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SPECTRAL PROPERTIES OF VOLTERRA-TYPE INTEGRAL OPERATORS ON FOCK–SOBOLEV SPACES

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Abstract. We study some spectral properties of Volterra-type integral operators V_g and I_g with holomorphic symbol g on the Fock–Sobolev spaces $\mathcal{F}^p_{\psi_m}$. We showed that V_g is bounded on $\mathcal{F}^p_{\psi_m}$ if and only if g is a complex polynomial of degree not exceeding two, while compactness of V_g is described by degree of g being not bigger than one. We also identified all those positive numbers p for which the operator V_g belongs to the Schatten $\mathcal{S}_p$ classes. Finally, we characterize the spectrum of V_g in terms of a closed disk of radius twice the coefficient of the highest degree term in a polynomial expansion of g.

1. Introduction

The boundedness and compactness properties of integral operators stand among the very well studied objects in operator related function-theories. They have been studied for a broad class of operators on various spaces of holomorphic functions including the Hardy spaces [1, 2, 18], Bergman spaces [19–21], Fock spaces [6, 7, 11, 13, 14, 16, 17], Dirichlet spaces [3, 9, 10], Model spaces [15], and logarithmic Bloch spaces [24]. Yet, they still constitute an active area of research because of their multifaceted implications. Typical examples of operators subjected to this phenomena are the Volterra-type integral operator V_g and its companion I_g, defined by

$$V_g f(z) = \int_0^z f(w)g'(w)dw \quad \text{and} \quad I_g f(z) = \int_0^z f'(w)g(w)dw,$$

where g is a holomorphic symbol. Applying integration by parts in any one of the above integrals gives the relation

$$V_g f + I_g f = M_g f - f(0)g(0), \tag{1.1}$$

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where $M_g f = gf$ is the multiplication operator of symbol g. On the classical Fock spaces with the Gaussian weight, some spectral structures of these operators were studied by several authors for example in [6, 11, 13, 14, 16]. On the other hand, when the weight decays faster than the classical Gaussian weight, they were recently studied in [7, 17]. From the results in these two later works, we observed that while the operator V_g enjoys a richer structure when it acts between weighted Fock spaces of faster decaying weights in contrast to its action on the classical Fock spaces, the analogues structures for I_g and M_g has got rather poorer. A natural question is then what happens to these structures when the weight decays slower than the classical Gaussian weight? The central aim of this paper is to investigate this situation. Prototype examples of spaces generated by such slower decaying weights, which we are interested in, are the Fock–Sobolev spaces as described below.

Let m be any nonnegative integer and $0 < p < \infty$. Then, the Fock–Sobolev spaces $\mathcal{F}_{(m,p)}$ consist of entire functions f such that $f^{(m)}$, the m-th order derivative of f, belongs to the classical Fock spaces $\mathcal{F}_p$; which consist of all entire functions f for which

$$\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) < \infty.$$

The Fock–Sobolev spaces were introduced in [5] where it was proved that f belongs to $\mathcal{F}_{(m,p)}$ if and only if the function $z \mapsto |z|^m f(z)$ belongs to $L^p(\mathbb{C}, e^{-p|z|^2/2})$. By closed graph theorem argument, we have that f belongs to $\mathcal{F}_{(m,p)}$ if and only if $z \mapsto (\beta + |z|)^m f(z)$ belongs to $L^p(\mathbb{C}, e^{-p|z|^2/2})$ for any positive number β. A consequence of this is that the norm in $\mathcal{F}_{(m,p)}$ is comparable to the quantity

$$\left(C_{(p,m)} \int_{\mathbb{C}} |f(z)|^p (1 + |z|)^{mp} e^{-\frac{p}{2}|z|^2} dA(z) \right)^{1/p}$$

for $0 < p < \infty$, and

$$C_{(m,p)} = (p/2)^{\frac{mp}{2}+1} \left(\pi \Gamma\Big(\frac{mp}{2} + 1 \Big) \right)^{-1},$$

where Γ denotes the Gamma function, dA denotes the usual Lebesgue area measure on $\mathbb{C}$, and we fix $\beta \simeq 1$ for simplicity. To put the spaces into weighted/generalized Fock spaces context, we may now set the sequence of the corresponding weight functions as

$$\psi_m(z) = \frac{1}{2}|z|^2 - m \log(1 + |z|), \tag{1.2}$$

and observe that the Fock–Sobolev spaces $\mathcal{F}_{(m,p)}$ are just the weighted Fock spaces $\mathcal{F}^p_{\psi_m}$ which consist of all entire functions f for which [1]

$$\int_{\mathbb{C}} |f(z)|^p e^{-p\psi_m(z)} dA(z) \simeq \|f\|^p_{(p,m)} < \infty.$$

We may now state our first main result.

Theorem 1.1. *Let g be an entire function on $\mathbb{C}$, $0 < p, q < \infty$, and if*

(i) $0 < p \le q < \infty$, *then* $V_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ *is*
  (a) *bounded if and only if* $g(z) = az^2 + bz + c,\ a, b, c \in \mathbb{C}$.
  (b) *compact if and only if* $g(z) = az + b,\ a, b, \in \mathbb{C}$.

(ii) $0 < q < p < \infty$, *then the following statements are equivalent*
  (a) $V_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ *is bounded*;
  (b) $V_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ *is compact*;
  (c) $g(z) = az + b$ *whenever* $\frac{q}{2} > \frac{p-q}{p}$, *and* $g=$ *constant otherwise.*

(iii) $0 < p < \infty$ *and* V_g *compact on* $\mathcal{F}^2_{\psi_m}$, *then* V_g *belongs to the Schatten* $\mathcal{S}_p(\mathcal{F}^2_{\psi_m})$ *classes for all* $p > 2$. *On the other hand, if* $0 < p < 2$, *then* V_g *belongs to* $\mathcal{S}_p(\mathcal{F}^2_{\psi_m})$ *if and only if* g *is the zero function.*

Theorem 1.2. *Let g be an entire function on $\mathbb{C}$, $0 < p, q < \infty$, and if*

(i) $0 < p \le q < \infty$, *then* $I_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ *is*
  (a) *bounded if and only if g is a constant function.*
  (b) *compact if and only if g is the zero function.*

(ii) $0 < q < p < \infty$, *then the following are equivalent.*
  (a) $I_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ *is bounded*;
  (b) $I_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ *is compact*;
  (c) *g is the zero function.*

It may be noted that when $m = 0$, the spaces $\mathcal{F}^p_{\psi_m}$ reduce to the classical Fock spaces $\mathcal{F}^p$, and for this particular case, the results were proved in [6, 13, 14]. In view of our current results, we conclude that there exists no richer boundedness and compactness structures for V_g and I_g on Fock-Sobolev spaces than those on the classical setting. As can be seen from (1.2), the Fock–Sobolev spaces are generated by making small perturbations of the weight function on the classical Fock spaces. It turns out that such perturbations play no role in the structure of the operators and rather extend the classical results to all the spaces $\mathcal{F}^p_{\psi_m}$ independent of the values of m.

In addition, the results show that there exists no nontrivial Volterra companion integral type operators I_g acting between any of the Fock–Sobolev spaces.

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[1] The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \le CV(z)$ holds for all z in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

Another observation worthwhile making is that when $g(z) = z$, the operator V_g reduces to the original Volterra operator

$$Vgf(z) = \int_0^z f(w)dwA(w).$$

By particular cases of the results above, we conclude that this operator is always bounded in its action on the Fock–Sobolev spaces.

1.1. Spectrum of the integral operators

In contrast to the fairly good understanding of the boundedness, compactness, and Schatten class membership of the Volterra-type integral operators on various Banach spaces, much less is known about their spectral. Recently, Constantin and Persson [8], determined the spectrum of V_g acting on generalized Fock spaces where the inducing weight function takes the particular form $|z|^A$, $A > 0$ and $1 \leq p < \infty$. Our next result describes the spectrum of the Volterra-type integral operators on Fock–Sobolev spaces in terms of a closed disk of radius involving the coefficient of the highest degree term in a polynomial expansion of g as precisely formulated below.

Theorem 1.3. (i) *Let* $p \geq 1$ *and* $V_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^p_{\psi_m}$ *be a bounded operator, i.e.,* $g(z) = az^2 + bz, \quad a, b, \in \mathbb{C}$. *Then*

$$\sigma(V_g) = \{\lambda \in \mathbb{C} : |\lambda| \leq 2|a|\} = \{0\} \cup \overline{\{\lambda \in \mathbb{C} \setminus \{0\} : e^{g(z)/\lambda} \notin \mathcal{F}^p_{\psi_m}\}}. \tag{1.3}$$

(ii) *Let* $p \geq 1$ *and* $I_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^p_{\psi_m}$ *be a bounded operator, i.e.,* $g(z) = c =$*constant. Then*

$$\sigma(I_g) = \{c\}.$$

The results here are also independent of the order m, and coincide with the corresponding results in the classical Fock spaces setting. Furthermore, for $m = 0$, the result in (1.3) follows also from the main result in [8] as a particular case.

2. Preliminaries

For each m, the spaces $\mathcal{F}^2_{\psi_m}$ are reproducing kernel Hilbert spaces with kernel $K_{(w,m)}$ and normalized reproducing kernel functions $k_{(w,m)}$ for a point w in $\mathbb{C}$. An explicit expression for $K_{(w,m)}$ is still unknown. On the other hand, for each w in $\mathbb{C}$ by Proposition 2.7 of [4], we have an important asymptotic relation

$$\|K_{(w,m)}\|^2_{(2,m)} \simeq e^{2\psi_m(w)}. \tag{2.1}$$

As noted before when $m = 0$, the space $\mathcal{F}^2_m$ reduces to the classical Fock space $\mathcal{F}^2$, and in this case we precisely have $\|K_{(z,0)}\|^2_{(2,0)} = e^{|z|^2}$ and $K_{(w,0)}(z) = e^{\overline{w}z}$. For other p's, Corollary 14 of [5] gives the one sided estimate

$$\|K_{(w,m)}\|_{(p,m)} \lesssim e^{\psi_m(w)}. \tag{2.2}$$

Because of the reproducing property of the kernel and Parseval identity, it further holds that

$$K_{(w,m)}(z) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(w)} \text{ and } \|K_{(w,m)}\|^2_{(2,m)} = \sum_{n=1}^{\infty} |e_n(w)|^2 \tag{2.3}$$

for any orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{F}^2_{\psi_m}$. This and the estimate in (2.1) will be repeatedly used in the sequel. Another important ingredient needed in the proofs of the results is the following pointwise estimate for the reproducing kernel functions.

Lemma 2.1. *There exists a small positive number δ such that for any $w \in \mathbb{C}$*

$$|K_{(m,w)}(z)| \gtrsim e^{\psi_m(z)+\psi_m(w)}$$

for all $z \in D(w,\delta)$, where $D(w,\delta)$ refers to the Euclidian disk of radius r and center w.

Proof. The lemma will follow from [22, Proposition 3.3] once we show that the weight function ψ_m satisfies the growth condition

$$c \lesssim \Delta\psi_m(z) \lesssim C \tag{2.4}$$

for all $z \in \mathbb{C}$ and some positive constants c and C. Thus, we consider

$$\psi_m(z) = \frac{|z|^2}{2} - m\log(\beta + |z|) \simeq \frac{|z|^2}{2} - \frac{m}{2}\log(\beta + |z|^2),$$

and a straightforward calculation gives that

$$\Delta\psi_m(z) = 2 - \frac{2m\beta}{(\beta + |z|^2)^2}.$$

Then, the required condition (2.4) holds for any choice of $\beta > m$ as $2(1-\frac{m}{\beta}) \leq \Delta\psi_m(z) \leq 2$. For simplicity, we will continue setting $\beta = 1$ throughout the rest of the paper. ☐

2.1. Littlewood–Paley type formula

Dealing with Volterra-type integral operators in normed spaces gets easier when the norms in the target spaces of the operators are described in terms of Littlewood–Paley type formula. The operators have been extensively studied in the spaces where such formulas are found to be accessible. The formulas will primarily help get rid of the integrals appearing in defining the operators. Our next key lemma does this job by characterizing the Fock–Sobolev spaces in terms of derivatives.

Lemma 2.2. *Let $0 < p < \infty$ and f be a holomorphic function on $\mathbb{C}$. Then*

$$\|f\|^p_{\mathcal{F}^p_{\psi_m}} \simeq |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p(1+|z|)^p e^{-p\psi_m(z)}}{\big(1+|z|+\big||z|^2+|z|-m\big|\big)^p} dA(z). \tag{2.5}$$

Proof. We plan to show that the estimate in (2.5) follows from the general estimate in Theorem 19 of [7]. To this end, it suffices to verify that the sequence of our weight functions ψ_m satisfy all the preconditions required in the theorem there, which are;

i) There should exist a positive r_0 for which $\psi_m'(r) \neq 0$ for all $r > r_0$. This, rather week requirement on the growth of ψ_m works fine as one can for example take

$$r_0 = \frac{1+\sqrt{1+4m}}{2}.$$

In addition, we have that $1+\psi_m'(z) \simeq \psi_m'(z)$ when $|z| \to \infty$.

ii) The estimates

$$\begin{aligned} \lim_{r\to\infty} \frac{re^{-p\psi_m(r)}}{\psi_m'(r)} &= 0 \\ \limsup_{r\to\infty} \frac{1}{r}\left(\frac{r}{\psi_m'(r)}\right)' &< p \quad \text{and} \\ \liminf_{r\to\infty} \frac{1}{r}\left(\frac{r}{\psi_m'(r)}\right)' &> -\infty, \end{aligned} \tag{2.6}$$

hold for all positive p. The first estimate in (2.6) follows easily since

$$\lim_{r\to\infty} \frac{re^{-p\psi_m(r)}}{\psi_m'(r)} = \lim_{r\to\infty} \frac{r+r^2}{r^2+r-m} e^{-p\psi_m(r)} = \lim_{r\to\infty} e^{-p\psi_m(r)} = 0.$$

On the other hand, a simple computation shows that

$$\frac{1}{r}\left(\frac{r}{\psi_m'(r)}\right)' = \frac{2r^2-2rm-m}{r(r^2+r-m)^2},$$

from which it follows that

$$\limsup_{r\to\infty} \frac{1}{r}\left(\frac{r}{\psi_m'(r)}\right)' = \limsup_{r\to\infty} \frac{2r^2-2rm-m}{r(r^2+r-m)^2} \leq 0 < p.$$

It remains to verify the last estimate in (2.6). But this is rather immediate as

$$\liminf_{r\to\infty} \frac{1}{r}\left(\frac{r}{\psi_m'(r)}\right)' = \liminf_{r\to\infty} \frac{2r^2-2rm-m}{r(r^2+r-m)^2} = 0 > -\infty.$$

□

We now state a key lemma on spectral properties of the operator M_g acting between Fock–Sobolev spaces. The lemma is interest of its own.

Lemma 2.3. *Let g be an analytic function on $\mathbb{C}$, $0 < p, q < \infty$, and if*

(i) *$q \geq p$, then $M_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ is bounded (compact) if and only if g is a constant (zero) function.*

(ii) *$q < p$, then $M_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ is bounded (compact) if and only if g is the zero function.*

(iii) $1 \leq p < \infty$ *and* $M_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^p_{\psi_m}$ *is a bounded map, that is* $g = \alpha =$ *constant, then*

$$\sigma_p(M_g) = \sigma(M_g) = \{\alpha\}.$$

Like that of the operator I_g, the lemma shows that there exists no nontrivial multiplication operators M_g acting between the Fock–Sobolev spaces. This is rather due to the relation in (1.1), as will be also explained in the proof of Theorem 1.2 in Section 3.1.

Proof. We observe that the multiplication operator M_g is a special case of weighted composition operators $uC_\phi f(z) = u(z)f(\phi(z))$; set $u = g$ and $\phi(z) = z$. Several properties of uC_ϕ have already been described in [12] from which some will be used in our subsequent considerations. Let us now assume that $0 < p \leq q < \infty$. Then by Theorem 3.1 of [12], $M_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ is bounded if and only

$$\sup_{w\in\mathbb{C}} B_m(|g|^q)(w) = \sup_{w\in\mathbb{C}} \int_{\mathbb{C}} |k_{(w,m)}(z)|^q |g(z)|^q e^{-q\psi_m(z)} dA(z) < \infty. \tag{2.7}$$

To arrive at the desired conclusion, we may proceed to investigate further the boundedness of the integral transform in (2.7). To this end, assuming this condition, and applying (2.1) and Lemma 2.1 we have

$$\begin{aligned} \infty > \sup_{w\in\mathbb{C}} \int_{\mathbb{C}} \frac{|k_{(w,m)}(z)|^q |g(z)|^q}{e^{q\psi_m(z)}} dA(z) &\geq \sup_{w\in\mathbb{C}} \int_{D(w,\delta)} \frac{|k_{(w,m)}(z)|^q}{e^{q\psi_m(z)}} |g(z)|^q dA(z) \\ &\gtrsim \sup_{w\in\mathbb{C}} \int_{D(w,\delta)} |g(z)|^q dA(z) \end{aligned} \tag{2.8}$$

for a small positive number δ. By subharmonicity of $|g|^q$, we further have

$$\infty > \sup_{w\in\mathbb{C}} \int_{D(w,\delta)} |g(z)|^q dA(z) \gtrsim \sup_{w\in\mathbb{C}} |g(w)|^q \tag{2.9}$$

for all $w \in \mathbb{C}$. From this we deduce that g is a bounded analytic function on $\mathbb{C}$. Then Liouville's classical theorem forces it to be a constant.

Conversely, if g is a constant, then the integral in (2.7) is obviously finite since all the Fock–Sobolev spaces $\mathcal{F}^p_{\psi_m}$ contain the reproducing kernels (see [5, Corollary 14]).

A similar analysis shows that when $0 < p \leq q < \infty$, M_g is compact if and only if g is the zero function.

(ii) When $0 < q < p < \infty$, then an application of Theorem 3.3 of [12] ensures that the boundedness and compactness properties of $M_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ are equivalent and this happens if and only if

$$\int_{\mathbb{C}} \big(B_m(|g|^q)(w)\big)^{\frac{p}{p-q}} dA(w)$$

$$(2.10) \qquad = \int_{\mathbb{C}} \left(\int_{\mathbb{C}} |k_{(w,m)}(z)|^p |g(z)|^p e^{-q\psi_m(z)} dA(z) \right)^{\frac{p}{p-q}} dA(w) < \infty.$$

Arguing as in the series of estimates leading to (2.8) and (2.9), condition (2.10) implies

$$\int_{\mathbb{C}} |g(w)|^p dA(w) \lesssim \int_{\mathbb{C}} (B_m(|g|^q)(w))^{\frac{p}{p-q}} dA(w) < \infty,$$

and from this we conclude that g is indeed the zero function.

(iii) By part (i) of the lemma, the only bounded multiplication operators are the multiplications by constant functions. It means that we are actually dealing with constant multiples of the identity operator, whose spectrum obviously consists of the multiplicative constant. □

Lemma 2.4. *Let $a, \lambda \in \mathbb{C}$, $g(z) = az^2$ and assume that $|\lambda| > 2|a|$. If f is an entire function such that $fe^{g/\lambda}$ belongs to $\mathcal{F}^p_{\psi_m}$, then*

(2.11)

$$\int_{\mathbb{C}} \left|e^{\frac{g(z)}{\lambda}} f(z)\right|^p e^{-p\psi_m(z)} dA(z) \lesssim |f(0)|^p + \int_{\mathbb{C}} \frac{\left|f'(z)e^{\frac{g(z)}{\lambda}}\right|^p}{(1+\psi_m'(z))^p} e^{-p\psi_m(z)} dA(z).$$

The proof of the lemma follows from a simple variant of the proof of Proposition 1 in [8]. We only need to set $\alpha = 1$ and replace $\alpha|z|^A$ in there by $\psi_m(z)$ and reset $w(z) = p\Re(g(z)/\lambda) - p\psi_m(z)$ and run the arguments.

3. Proof of the main results

We now turn to the proofs of the main results of the paper. Let $0 < p, q < \infty$ and μ be a positive Borel measure on $\mathbb{C}$. We call μ a (p, q) Fock–Carleson measure if the inequality

$$\int_{\mathbb{C}} |f(z)|^q e^{-\frac{q}{2}|z|^2} d\mu(z) \lesssim \|f\|^q_{(p,m)},$$

holds, and we call it a vanishing (p, q) Fock–Carleson measure if

$$\lim_{n\to\infty} \int_{\mathbb{C}} |f_n(z)|^q e^{-\frac{q}{2}|z|^2} d\mu(z) = 0$$

for every uniformly bounded sequence f_n in $\mathcal{F}^p_{\psi_m}$ that converges to zero uniformly on compact subset of $\mathbb{C}$ as $n \to \infty$. These measures have been completely identified in [12].

We observe that by first setting

$$d\mu_{(g,q)}(z) = \frac{|g'(z)|^q (1+|z|)^{qm+q}}{\left(1+|z|+\left||z|^2+|z|-m\right|\right)^q} dA(z),$$

and applying (2.5), we may write the norm of $V_g f$ as

$$\|V_g f\|^q_{(q,m)} \simeq \int_{\mathbb{C}} \frac{|g'(z)|^q |f(z))|^q (1+|z|)^{mq+q}}{\left(1+|z|+\left||z|^2+|z|-m\right|\right)^q} e^{-\frac{q}{2}|z|^2} dA(z)$$

$$= \int_{\mathbb{C}} |f(z)|^q e^{-\frac{q}{2}|z|^2} d\mu_{(g,q)}(z).$$

In view of this, it follows that $V_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ is bounded (compact) if and only if $\mu_{(g,q)}$ is a (p,q) (vanishing) Fock-Carleson measure. Consequently, if $0 \le p \le q < \infty$, then Theorem 2.1 of [12] ensures that $\mu_{(g,q)}$ is a (p,q) Fock-Carleson measure if and only if $\tilde{\mu}_{(t,mq)}$ is bounded for some or any positive t where

$$\tilde{\mu}_{(t,mq)}(w) = \int_{\mathbb{C}} \frac{e^{-\frac{t}{2}|z-w|^2}}{(1+|z|)^{mq}} d\mu_{(g,q)}(z).$$

Having singled out this equivalent reformulation, our next task will be to investigate the new formulation further, namely boundedness of the transform $\tilde{\mu}_{(t,mq)}$. Let us first assume its boundeness, and show that g is a complex polynomial of degree not exceeding two. To this end,

$$\infty > \sup_{w\in\mathbb{C}} \int_{\mathbb{C}} \frac{e^{-\frac{t}{2}|z-w|^2}}{(1+|z|)^{mq}} d\mu_{(g,q)}(z) = \sup_{w\in\mathbb{C}} \int_{\mathbb{C}} \frac{e^{-\frac{t}{2}|z-w|^2}|g'(z)|^q(1+|z|)^q}{\big(1+|z|+\big||z|^2+|z|-m\big|\big)^q} dA(z)$$
$$\gtrsim \sup_{w\in\mathbb{C}} \int_{D(w,1)} \frac{|g'(z)|^q(1+|z|)^q}{\big(1+|z|+\big||z|^2+|z|-m\big|\big)^q} dA(z) = S.$$

Observe that whenever z belongs to the disk $D(w,1)$, then

$$\begin{cases} 1+|z| \simeq |+|w| \\ 1+|z|+\big||z|+|z|^2-m\big| \simeq 1+|w|+\big||w|+|w|^2-m\big|. \end{cases} \tag{3.1}$$

This together with the subharmonicity of $|g'|^q$ implies that

$$S \gtrsim \frac{|g'(w)|^q(1+|w|)^q}{\big(1+|w|+\big||w|^2+|w|-m\big|\big)^q}$$

for all $w \in \mathbb{C}$ and our assertion follows.

On the other hand, if $g(z) = az^2+bz+c,\ a,b,c \in \mathbb{C}$, then

$$\sup_{w\in\mathbb{C}} \tilde{\mu}_{(t,mq)}(w) = \sup_{w\in\mathbb{C}} \int_{\mathbb{C}} e^{-\frac{t}{2}|z-w|^2} \frac{|2az+b|^q(1+|z|)^q}{\big(1+|z|+\big||z|^2+|z|-m\big|\big)^q} dA(z)$$
$$\lesssim \sup_{w\in\mathbb{C}} \int_{\mathbb{C}} e^{-\frac{t}{2}|z-w|^2} dA(z) < \infty,$$

and completes the proof of part (a) of (i) in the theorem.

Similarly, for part (b), for $0 < p \le q < \infty$, by Theorem 2.2 of [12], $\mu_{(g,q)}$ is a (p,q) vanishing Fock-Carleson measure if and only if $\tilde{\mu}_{(t,mq)}(z) \to 0$ as $|z| \to \infty$. We may first assume this vanishing property and show that g is a complex polynomial of degree not exceeding one. For this, following the same arguments as above, we easily see from our assumption that

$$\lim_{|w|\to\infty} \frac{|g'(w)|(1+|w|)}{1+|w|+\big||w|^2+|w|-m\big|} = 0,$$

and this obviously holds only if g' is a constant as asserted.

Conversely, if $g(z) = az + b,\ a, b, \in \mathbb{C}$, then

$$\lim_{|w|\to\infty} \tilde{\mu}_{(q,mq)}(w) = \lim_{|w|\to\infty} \int_{\mathbb{C}} e^{-\frac{q}{2}|z-w|^2} \frac{|a|^q(1+|z|)^q}{\big(1+|z|+\big||z|^2+|z|-m\big|\big)^q} dA(z)$$
$$\lesssim \lim_{|w|\to\infty} \int_{\mathbb{C}} \frac{e^{-\frac{q}{2}|z-w|^2}}{(1+|z|)^q} dA(z) \simeq \lim_{|w|\to\infty} (1+|w|)^{-q} = 0.$$

ii) If $0 < q < p < \infty$, then by Theorem 2.3 of [12] again, $V_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ is bounded (compact) if and only if $\tilde{\mu}_{(t,mq)}$ belongs to $L^{\frac{p}{p-q}}(\mathbb{C}, dA)$. We plan to show that this holds if and only if g is of at most degree one and $q > \frac{2p}{p+2}$. To this end, applying (3.1) and subharmonicity of $|g'(w)|^{\frac{pq}{p-q}}$, we infer

$$\int_{\mathbb{C}} |g'(w)|^{\frac{pq}{p-q}} \left(\frac{1+|w|}{1+|w|+\big||w|^2+|w|-m\big|}\right)^{\frac{pq}{p-q}} dA(w)$$
$$\lesssim \int_{\mathbb{C}} \left(\int_{D(w,1)} |g'(z)|^q e^{-\frac{q}{2}|z-w|^2} \left(\frac{1+|z|}{1+|z|+\big||z|^2+|z|-m\big|}\right)^q dA(z)\right)^{\frac{p}{p-q}} dA(w)$$
$$\le \int_{\mathbb{C}} \left(\int_{\mathbb{C}} |g'(z)|^q e^{-\frac{q}{2}|z-w|^2} \left(\frac{1+|z|}{1+|z|+\big||z|^2+|z|-m\big|}\right)^q dA(z)\right)^{\frac{p}{p-q}} dA(w)$$
$$\simeq \int_{\mathbb{C}} \tilde{\mu}^{\frac{p}{p-q}}_{(q,mq)}(w) dA(w) < \infty,$$

from which we conclude that g' must be a constant. In addition, if g' is a nonzero constant, the above holds only if $\frac{pq}{p-q} > 2$.

Conversely, assuming that $g' \simeq \alpha =$ constant we have

$$\int_{\mathbb{C}} \tilde{\mu}^{\frac{p}{p-q}}_{(q,mq)}(w) dA(w)$$
$$\simeq \int_{\mathbb{C}} \left(\int_{\mathbb{C}} \left(\frac{|\alpha| e^{-\frac{1}{2}|z-w|^2}(1+|z|)}{1+|z|+\big||z|^2+|z|-m\big|}\right)^q dA(z)\right)^{\frac{p}{p-q}} dA(w)$$
$$\lesssim \int_{\mathbb{C}} \left(\int_{\mathbb{C}} \frac{|\alpha|^q e^{-\frac{q}{2}|z-w|^2}}{(1+|z|)^q} dA(z)\right)^{\frac{p}{p-q}} dA(w)$$
$$\simeq \int_{\mathbb{C}} |\alpha|^q (1+|w|)^{-\frac{pq}{p-q}} dA(w) < \infty,$$

where the last integral converges since either $\frac{pq}{p-q} > 2$ or $\alpha = 0$ by our assumption.

(iii) Let us now turn to the Schatten class membership of V_g. We recall that a compact operator V_g belongs to the Schatten $\mathcal{S}_p(\mathcal{F}^2_{\psi_m})$ class if and only if the sequence of the eigenvalues of the positive operator $(V_g^* V_g)^{1/2}$ is ℓ^p summable.

In particular when $p \geq 2$, this happens if and only if

$$\sum_{n=1}^{\infty} \|V_g e_n\|_{(2,m)}^p < \infty \tag{3.2}$$

for any orthonormal basis (e_n) of $\mathcal{F}_{\psi_m}^2$ (see [23, Theorem 1.33]). Let us assume that V_g is compact, that is g' is a constant. Then for $p > 2$, applying (3.2), (2.5), and Hölder's inequality, and subsequently (2.3) and (2.1), we compute

$$\begin{aligned}
&\sum_{n=1}^{\infty} \|V_g e_n\|_{(2,m)}^p \\
&\simeq \sum_{n=1}^{\infty} \left(\int_{\mathbb{C}} \left(\frac{|e_n(z)| e^{-\psi_m(z)} (1+|z|)}{1+|z|+\big||z|^2+|z|-m\big|} \right)^2 dA(z) \right)^{\frac{p}{2}} \\
&\leq \sum_{n=1}^{\infty} \left(\int_{\mathbb{C}} \frac{|e_n(z)|^2 (1+|z|)^2}{e^{2\psi_m(z)}} dA(z) \right)^{\frac{p-2}{2}} \int_{\mathbb{C}} \frac{|e_n(z)|^2 e^{-2\psi_m(z)} \big(1+|z|\big)^2}{\big(1+|z|+\big||z|+|z|^2-m\big|\big)^p} dA(z) \\
&\simeq \sum_{n=1}^{\infty} \int_{\mathbb{C}} \frac{|e_n(z)|^2 e^{-2\psi_m(z)} \big(1+|z|\big)^2}{\big(1+|z|+\big||z|+|z|^2-m\big|\big)^p} dA(z) \\
&\simeq \int_{\mathbb{C}} \Big(1+|z|+\big||z|+|z|^2-m\big|\Big)^{-p} dA(z) < \infty.
\end{aligned}$$

On the other hand, if $p = 2$ and $g' = \alpha = \text{constant}$, then

$$\begin{aligned}
\sum_{n=1}^{\infty} \|V_g e_n\|_{(2,m)}^2 &\simeq \sum_{n=1}^{\infty} \int_{\mathbb{C}} \left(\frac{|g'(z)||e_n(z)| e^{-\psi_m(z)} (1+|z|)}{1+|z|+\big||z|^2+|z|-m\big|} \right)^2 dA(z) \\
&\simeq \int_{\mathbb{C}} |\alpha|^2 \big(1+|z|+\big||z|^2+|z|-m\big|\big)^{-2} dA(z).
\end{aligned}$$

The last integral above is finite if and only if $\alpha = 0$, and hence g is a constant. The same conclusion holds for the case when $0 < p < 2$ by the monotonicity property of Schatten class membership, in the sense that $\mathcal{S}_p(\mathcal{F}_{\psi_m}^2) \subseteq \mathcal{S}_2(\mathcal{F}_{\psi_m}^2)$, for all $p \leq 2$.

Remark 1. Because of Lemma 2.2, it is tempting to prove Theorem 1.1 by first setting

$$\begin{aligned}
\int_{\mathbb{C}} |V_g f(z)|^q e^{-q\psi_m(z)} dA(z) &\simeq \int_{\mathbb{C}} |f(z)|^q \frac{|g'(z)|^q (1+|z|)^q e^{-q\psi_m(z)}}{\big(1+|z|+\big||z|+|z|^2-m\big|\big)^q} dA(z) \\
&\simeq \int_{\mathbb{C}} |M_h f(z)|^q e^{-q\psi_m(z)} dA(z),
\end{aligned}$$

where

$$h(z) = \frac{|g'(z)|(1+||z)}{1+|z|+\big||z|+|z|^2-m\big|},$$

and then apply Lemma 2.3 with h as the multiplier function. Unfortunately, this approach is not valid as the lemma on the multiplication operator can not be directly applied; since the analyticity property of g is heavily used in its proof, while h fails to be analytic in here.

3.1. Proof of Theorem 1.2

We note that relation (1.1) ensures that if any two of the operators are bounded (compact), so is the third one. In view of this, $I_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ is bounded (compact) if both M_g and V_g are bounded (compact). By Theorem 1.1 and Lemma 2.3, this happens if and only if g is a constant function. This obviously gives the sufficiency part of the conditions in the theorem. We proceed to show that it is also necessary. First from Lemma 2.1, Cauchy–Schawarz inequality and (2.1), observe that for each $z \in D(w, \delta)$ and a small positive δ;

$$|K_{(w,m)}(z)| \simeq e^{\psi_m(z)+\psi_m(w)}.$$

Now assuming that $I_g : \mathcal{F}^p_{\psi_m} \to \mathcal{F}^q_{\psi_m}$ is bounded and $0 < p \leq q < \infty$. Then, applying (2.5), (2.2) we have

$$\begin{aligned}
e^{q\psi_m(w)} &\gtrsim \int_{\mathbb{C}} |I_g K_{(w,m)}(z)|^q e^{-q\psi_m(z)} dA(z)\\
&\simeq \int_{\mathbb{C}} \frac{|K'_{(w,m)}(z)|^q |g(z)|^q (1+|z|)^q}{\big(1+|z|+\big||z|+|z^2|-m\big|\big)^q} e^{-q\psi_m(z)} dA(z)\\
&\geq \int_{D(w,\delta)} \frac{|K'_{(w,m)}(z)|^q |g(z)|^q (1+|z|)^q}{\big(1+|z|+\big||z|+|z^2|-m\big|\big)^q} e^{-q\psi_m(z)} dA(z)\\
&\gtrsim \int_{D(w,\delta)} \frac{e^{q\psi_m(w)} |\psi'_m(z)|^q |g(z)|^q (1+|z|)^q}{\big(1+|z|+\big||z|+|z^2|-m\big|\big)^q} dA(z) = S_1.
\end{aligned}$$

On the other hand $\psi'_m(z) \simeq \psi_m(w)$ for each $z \in D(w, \delta)$. Applying this, (3.1), and the subharmonicity of $|g|^q$, we estimate S_1 from below as

$$S_1 \gtrsim \frac{e^{q\psi_m(w)} |\psi'_m(w)|^q |g(w)|^q (1+|w|)^q}{\big(1+|w|+\big||w|+|w^2|-m\big|\big)^q} \simeq \frac{e^{q\psi(w)} \Big(|w|+|w|^2-m\Big)^q |g(w)|^q}{\big(1+|w|+\big||w|+|w^2|-m\big|\big)^q},$$

from which and taking further simplifications, we infer

$$|g(w)| \lesssim \frac{1+|w|}{|w|+|w|^2-m} + 1,$$

and hence g is a bounded analytic function. By Liouville's classical theorem, g turns out to be a constant function.

3.2. Proof of Theorem 1.3

Recall that $\lambda \in \mathbb{C}$ belongs to the spectrum $\sigma(T)$ of a bounded operator T on a Banach space if $\lambda I - T$ fails to be invertible, where I is the identity operator on the space. The point spectrum $\sigma_p(T)$ of T consists of its eigenvalues. We now turn to the spectrum of V_g in particular, and assume that V_g is bounded on $\mathcal{F}^p_{\psi_m}$ and hence $g(z) = az^2 + bz + c, \quad a, b, c \in \mathbb{C}$. By linearity of integrals we may first make a splitting $\lambda I - V_g = (\lambda I - V_{g_1}) - V_{g_2}$ where $g_1(z) = az^2$ and $g_2(z) = bz + c$. A simple analysis shows that $\lambda I - V_g$ and $\lambda I - V_{g_1}$ are injective maps. On the other hand, by part (i) of the result, V_{g_2} is compact and hence $\sigma(V_{g_2}) = \{0\}$. Thus, we shall investigate the case with V_{g_1}. We may first observe that if $\lambda \neq 0$, then the equation $\lambda f - V_g f = h$ has the unique analytic solution

(3.3)
$$f(z) = (\lambda I - V_{g_1})^{-1} h(z) = \frac{1}{\lambda} h(0) e^{\frac{g_1(z)}{\lambda}} + \frac{1}{\lambda} e^{\frac{g_1(z)}{\lambda}} \int_0^z e^{-\frac{g_1(w)}{\lambda}} h'(w) dA(w),$$

where I is the identity operator. This can easily be seen by solving an initial valued first order linear ordinary differential equation

$$\lambda y' - g_1' y = h', \quad \lambda f(0) = h(0).$$

Recall that $(\lambda I - V_{g_1})^{-1} h(z) = R_{(g_1,\lambda)} h(z)$ is the Resolvent operator of V_{g1} at λ. It follows that $\lambda \in \mathbb{C}$ belongs to the resolvent of V_{g_1} whenever $R_{(g_1,\lambda)}$ is a bounded operator. Since we assumed that V_{g_1} is bounded and as $\mathcal{F}^p_{\psi_m}$ contain the constants, setting $h = 1$ in (3.3) shows that $R_{(g_1,\lambda)} 1 = e^{g_1(z)/\lambda} \in \mathcal{F}^p_{\psi_m}$ for each λ in the resolvent set of V_{g_1}. From this, we obviously deduce

$$\sigma(V_{g_1}) \supseteq \{0\} \cup \overline{\{\lambda \in \mathbb{C} \setminus \{0\} : e^{g_1(z)/\lambda} \notin \mathcal{F}^p_{\psi_m}\}}.$$

On the other hand, if $|\lambda| > 2|a|$, then we set polar coordinates for $z = re^{i\theta}, \;\; a = |a|e^{i\theta_1}, \;\; \lambda = |\lambda| e^{i\theta_2}$, and estimate

$$\begin{aligned}
&\int_{\mathbb{C}} |R_{(g_1,\lambda)} 1(z)|^p e^{-p\psi_m(z)} dA(z) = \int_{\mathbb{C}} e^{p\Re\left(\frac{az^2}{\lambda}\right) - p\psi_m(z)} dA(z) \\
&= \int_0^\infty \int_0^{2\pi} e^{p\left(\frac{|a|}{|\lambda|}\cos(\theta+\theta_1-\theta_1) - \frac{1}{2}\right) r^2 + mp\log(1+r)} r d\theta dr \\
&\lesssim \int_0^\infty e^{p\left(\frac{|a|}{|\lambda|} - \frac{1}{2}\right) r^2 + (m+1)p\log(1+r)} dr \lesssim \int_0^\infty e^{p\left(\frac{|a|}{|\lambda|} - \frac{1}{2}\right) r^2 + p(m+1)r} dr \\
&\leq \sqrt{\frac{2\pi|\lambda|}{p(2|a| - |\lambda|)}} e^{\frac{2|\lambda|(pm+p)^2}{p(2|a|-|\lambda|)}} < \infty.
\end{aligned}$$

This means that the spectrum of V_{g_1} contains the closed disc $\overline{D(0, 2|a|)}$. We remain to show that $R_{(g_1,\lambda)}$ is bounded for all $\lambda \in \mathbb{C}$ such that $|\lambda| \leq 2|a|$. To this end, applying Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}
&\int_{\mathbb{C}} |R_{(g_1,\lambda)} f(z)|^p e^{-p\psi_m(z)} dA(z) \\
&\leq 2^p |f(0)|^p \int_{\mathbb{C}} |e^{\frac{g_1(z)}{\lambda}}|^p e^{-p\psi_m(z)} dA(z)
\end{aligned}$$

$$+ 2^p \int_{\mathbb{C}} \left| e^{-\frac{g_1(z)}{\lambda}} \int_0^z e^{-\frac{g_1(w)}{\lambda}} f'(w) dA(w) \right|^p e^{-p\psi_m(z)} dA(z)$$
$$\lesssim \|f\|_{(p,m)}^p + \int_{\mathbb{C}} \frac{|f'(z)|^p e^{-p\psi_(z)}}{(1+\psi_m'(z))} dA(z) \lesssim \|f\|_{(p,m)}^p,$$

and completes the proof of part (i).

The proof of part (ii) is rather straightforward. If λ belongs to the point spectrum of I_g, then there exists a nonzero function $f \in \mathcal{F}_{\psi_m}^p$ for which

$$\lambda f(z) = \int_0^z c f'(w) dA(w). \tag{3.4}$$

It follows from this that $\lambda f'(z) = cf'(z)$ which holds either $\lambda = c$ or $f' = 0$. The later leads to a contradiction because of the relation in (3.4). Thus, we must have $\lambda = c$. This implies

$$\{c\} \subseteq \sigma(I_g).$$

To show the converse inclusion, it suffices to show that the resolvent operator $R_{(\lambda,g)}$ of I_g at point λ is bounded on $\mathcal{F}_{\psi_m}^p$ for each $\lambda \neq c$. To this end, from the relation $\lambda f - I_g f = h$, it follows that

$$\lambda f' - cf' = h'.$$

Solving this linear ordinary differential equation gives the explicit expression for the resolvent operator

$$f(z) = R_\lambda h(z) = \frac{h(z)}{\lambda - c},$$

which obviously is bounded on $\mathcal{F}_{\psi_m}^p$ and completes the required proof.

3.3. The differential operator D

The differential operator $Df = f'$ has become a prototype example of unbounded operators in many Banach spaces. Its unboundedness in the classical Flock spaces with Gaussian weight and in weighted Fock spaces where the weight decays faster than the Gaussian weight was recently verified in [17]. Another natural question would be then what happens when the weight decays slower than the Gaussian weight in which the Fock–Sobolev spaces constitute typical examples. In what follows we will verify that the action of the operator remains unbounded. If $D : \mathcal{F}_{\psi_m}^p \to \mathcal{F}_{\psi_m}^q$ were indeed bounded, then applying D to the sequence of the reproducing kernels, using estimates (2.1) and (2.2), and subharmonicity of $|K'_{(w,m)}|^q$, we would find

$$e^{q\psi_m(w)} \gtrsim \|K'_{(w,m)}\|_{(p,m)}^q \|D\|^q \geq \int_{\mathbb{C}} |K'_{(w,m)}(z)|^q e^{-q\psi_m(z)} dA(z)$$
$$\geq \int_{D(w,1)} |K'_{(w,m)}(z)|^q e^{-q\psi_m(z)} dA(z)$$
$$\gtrsim |K'_{(w,m)}(w)|^q e^{-q\psi_m(w)} \simeq \psi'(w) e^{q\psi_m(w)}$$

and from this we conclude $||w| + |w|^2 - m| \lesssim 1 + |w|$, resulting the desired contradiction when $|w| \to \infty$.

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