alternating projection method (1.7) converges weakly to some point in $C_1 \cap C_2$. The question on whether $\{x_n\}$ converges strongly or not, was presented by Hundal [11].

Let A and B be two maximal monotone operators in a Hilbert space H. In 2005, base on the alternating projection method of von Neumann [19] and Bregmann [5], Bauschke et al. [3] proved that the sequence $\{x_n\}$ generated by the method of alternating resolvent

$$\begin{cases} x_{2n+1} = J_\lambda^A(x_{2n}), \ n = 0, 1, 2, \ldots, \\ x_{2n} = J_\lambda^B(x_{2n-1}), \ n = 1, 2, \ldots, \end{cases} \tag{1.8}$$

for $\lambda > 0$, converges weakly to a point of $F(J_\lambda^A J_\lambda^B)$ which is the fixed point set of the composition $J_\lambda^A J_\lambda^B$.

The purpose of this paper is to introduce a new iteration algorithm of prox-Tikhonov regularization method with alternating resolvents which converges strongly to a solution of problem for finding a common zero of two accretive operators. Our results are the improvements and generalizations of the corresponding works of Lehdili and Moudafi [16], Song and Yang [26], Xu [31] and some others.

2. Preliminaries

Let E be a real Banach space with the dual space E^*. For the sequence $\{x_n\}$ in E, we denote by '$x_n \to x$' (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x. We use $\omega_w(\{x_n\})$ to denote the set of weakly cluster point of the sequence $\{x_n\}$.

The normalized duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{f \in E^* : \ \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \ \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single-valued. Afterwards, we denote the single-valued normalized duality mapping by j.

A Banach space E is said to have a weakly sequential continuous normalized duality mapping, if J is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, then $j(x_n) \overset{*}{\rightharpoonup} j(x)$ [6].

We use S_E to denote the unit sphere $S_E = \{x \in E : \ \|x\| = 1\}$ and $F(T)$ to denote the set of fixed points of the mapping $T : \ C \subseteq E \to E$.

A Banach space E is said to be strictly convex if $x, y \in S_E$ with $x \neq y$, and, for all $t \in (0,1)$,

$$\|(1-t)x + ty\| < 1.$$

A Banach space E is said to be uniformly convex if for any $\varepsilon \in (0,2]$ and the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x-y\| \geq \varepsilon$, there exists a $\delta = \delta(\varepsilon) \geq 0$ such that

$$\frac{\|x+y\|}{2} \leq 1 - \delta.$$

A Banach space E is said to be smooth provided the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in S_E. In this case, the norm of E is said to be Gâteaux differentiable. It is said to be uniformly Gâteaux differentiable if for each $y \in S_E$, this limit attained uniformly for $x \in S_E$.

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$, where the modulus of smoothness $\rho_E : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(t) = \sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x \in S_E, \|y\| \leq t\Big\}.$$

It is well known that every uniformly smooth Banach space has a uniformly Gâteaux differentiable norm.

A Banach space E is said to satisfy the Opial's condition [20] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. We know that if E admits a weakly sequentially continuous normalized duality mapping, then E satisfies the Opial's condition.

Recall that a mapping T defined on a subset C of E is said to be demiclosed if for any sequence $\{x_n\}$ in C the following implication holds:

$$x_n \rightharpoonup x \ \text{ and } \ \|T(x_n) - y\| \to 0,$$

implies

$$x \in C \text{ and } T(x) = y.$$

Lemma 2.1 ([7]). *Let C be a nonempty closed and convex subset of a reflexive Banach space E which satisfies the Opial's condition. Suppose $T : C \to E$ is a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C.*

A closed convex subset C of E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a nonempty closed and convex subset M of C into itself has a fixed point in M.

A subset C of E is called a retract of E if there is a continuous mapping Q from E onto C such that $Qx = x$ for all $x \in C$ and we call such Q a retraction of E onto C. It follows that if Q is a retraction, then $Qy = y$ for all y in the range of Q. A retraction Q is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx$$

for all $x \in E$ and $t \geq 0$. If a sunny retraction Q is also nonexpansive, then C is said to be a sunny nonexpansive retract of E [8]. We know that in a Hilbert space H, the sunny nonexpansive retract mapping form H onto a closed convex subset $C \subset H$ is a metric projection and denoted by P_C.

An accretive operator A defined on a Banach space E is said to satisfy the range condition if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A. We know that for an accretive operator A which satisfies the range condition, $A^{-1}0 = F(J_\lambda^A)$ for all $\lambda > 0$.

The following lemmas will be needed in the sequel for the proof of main theorems.

Lemma 2.2 ([2]). *Let $A: D(A) \to 2^E$ be an accretive operator. Then for $\lambda, \mu > 0$, and $x \in E$, we have*

$$J_\lambda^A x = J_\mu^A \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^A x \right).$$

Lemma 2.3 ([1]). *Let E be a uniformly convex Banach space. Then for all $x, y \in E$ with $\max\{\|x\|, \|y\|\} \leq R$ and for all $j_x \in J(x)$, $j_y \in J(y)$, there exists an increasing function $g: \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$, $g(t) > 0$ for all $t > 0$ such that*

$$\langle x - y, j_x - j_y \rangle \geq g(\|x - y\|)\|x - y\|.$$

Lemma 2.4 ([8]). *Let C be a convex subset of a smooth Banach space E, D a nonempty subset of C and P a retraction from C onto D. Then the following statements are equivalent*:

(i) *P is a sunny nonexpansive mapping.*
(ii) *$\langle x - Px, j(z - Px) \rangle \leq 0$ for all $x \in C$, $z \in D$.*
(iii) *$\langle x - y, j(Px - Py) \rangle \geq \|Px - Py\|^2$ for all $x, y \in C$.*

Lemma 2.5 ([21]). *Let E be a Banach space. Then for every $x, y \in E$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle$$

for all $j(x + y) \in J(x + y)$.

Lemma 2.6 ([30]). *Let C a nonempty closed and convex subset of a Banach space E with a uniformly Gâteaux differentiable norm, $f: C \to C$ a continuous mapping, $T: C \to C$ a nonexpansive mapping, and $\{x_n\}$ a bounded sequence in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Suppose that $\{z_t\}$ is a path in C defined by $z_t = tfz_t + (1 - t)Tz_t$, $t \in (0, 1)$ such that $z_t \to z$ as $t \to 0$. Then we have*

$$\limsup_{n\to\infty} \langle fz - z, j(x_n - z) \rangle \leq 0.$$

Lemma 2.7 ([32])**.** *Let $\{a_n\}, \{b_n\}, \{c_n\}$, and $\{\sigma_n\}$ be sequences of positive numbers satisfying the inequality*:

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n + \sigma_n,\ b_n < 1.$$

If $\sum_{n=0}^{\infty} b_n = +\infty$, $\limsup_{n\to\infty} c_n \leq 0$ and $\sum_{n=0}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} a_n =$ 0.

3. Main results

Fiest we need the following conditions:

Condition A. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences of positive numbers satisfying the conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(2) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to 0} \alpha_{n+1}/\alpha_n = 1$;
(3) (a) for all n, $\beta_n \geq r > 0$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(b) for all n, $\gamma_n \geq r > 0$ and $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Now we are in a position to introduce and prove the main theorems.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a reflexive Banach space E with a weakly sequentially continuous duality mapping j. Let $A :\ D(A) \subseteq C \to 2^E$ and $B :\ D(B) \subseteq C \to 2^E$ be accretive operators such that*

$$S := A^{-1}0 \cap B^{-1}0 \neq \emptyset,$$

$\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$ and $\overline{D(B)} \subset C \subset \cap_{r>0} R(I + rB)$. Let $\{\alpha_n\}$ be a control sequence of positive numbers satisfying (1) *in Condition A, and let $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[\varepsilon, \infty)$ for $\varepsilon > 0$. Let $\{x_n\}$ be an alternating resolvent algorithm generated by*

$$\begin{cases} x_{2n+1} = J^A_{\beta_n}(\alpha_n u + (1 - \alpha_n)x_{2n}),\ n = 0, 1, \dots, \\ x_{2n} = J^B_{\gamma_n}(x_{2n-1}),\ n = 1, 2, \dots, \end{cases} \tag{3.1}$$

where u and x_0 are arbitrary elements in C. If the sequence $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $Q_S u$, where $Q_S :\ C \to S$ is a sunny nonexpansive retraction from C onto S.

Proof. First, we show that $\{x_n\}$ is bounded. Let $p \in S$. Then from (3.1), we have

$$\begin{aligned} \|x_{2n+1} - p\| &\leq \|\alpha_n(u - p) + (1 - \alpha_n)(x_{2n} - p)\| \\ &\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|x_{2n} - p\| \\ &\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|x_{2n-1} - p\|. \end{aligned} \tag{3.2}$$

Applying induction to (3.2), then we have

$$\|x_{2n+1} - p\| \leq \Big(1 - \prod_{k=0}^{n}(1 - \alpha_k)\Big)\|u - p\| + \prod_{k=0}^{n}(1 - \alpha_k)\|x_0 - p\|.$$

Therefore, the sequence $\{x_{2n+1}\}$ is bounded, and so is the sequence $\{x_{2n}\}$. Consequently $\{x_n\}$ is bounded.

Next, we show that $\|x_{2n+1} - J_r^A(x_{2n+1})\| \to 0$ and $\|x_{2n} - J_r^B(x_{2n})\| \to 0$, where $0 < r < \varepsilon$. Since

$$\|x_{2n+1} - J_{\beta_n}^A(x_{2n})\| = \alpha_n \|u - x_{2n}\| \to 0 \tag{3.3}$$

and

$$\|J_{\beta_n}^A(x_{2n+1}) - J_{\beta_n}^A(x_{2n})\| \le \|x_{2n+1} - x_{2n}\| \to 0, \tag{3.4}$$

we have

$$\begin{aligned}\|x_{2n+1} - J_{\beta_n}^A(x_{2n+1})\| &\le \|x_{2n+1} - J_{\beta_n}^A(x_{2n})\| + \|J_{\beta}^A(x_{2n+1}) - J_{\beta_n}^A(x_{2n})\| \\ &\to 0.\end{aligned} \tag{3.5}$$

By the resolvent identity in Lemma 2.2, we obtain

$$\begin{aligned}&\|J_{\beta_n}^A(x_{2n+1}) - J_r^A(x_{2n+1})\| \\ &= \|J_r^A\big(\frac{r}{\beta_n}x_{2n+1} + (1 - \frac{r}{\beta_n})J_{\beta_n}^A(x_{2n+1})\big) - J_r^A(x_{2n+1})\| \\ &\le \big(1 - \frac{r}{\beta_n}\big)\|x_{2n+1} - J_{\beta_n}^A(x_{2n+1})\| \to 0.\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we get

$$\|x_{2n+1} - J_r^A(x_{2n+1})\| \to 0.$$

Similarly, we have

$$\|x_{2n} - J_r^B(x_{2n})\| \to 0.$$

By Lemma 2.1, $\omega_w(\{x_{2n+1}\}) \subset A^{-1}0$ and $\omega_w(\{x_{2n}\}) \subset B^{-1}0$, and so $\omega_w(\{x_n\}) \subset S$.

Let $\{y_n\}$ be the sequence defined by

$$\begin{cases} y_{2n+1} = \alpha_n u + (1 - \alpha_n)x_{2n}, \ n = 0, 1, \ldots, \\ y_{2n} = x_{2n-1}, \ n = 1, 2, \ldots. \end{cases} \tag{3.7}$$

Then it is clear that $\|y_n - x_n\| \to 0$ and $\omega_w(\{y_n\}) \subset S$. Form the last inclusion, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u - Q_S u, j(y_n - Q_S u)\rangle &= \lim_{k\to\infty}\langle u - Q_S u, j(y_{n_k} - Q_S u)\rangle \\ &= \langle u - Q_S u, j(\overline{x} - Q_S u)\rangle \le 0,\end{aligned} \tag{3.8}$$

where $\overline{x} = w - \lim_{k\to\infty} y_{n_k} \in S$.

Let $q = Q_S u \in S$. From (3.7), we have

$$\begin{aligned}\|y_{2n+1} - q\|^2 &\le \|(1 - \alpha_n)(x_{2n} - q) + \alpha_n(u - q)\|^2 \\ &\le (1 - \alpha_n)^2\|x_{2n} - q\|^2 + 2\alpha_n\langle u - q, j(y_{2n+1} - q)\rangle \\ &\le (1 - \alpha_n)^2\|x_{2n-1} - q\|^2 + 2\alpha_n\langle u - q, j(y_{2n+1} - q)\rangle \\ &\le (1 - \alpha_n)\|y_{2n-1} - q\|^2 + \sigma_n,\end{aligned}$$

where $\sigma_n = 2\alpha_n\langle u - q, j(y_{2n+1} - q)\rangle$. From (3.8), we have

$$\limsup_{n\to\infty} \frac{\sigma_n}{\alpha_n} = \limsup_{n\to\infty}\langle u - q, j(y_{2n+1} - q)\rangle \le 0.$$

Hence, by Lemma 2.7, we get $y_{2n+1} \to q$. Moreover, since $x_{2n+1} - x_{2n} \to 0$ and $y_n - x_n \to 0$, we also have $x_{2n+1} \to q$ and $x_{2n} \to q$. Consequently, the sequence $\{x_n\}$ converges strongly to a point $q = Q_S u$ as $n \to \infty$. $\square$

The following theorem was proved by Jung [12,13].

Theorem 3.2 ([12,13]). *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let C be a closed convex subset of E and let T be a nonexpansive mapping from C into itself with $F(T) \neq \emptyset$. If one of the following assumptions holds*:

(H1) *every weakly compact convex subset of E has the fixed point property for nonexpansive mappings*;

(H2) *E is strictly convex,*

then the net $\{x_t\}$ defined by $x_t = tfx_t + (1-t)Tx_t$, where $f : C \to C$ is a contraction mapping and $t \in (0,1)$, converges strongly to a point $x^* \in F(T)$ such that $Q_{F(T)}fx^* = x^*$.

Remark 3.1. In Theorem 3.2, if $f(x) = u$ for all $x \in C$, then $x^* = Q_{F(T)}u$, where $Q_{F(T)} :\ C \longrightarrow F(T)$ is a sunny nonexpansive retraction from C onto $F(T)$.

Theorem 3.3. *Let C be a nonempty closed and convex subset of a reflexive and strictly convex Banach space E with a uniformly Gâteaux differentiable norm. Let $A :\ D(A) \subset C \to 2^E$ and $B :\ D(B) \subset C \to 2^E$ be accretive operators such that*

$$S := A^{-1}0 \cap B^{-1}0 \neq \emptyset,$$

$\overline{D(A)} \subset C \subset \cap_{r>0} R(I+rA)$ and $\overline{D(B)} \subset C \subset \cap_{r>0} R(I+rB)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be the sequences as in Theorem 3.1. If the sequence $\{x_n\}$ generated by (3.1) *is asymptotically regular, then $\{x_n\}$ converges strongly to $Q_S u$.*

Proof. Let $T_1 = J_r^A$, $T_2 = J_r^B$ and $T = \frac{1}{2}(T_1 + T_2)$, where $J_r^A,\ J_r^B$ are defined as in the proof of Theorem 3.1. Since E is a strictly convex Banach space, $S = F(T)$. Let $\{y_n\}$ be the sequence defined by (3.7). Then we have

$$\begin{aligned} \|y_n - J_r^A(y_n)\| &\le \|y_n - x_n\| + \|x_n - J_r^A(x_n)\| \\ &\quad + \|J_r^A(x_n) - J_r^A(y_n)\| \\ &\le 2\|x_n - y_n\| + \|x_n - J_r^A(x_n)\|. \end{aligned} \tag{3.9}$$

We show that $\|x_n - J_r^A(x_n)\| \to 0$. Indeed,

$$\begin{aligned} \|x_{2n} - J_r^A(x_{2n})\| &\le \|x_{2n} - x_{2n+1}\| + \|x_{2n+1} - J_r^A(x_{2n+1})\| \\ &\quad + \|J_r^A(x_{2n+1}) - J_r^A(x_{2n})\| \\ &\le 2\|x_{2n} - x_{2n+1}\| + \|x_{2n+1} - J_r^A(x_{2n+1})\| \to 0. \end{aligned} \tag{3.10}$$

From (3.9), (3.10) and $\|x_n - y_n\| \to 0$, we get $\|y_n - J_r^A(y_n)\| \to 0$. Similarly, we get $\|y_n - J_r^B(y_n)\| \to 0$. So, we have

$$\|y_n - T(y_n)\| \leq \frac{1}{2}\Big(\|y_n - J_r^A(y_n)\| + \|y_n - J_r^B(y_n)\|\Big) \to 0.$$

Apply Theorem 3.2 and Lemma 2.6 which $f(x) = u$ for all $x \in C$, we have

$$\limsup_{n\to 0}\langle u - Q_S u, j(y_n - Q_S u)\rangle \leq 0.$$

The rest of the proof follows the pattern of Theorem 3.1. $\square$

Lemma 3.1. *Let E be a uniformly convex Banach space. If $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy* (1), (2), (3) *in Condition A, then the sequence $\{x_n\}$ generated by* (3.1) *is asymptotically regular.*

Proof. Case 1. $\beta_{n+1} \leq \beta_n$: Using the resolvent identity, we write (3.1) as

$$x_{2n+1} = J^A_{\beta_{n+1}}\Big(\frac{\beta_{n+1}}{\beta_n}\big(\alpha_n u + (1-\alpha_n)x_{2n}\big) + \big(1 - \frac{\beta_{n+1}}{\beta_n}\big)x_{2n+1}\Big).$$

Therefore,

$$\begin{aligned}
\|x_{2n+3} - x_{2n+1}\| &\leq \Big\|\frac{\beta_{n+1}}{\beta_n}(1-\alpha_n)(x_{2n+2} - x_{2n}) \\
&\quad + \big(1 - \frac{\beta_{n+1}}{\beta_n}\big)(x_{2n+2} - x_{2n+1}) \\
&\quad + \big(\alpha_{n+1} - \frac{\beta_{n+1}\alpha_n}{\beta_n}\big)(u - x_{2n+2})\Big\| \\
&\leq (1-\alpha_n)\|x_{2n+2} - x_{2n}\| \\
&\quad + \Big(\big|1 - \frac{\beta_{n+1}}{\beta_n}\big| + \big|\alpha_{n+1} - \frac{\beta_{n+1}\alpha_n}{\beta_n}\big|\Big)K \\
&\leq (1-\alpha_n)\|x_{2n+2} - x_{2n}\| \\
&\quad + \Big(\frac{2}{r}|\beta_{n+1} - \beta_n| + |\alpha_{n+1} - \alpha_n|\Big)K,
\end{aligned} \tag{3.11}$$

where K is a positive constant such that

$$\sup_{n\geq 0}\Big\{\|u - x_{2n+2}\| + \|x_{2n+2} - x_{2n+1}\|\Big\} \leq K.$$

Case 2. $\beta_{n+1} > \beta_n$: We have

$$\begin{aligned}
x_{2n+3} &= J^A_{\beta_{n+1}}(\alpha_{n+1}u + (1-\alpha_{n+1})x_{2n+2}) \\
&= J^A_{\beta_n}\Big(\frac{\beta_n}{\beta_{n+1}}(\alpha_{n+1}u + (1-\alpha_{n+1})x_{2n+2}) + (1 - \frac{\beta_n}{\beta_{n+1}})x_{2n+3}\Big).
\end{aligned}$$

So, we have

(3.12)

$$
\begin{aligned}
\|x_{2n+3} - x_{2n+1}\| \leq & \Big\| \Big(\frac{\beta_n}{\beta_{n+1}} (\alpha_{n+1} u + (1-\alpha_{n+1}) x_{2n+2}) + (1 - \frac{\beta_n}{\beta_{n+1}}) x_{2n+3} \Big) \\
& \quad - (\alpha_n u + (1-\alpha_n) x_{2n}) \Big\| \\
\leq & \frac{\beta_n}{\beta_{n+1}} (1-\alpha_{n+1}) \|x_{2n+2} - x_{2n}\| \\
& + \Big| \frac{\beta_n}{\beta_{n+1}} \alpha_{n+1} - \alpha_n \Big| \|u - x_{2n}\| \\
& + \Big| 1 - \frac{\beta_n}{\beta_{n+1}} \Big| \|x_{2n} - x_{2n+3}\| \\
\leq & (1-\alpha_n) \|x_{2n+2} - x_{2n}\| \\
& + 2 \Big(\frac{1}{r} |\beta_{n+1} - \beta_n| + |\alpha_{n+1} - \alpha_n| \Big) M,
\end{aligned}
$$

where M is a positive constant such that

$$
\sup_{n \geq 0} \Big\{ \|u - x_{2n}\| + \|x_{2n+3} - x_{2n}\| + \|x_{2n+2} - x_{2n}\| \Big\} \leq M.
$$

From (3.11) and (3.12), we get

(3.13)

$$
\|x_{2n+3} - x_{2n+1}\| \leq (1-\alpha_n) \|x_{2n+2} - x_{2n}\| + 2 \Big(\frac{1}{r} |\beta_{n+1} - \beta_n| + |\alpha_{n+1} - \alpha_n| \Big) M',
$$

where $M' = \max\{K, M\}$.

On the other hand, we have

$$
\begin{aligned}
\|x_{2n+2} - x_{2n}\| = & \Big\| J^B_{\gamma_n} \Big(\frac{\gamma_n}{\gamma_{n+1}} x_{2n+1} + (1 - \frac{\gamma_n}{\gamma_{n+1}}) x_{2n+2} \Big) - J^B_{\gamma_n} (x_{2n-1}) \Big\| \\
\leq & \Big\| (x_{2n+1} - x_{2n-1}) + \big(1 - \frac{\gamma_n}{\gamma_{n+1}}\big) (x_{2n+2} - x_{2n+1}) \Big\| \\
\leq & \|x_{2n+1} - x_{2n-1}\| + \big(1 - \frac{\gamma_n}{\gamma_{n+1}}\big) K \\
\leq & \|x_{2n+1} - x_{2n-1}\| + \frac{1}{r} |\gamma_{n+1} - \gamma_n| M'.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|x_{2n+3} - x_{2n+1}\| \leq & (1-\alpha_n) \|x_{2n+1} - x_{2n-1}\| \\
& + 2 \Big(\frac{1}{r} |\beta_{n+1} - \beta_n| + |\alpha_{n+1} - \alpha_n| + \frac{1}{r} |\gamma_{n+1} - \gamma_n| \Big) M'.
\end{aligned}
$$

By Lemma 2.7, we obtain

$$
\|x_{2n+1} - x_{2n-1}\| \to 0.
$$

As a matter of fact, $\|x_{2n+2} - x_{2n}\| \to 0$ as well.

Now from (3.1), we have

$$x_{2n+1} - x_{2n+2} + \beta_n A x_{2n+1} \ni \alpha_n(u - x_{2n}) + x_{2n} - x_{2n+2}.$$

Let $p \in S$. By accretivity of A, we get

$$\langle x_{2n+1} - x_{2n+2}, j(x_{2n+1} - p)\rangle \le \alpha_n K' + L\|x_{2n} - x_{2n+2}\| \tag{3.14}$$

for some positive constants K' and L. Similarly, multiplying the inclusion relation

$$x_{2n+2} - x_{2n+1} + \gamma_{n+1} B x_{2n+2} \ni 0$$

scalarly by $j(x_{2n+2} - p)$ and using accretivity of B, we have

$$\langle x_{2n+2} - x_{2n+1}, j(x_{2n+2} - p)\rangle \le 0. \tag{3.15}$$

Adding inequalities (3.14) and (3.15), and passing to the limit in the resulting yields

$$\langle x_{2n+2} - x_{2n+1}, j(x_{2n+2} - p) - j(x_{2n+1} - p)\rangle \to 0. \tag{3.16}$$

By Lemma 2.3, we get

$$g(\|x_{2n+2} - x_{2n+1}\|)\|x_{2n+2} - x_{2n+1}\| \to 0.$$

By the properties of g, we obtain

$$\|x_{2n+2} - x_{2n+1}\| \to 0.$$

Hence, $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. □

Now, we study the strong convergence of the sequence $\{z_n\}$ defined by

$$\begin{cases} z_{2n+1} = J^A_{\beta_n}(\alpha_n f(z_{2n}) + (1-\alpha_n) z_{2n}), \ n = 0, 1, \dots, \\ z_{2n} = J^B_{\gamma_n}(z_{2n-1}), \ n = 1, 2, \dots, \end{cases} \tag{3.17}$$

where $f : C \to C$ is a contraction mapping from C into itself with the contraction coefficient $c \in [0, 1)$.

Theorem 3.4. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space E with either a weakly sequentially continuous duality mapping j or a uniformly Gâteaux differentiable norm, and let $f : C \to C$ be a contraction mapping from C into itself. Let $A : D(A) \subset C \to 2^E$ and $B : D(B) \subset C \to 2^E$ be accretive operators such that*

$$S := A^{-1}0 \cap B^{-1}0 \neq \emptyset,$$

$\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$ and $\overline{D(B)} \subset C \subset \cap_{r>0} R(I + rB)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences satisfying (1), (2), (3) *in Condition A. Then the sequence $\{z_n\}$ generated by* (3.17) *converges strongly to an element $x^* \in S$ such that $Q_S f(x^*) = x^*$.*

Proof. Let x^* be a unique fixed point of $Q_S f$, that is, $Q_S f(x^*) = x^*$. From Lemma 3.1, Theorem 3.1, Theorem 3.3, and replacing u by $f(x^*)$ in (3.1), we know that the sequence $\{x_n\}$ converges strongly to $Q_S f(x^*) = x^*$.

Now, we will prove that $\|z_n - x_n\| \to 0$ as $n \to \infty$. Assume that

$$\limsup_{n\to\infty} \|z_{2n+1} - x_{2n+1}\| > 0.$$

Then we can choose ε with $\varepsilon \in (0, \limsup_{n\to\infty} \|z_{2n+1} - x_{2n+1}\|)$. Since $\|x_n - x^*\| \to 0$, there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$,

$$\|x_{2n+1} - x^*\| < \Big(\frac{1-c}{c}\Big)\varepsilon.$$

We divide the following two cases:

(i) There exists $n_2 \in \mathbb{N}$ with $n_2 \geq n_1$ such that $\|z_{2n_2+1} - x_{2n_2+1}\| \leq \varepsilon$.

(ii) $\|z_{2n+1} - x_{2n+1}\| > \varepsilon$ for all $n \geq n_1$.

In the case of (i), we have

$$\begin{aligned}\|z_{2n_2+3} - x_{2n_2+3}\| &\leq (1-\alpha_{n_2+1})\|z_{2n_2+2} - x_{2n_2+2}\| \\ &\quad + \alpha_{n_2+1}\|f(z_{2n_2+2}) - f(x^*)\| \\ &\leq [1-\alpha_{n_2+1}(1-c)]\|z_{2n_2+2} - x_{2n_2+2}\| \\ &\quad + \alpha_{n_2+1}c\|x_{2n_2+2} - x^*\| \\ &\leq [1-\alpha_{n_2+1}(1-c)]\|z_{2n_2+1} - x_{2n_2+1}\| \\ &\quad + \alpha_{n_2+1}c\|x_{2n_2+1} - x^*\| \\ &\leq \varepsilon.\end{aligned}$$

By induction, we can show that $\|z_{2n+1} - x_{2n+1}\| \leq \varepsilon$ for all $n \geq n_2$. This contradicts to $\varepsilon < \limsup_{n\to\infty} \|z_{2n+1} - x_{2n+1}\|$.

In the case of (ii), for each $n \geq n_1$, we have

$$\begin{aligned}\|z_{2n+3} - x_{2n+3}\| &\leq (1-\alpha_{n+1})\|z_{2n+2} - x_{2n+2}\| \\ &\quad + \alpha_{n+1}\|f(z_{2n+2}) - f(x^*)\| \\ &\leq [1-\alpha_{n+1}(1-c)]\|z_{2n+2} - x_{2n+2}\| \\ &\quad + \alpha_{n_2+1}c\|x_{2n+2} - x^*\| \\ &\leq [1-\alpha_{n+1}(1-c)]\|z_{2n+1} - x_{2n+1}\| \\ &\quad + \alpha_{n+1}c\|x_{2n+1} - x^*\|.\end{aligned}$$

So by Lemma 2.7, we get $\lim_{n\to\infty} \|z_{2n+1} - x_{2n+1}\| = 0$. This is a contradiction. Therefore $\lim_{n\to\infty} \|z_{2n+1} - x_{2n+1}\| = 0$. Similarly, we have $\lim_{n\to\infty} \|z_{2n} - x_{2n}\| = 0$. So $\lim_{n\to\infty} \|z_n - x_n\| = 0$. Thus we obtain

$$\lim_{n\to\infty} \|z_n - x^*\| \leq \lim_{n\to\infty} \|z_n - x_n\| + \lim_{n\to\infty} \|x_n - x^*\| = 0.$$

This completes the proof. ◻

Remark 3.2. If $B = 0$, then from (3.1), we get $z_{2n} = z_{2n-1}$. So, from Lemma 3.1, in Theorem 3.4 we can remove the condition that E is uniformly convex. Hence, we have the following corollary (see Theorem 3.5 in [23]).

Corollary 3.5 ([23]). *Let C be a nonempty closed and convex subset of a reflective Banach space E with either a weakly sequentially continuous duality mapping j or a uniformly Gâteaux differentiable norm such that every weakly compact convex subset of E has fixed point property for nonexpansive mappings, and let $f : C \to C$ be a contraction mapping from C into itself. Let $A : D(A) \subset C \to 2^E$ be an accretive operator such that*

$$S := A^{-1}0 \neq \emptyset,$$

and $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences satisfying (1), (2), (3)-(a) *in Condition A. Then the sequence $\{z_n\}$ generated by*

$$z_{n+1} = J^A_{\beta_n}(\alpha_n f(z_n) + (1 - \alpha_n)z_n), \; z_0 \in C, \; n = 0, 1, 2, \ldots, \tag{3.18}$$

converges strongly to an element $x^* \in S$ such that $Q_S f(x^*) = x^*$.

Remark 3.3. This corollary is a generalization of the result of Sahu and Yao (see Theorem 3.5 in [25]).

We know that every m-accretive operator satisfies the range condition. So we have the following result.

Theorem 3.6. *Let E be a uniformly convex Banach space with either a weakly sequentially continuous duality mapping j or a uniformly Gâteaux differentiable norm and let $f : E \to E$ be a contraction mapping in E. Let $A : E \to 2^E$ and $B : E \to 2^E$ be m-accretive operators such that*

$$S := A^{-1}0 \cap B^{-1}0 \neq \emptyset.$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences satisfying (1), (2), (3) *in Condition A, and $\{e_n\}$ and $\{e'_n\}$ be the sequences of errors satisfying the conditions*:

(4) (a) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$;
(b) $\sum_{n=0}^{\infty} \|e'_n\| < \infty$ *or* $\lim_{n\to\infty} \|e'_n\|/\alpha_n = 0$.

Then the sequence $\{u_n\}$ generated by $u_0 \in E$ and

$$\begin{cases} u_{2n+1} = J^A_{\beta_n}(\alpha_n f(u_{2n}) + (1 - \alpha_n)u_{2n} + e_n), \; n = 0, 1, \ldots, \\ u_{2n} = J^B_{\gamma_n}(u_{2n-1} + e'_n), \; n = 1, 2, \ldots, \end{cases} \tag{3.19}$$

converges strongly to an element $x^* \in S$ such that $Q_S f(x^*) = x^*$,

Proof. We have

$$\begin{aligned} \|u_{2n+1} - z_{2n+1}\| &\le \alpha_n \|f(u_{2n}) - f(z_{2n})\| + (1 - \alpha_n)\|u_{2n} - z_{2n}\| + \|e_n\| + \|e'_n\| \\ &\le [1 - \alpha_n(1 - c)]\|u_{2n} - z_{2n}\| + \|e_n\| + \|e'_n\| \\ &\le [1 - \alpha_n(1 - c)]\|u_{2n-1} - z_{2n-1}\| + \|e_n\| + \|e'_n\|. \end{aligned}$$

By Lemma 2.7, we have $\|u_{2n+1} - z_{2n+1}\| \to 0$. Similarly, $\|u_{2n} - z_{2n}\| \to 0$. So we have $\|u_n - z_n\| \to 0$ as $n \to \infty$. Thus we obtain

$$\lim_{n\to\infty} \|u_n - x^*\| \leq \lim_{n\to\infty} \|u_n - z_n\| + \lim_{n\to\infty} \|z_n - x^*\| = 0.$$

This completes the proof. □

Apply Theorem 3.6 with $Bx = 0$ for all $x \in E$, we obtain the following corollary (see [23]).

Corollary 3.7 ([23]). *Let E be a reflective Banach space with either a weakly sequentially continuous duality mapping j or a uniformly Gâteaux differentiable norm such that every weakly compact convex subset of E has fixed point property for nonexpansive mappings and let $f : E \to E$ be a contraction mapping in E. Let $A : E \to 2^E$ be an m-accretive operator such that*

$$S := A^{-1}0 \neq \emptyset.$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive numbers satisfying (1), (2), (3)-(a) *in Condition A, and $\{e_n\}$ be a sequence of errors in E satisfying the condition* (4)-(a) *in Theorem 3.6. Then the sequence $\{u_n\}$ generated by $u_0 \in E$ and*

$$u_{n+1} = J^A_{\beta_n}(\alpha_n f(u_n) + (1 - \alpha_n)u_n + e_n),\ n = 0, 1, \ldots, \tag{3.20}$$

converges strongly to an element $x^* \in S$ such that $Q_S f(x^*) = x^*$.

Remark 3.4. This corollary contains the result of Sahu and Yao (see Theorem 3.7 in [25]).

And also, we can get the same results in Hilbert spaces.

Corollary 3.8. *Let H be a Hilbert space. Let $A : H \to 2^H$ and $B : H \to 2^H$ be maximal monotone operators such that*

$$S := A^{-1}0 \cap B^{-1}0 \neq \emptyset.$$

Let $f : H \to H$ be a contraction mapping in H. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences satisfying (1), (2), (3) *in Condition A, and $\{e_n\}$ and $\{e'_n\}$ be the sequences of errors in H satisfying the condition* (4) *in Theorem 3.6. Then the sequence $\{u_n\}$ generated by* (3.19) *converges strongly to an element $x^* \in S$ such that $P_S f(x^*) = x^*$, where $P_S : H \to S$ is a metric projection from H onto S.*

Corollary 3.9. *Let H be a Hilbert space. Let $A : H \to 2^H$ be a maximal monotone operator such that*

$$S := A^{-1}0 \neq \emptyset.$$

Let $f : H \to H$ be a contraction mapping in H. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive numbers satisfying (1), (2), (3)-(a) *in Condition A, and $\{e_n\}$ be a sequence of errors in H satisfying the condition* (4)-(a) *in Theorem 3.6. Then the sequence $\{u_n\}$ generated by* (3.20) *converges strongly to an element $x^* \in S$ such that $P_S f(x^*) = x^*$.*

Remark 3.5. Corollary 3.9 is more general than the results of Xu [32], Song and Yang [26]. When $f(x) = u$ for all $x \in H$, we obtain Theorem 3.2 of Xu in [32].

4. Applications

In this section, we give some applications in the framework of Hilbert spaces. First, we give an application for the convex minimization problem.

Theorem 4.1. *Let H be a Hilbert space and let $f_1, f_2 : H \to (-\infty, \infty]$ be two proper lower semicontinuous convex functions such that*

$$S := (\partial f_1)^{-1}0 \cap (\partial f_2)^{-1}0 \neq \emptyset.$$

Let $f : H \longrightarrow H$ be a contraction mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences satisfying (1), (2), (3) *in Condition A. Then the sequence $\{x_n\}$ generated by $x_0 \in H$ and*

$$\tag{4.1} \begin{cases} y_n = \alpha_n f(x_{2n}) + (1-\alpha_n)x_{2n}, \ n = 0, 1, \dots, \\ x_{2n+1} = \operatorname{argmin}_{x\in H}\left\{ f_1(x) + \dfrac{1}{2\beta_n}\|y_n - x\|^2 \right\}, \ n = 0, 1, \dots, \\ x_{2n} = \operatorname{argmin}_{x\in H}\left\{ f_2(x) + \dfrac{1}{2\gamma_n}\|x_{2n-1} - x\|^2 \right\}, \ n = 1, 2, \dots, \end{cases}$$

converges strongly to an element $x^* \in S$ such that $P_S f(x^*) = x^*$.

Proof. By the Rockafellar theorem [22], the subdifferential mappings ∂f_1 and ∂f_2 are maximal monotone operators in H. So,

$$x_{2n+1} = \operatorname{argmin}_{x\in H}\left\{ f_1(x) + \frac{1}{2\beta_n}\|x - y_n\|^2 \right\}$$

is equivalent to $\beta_n \partial f_1(x_{2n+1}) + x_{2n+1} \ni y_n$, that is,

$$x_{2n+1} = J^A_{\beta_n}\big(\alpha_n f(x_{2n}) + (1-\alpha_n)x_{2n}\big),$$

with $A = \partial f_1$ and

$$x_{2n} = \operatorname{argmin}_{x\in H}\left\{ f_2(x) + \frac{1}{2\gamma_n}\|x_{2n-1} - x\|^2 \right\}$$

is equivalent to $\gamma_n \partial f_2(x_{2n}) + x_{2n} \ni x_{2n-1}$, that is,

$$x_{2n} = J^B_{\gamma_n}(x_{2n-1}),$$

with $B = \partial f_2$. Using Corollary 3.8, $\{x_n\}$ converges strongly to an element $x^* \in S$. This completes the proof. $\square$

Corollary 4.2. *Let H be a Hilbert space and let $h : H \to (-\infty, \infty]$ be a proper lower semicontinuous convex function such that*

$$S := (\partial h)^{-1}0 \neq \emptyset.$$

Let $f : H \longrightarrow H$ be a contraction mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences satisfying (1), (2), (3)-(a) *in Condition A. Then the sequence $\{x_n\}$ generated by $x_0 \in H$ and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \ n = 0, 1, \dots, \\ x_{n+1} = \mathrm{argmin}_{x\in H}\Big\{ h(x) + \dfrac{1}{2\beta_n}\|y_n - x\|^2 \Big\}, \ n = 0, 1, \dots, \end{cases} \tag{4.2}$$

converges strongly to an element $x^* \in S$ such that $P_S f(x^*) = x^*$.

Let C be a closed convex subset of a Hilbert space H. Recall that the indicator function of C is defined by

$$i_C(x) = \begin{cases} 0 \text{ if } x \in C; \\ \infty \text{ if } x \notin C \end{cases}$$

and the normal cone for C at a point $x \in C$ is defined by

$$N_C(x) = \big\{ z \in H : \ \langle y - x, z\rangle \leq 0 \text{ for all } y \in C \big\}.$$

We know that i_C is a proper lower semicontinuous convex function and argmin $i_C = C$. So the problem of finding a common element of two closed convex subsets C_1 and C_2 in a Hilbert space H is equivalent to the problem of finding an element of the set $S = \text{argmin } i_{C_1} \cap \text{argmin } i_{C_2}$. From Theorem 4.1, we have the following result.

Theorem 4.3. *Let C_1 and C_2 be two nonempty closed and convex subsets of a Hlbert space H such that*

$$S := C_1 \cap C_2 \neq \emptyset.$$

Let $f : H \to H$ be a contraction mapping. Let $\{\alpha_n\}$ be a sequence satisfying (1), (2) *in Condition A. Then the sequence $\{x_n\}$ generated by $x_0 \in H$,*

$$\begin{cases} x_{2n+1} = P_{C_1}\big(\alpha_n f(x_{2n}) + (1-\alpha_n)x_{2n}\big), \ n = 0, 1, 2, \dots, \\ x_{2n} = P_{C_2}(x_{2n-1}), \ n = 1, 2, \dots, \end{cases} \tag{4.3}$$

converges strongly to $x^* \in S$ such that $P_S f(x^*) = x^*$.

Proof. First, we have $S = \text{argmin } i_{C_1} \cap \text{argmin } i_{C_2}$. So, apply Theorem 4.1 for $f_1 = i_{C_1}$ and $f_2 = i_{C_2}$, and using the equality $(I + r\partial i_C)^{-1} = (I + rN_C)^{-1} = P_C$ for all closed convex subset in H and for all $r > 0$, we can prove this theorem. $\square$

Remark 4.1. Theorem 4.3 is an extension of the Bregman's results in [5].

We next apply Corollary 3.8 to find a common solution of the convex minimization problem and the variational inequality problem.

Let C be a nonempty closed and convex subset of a Hilbert space H and let $A : C \to H$ be a single-valued monotone operator which is hemicontinuous. A point $u \in C$ is said to be a solution of the variational inequality for A if

$$\langle y - u, Au \rangle \geq 0$$

holds for all $y \in C$. We denote by $VI(C, A)$ the set of all solutions of the variational inequality for A.

Theorem 4.4. *Let C be a nonempty closed and convex subset of a Hilbert space H, $h : H \to (-\infty, \infty]$ be a proper lower semicontinuous convex function, and $A : C \to H$ be a single-valued monotone operator and hemicontinuous such that*

$$S := (\partial h)^{-1}0 \cap VI(C, A) \neq \emptyset.$$

Let $f : H \to H$ be a contraction mapping in H. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences satisfying (1), (2), (3) *in Condition A. Then the sequence $\{x_n\}$ generated by $x_0 \in H$ and*

$$\begin{cases} y_n = \alpha_n f(x_{2n}) + (1 - \alpha_n)x_{2n}, \ n = 0, 1, \ldots, \\ x_{2n+1} = \operatorname{argmin}_{x \in H}\left\{ h(x) + \dfrac{1}{2\beta_n}\|y_n - x\|^2 \right\}, \ n = 0, 1, \ldots, \\ x_{2n} = VI(C, \gamma_n A + I - x_{2n-1}), \ n = 1, 2, \ldots, \end{cases} \tag{4.4}$$

converges strongly to $x^* \in S$ such that $P_S f(x^*) = x^*$.

Proof. Define a mapping $T \subset H \times H$ by

$$Tx = \begin{cases} Ax + N_C(x), \ x \in C \\ \emptyset, \ x \notin C. \end{cases}$$

By the Rockafellar theorem [23], we know that T is maximal monotone and $T^{-1}0 = VI(C, A)$. Note that

$$x_{2n} = VI(C, \gamma_n A + I - x_{2n-1})$$

if and only if

$$\langle y - x_{2n}, \gamma_n A x_{2n} + x_{2n} - x_{2n-1} \rangle \geq 0$$

for all $y \in C$, that is,

$$-\gamma_n A x_{2n} - x_{2n} + x_{2n-1} \in \gamma_n N_C(x_{2n}).$$

This implies that $x_{2n} = J^T_{\gamma_n}(x_{2n-1})$. By Corollary 3.8 and the proof of Theorem 4.1, we obtain the proof of this theorem. □

From Corollary 3.8 and the proof of Theorem 4.3, we have the following theorem.

Theorem 4.5. *Let C_1 and C_2 be two nonempty closed and convex subsets of a Hilbert space H and let $A_i : C_i \to H$ $(i = 1, 2)$ be two single-valued monotone and hemicontinuous operators such that*

$$S := VI(C_1, A_1) \cap VI(C_2, A_2) \neq \emptyset.$$

Let $f : H \to H$ be a contraction mapping in H. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences satisfying (1), (2), (3) *in Condition A. Then the sequence $\{x_n\}$ generated by $x_0 \in H$ and*

$$\begin{cases} y_n = \alpha_n f(x_{2n}) + (1-\alpha_n)x_{2n},\ n = 0, 1, \dots, \\ x_{2n+1} = VI(C_1, \beta_n A_1 + I - y_n),\ n = 0, 1, \dots, \\ x_{2n} = VI(C_2, \gamma_n A_2 + I - x_{2n-1}),\ n = 1, 2, \dots, \end{cases} \tag{4.5}$$

converges strongly to $x^* \in S$ such that $P_S f(x^*) = x^*$.

Corollary 4.6. *Let C be a nonempty closed and convex subset of a Hilbert space H and let $A : C \to H$ be a single-valued monotone and hemicontinuous operator such that*

$$S := VI(C, A) \neq \emptyset.$$

Let $f : H \to H$ be a contraction mapping in H. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences satisfying (1), (2), (3)-(a) *in Condition A. Then the sequence $\{x_n\}$ generated by $x_0 \in H$ and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n,\ n = 0, 1, \dots, \\ x_{n+1} = VI(C, \beta_n A + I - y_n),\ n = 0, 1, \dots, \end{cases} \tag{4.6}$$

converges strongly to $x^* \in S$ such that $P_S f(x^*) = x^*$.

Finally, we give an application for the equilibrium problem. Let C be a nonempty, closed and convex subset of a Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem: Finding $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C. \tag{4.7}$$

We denote by $EP(F)$ the set of all solutions of equilibrium problem (4.7). To study the equilibrium problem, we may assume that bifunction F satisfies the following restrictions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) For each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;
(A4) For each $x \in C$, $y \longmapsto F(x, y)$ is convex and lower semi-continuous.

We need the following lemma for the next theorem.

Lemma 4.1 ([27])**.** *Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4)*, and let A_F be a multi-valued mapping of H into itself defined by*

$$A_F x = \begin{cases} \big\{z \in H :\ F(x, y) \geq \langle y - x, z\rangle,\ \forall y \in C\big\},\ x \in C, \\ \emptyset,\ x \notin C. \end{cases}$$

Then A_F is a maximal monotone operator with the domain $D(A_F) \subset C$, $EP(F) = A_F^{-1}0$, and the resolvent $T_r = (I + rA_F)^{-1}$ is defined by

$$T_r x = \left\{ z \in C : \ F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}, \ \forall x \in H.$$

From Lemma 4.1 and Corollary 3.8, we have the following theorem.

Theorem 4.7. *Let C be a nonempty closed and convex subset of a Hilbert space H. Let $F_m : C \times C \to \mathbb{R}$, $m = 1, 2$ be two bifunctions satisfying* (A1)-(A4) *such that*

$$S := EP(F_1) \cap EP(F_2) \neq \emptyset.$$

Let $f : H \to H$ be a contraction mapping in H. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences satisfying (1), (2), (3) *in Condition A. Then the sequence $\{x_n\}$ generated by $x_0 \in H$ and*

$$\begin{cases} x_{2n+1} = T_{\beta_n}\big(\alpha_n f(x_{2n}) + (1 - \alpha_n)x_{2n}\big) \ n = 0, 1, \dots, \\ x_{2n} = T_{\gamma_n}(x_{2n-1}), \ n = 1, 2, \dots, \end{cases} \tag{4.8}$$

converges strongly to $x^* \in S$ such that $P_S f(x^*) = x^*$.

5. Numerical examples

In this section, we have tested the proposed algorithm in Theorem 4.1.

The algorithm is implemented in Matlab 7.0 running on a HP Compaq 510, Core(TM) 2 Duo CPU. T5870 with 2.0 GHz and 2GB RAM.

Example 5.1. Consider the problem for finding an element

$$x^* \in S := (\partial f_1)^{-1}0 \cap (\partial f_2)^{-1}0 \neq \emptyset,$$

that is,

$$x^* \in S = \ \operatorname{argmin}_{x\in\mathbb{R}^3} f_1(x) \cap \ \operatorname{argmin}_{x\in\mathbb{R}^3} f_2(x),$$

where f_1 and f_2 are defined by

$$f_i(x) = \langle A_i x, x\rangle + \langle B_i, x\rangle + C_i, \ i = 1, 2$$

with

$$A_1 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -4 & -4 & 4 \end{pmatrix}, \ B_2 = \begin{pmatrix} -4 & -4 & 0 \end{pmatrix}, \ C_1, \ C_2 \text{ are any constants.}$$

It is easy to show that f_1 and f_2 are proper continuous convex functions on $\mathbb{R}^3$, and

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 + x_2 = 2, \ x_3 = 0\}.$$

Apply Theorem 4.1 for $f(x) = u$ for all $x \in \mathbb{R}^3$ with $u = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$ and $\alpha_n = 1/(n + 1)$, $\beta_n = \gamma_n = 1$ for all $n \geq 0$, we obtain the following table of numerical results.

Table 1. The exact solution $P_S u = \begin{pmatrix} 0.5 & 1.5 & 0 \end{pmatrix}$

n	x_1^n	x_2^n	x_3^n	n	x_1^n	x_2^n	x_3^n
100	0.498749	1.498750	−0.017499	600	0.499791	1.499791	−0.002916
200	0.499374	1.499375	−0.008749	700	0.499821	1.499821	−0.002499
300	0.499583	1.499583	−0.005833	800	0.499843	1.499843	−0.002187
400	0.499687	1.499687	−0.004374	900	0.499861	1.499861	−0.001944
500	0.499749	1.499750	−0.003499	1000	0.499874	1.499875	−0.001749

Example 5.2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x) = \langle A_1 x, x\rangle + \langle B_1, x\rangle + C_1,$$

where

$$A_1 = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \ B_1 = \begin{pmatrix} 4 & -8 \end{pmatrix}, \ C_1 \text{ is any constant.}$$

Consider the problem: Finding an element

$$x^* \in S := \text{argmin}_{x\in C} f(x),$$

where

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 + x_2 \geq 2\}.$$

It is easy to see that f is a convex function on $\mathbb{R}^2$ and

$$\text{argmin}_{x\in C} f(x) = \{(x_1, x_2) : \ x_1 - 2x_2 + 2 = 0, \ x_1 \geq 2/3\}.$$

We know that

$$x^* \in \text{argmin}_{x\in C} f(x)$$

if and only if

$$\bigtriangledown f(x^*) + N_C(x^*) \ni 0,$$

that is, x^* is a solution of the following inequality

$$\langle y - x, \bigtriangledown f(x)\rangle \geq 0, \ \forall y \in C.$$

Apply Corollary 4.6 with $\alpha_n = \frac{1}{n+1}$, $\beta_n = 1$ for all n, $f(x) = u = \begin{pmatrix} 1 & 4 \end{pmatrix}$, $x_0 = \begin{pmatrix} 0 & 0 \end{pmatrix}$ and using the projected gradient method [9, 17] with $\mu = 1/80$ to solve the inequality

$$x_{n+1} = VI(C, A + I - y_n), \ n = 0, 1, \ldots,$$

we obtain the following table of numerical results. Note that $A = \bigtriangledown f$ and $A + I - y_n$ is 11-Lipschitz and η-strongly monotone over $\mathbb{R}^2$, with $\eta \geq 1$.

Table 2. The exact solution $P_S u = \begin{pmatrix} 2 & 2 \end{pmatrix}$

n	x_1^n	x_2^n	n	x_1^n	x_2^n
100	1.999000	2.002000	600	1.999833	2.000333
200	1.999500	2.001000	700	1.999857	2.000285
300	1.999666	2.000666	800	1.999875	2.000250
400	1.999750	2.000500	900	1.999888	2.000222
500	1.999800	2.000400	1000	1.999900	2.000200

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