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GRADED INTEGRAL DOMAINS AND PRÜFER-LIKE DOMAINS

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Abstract. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an integral domain graded by an arbitrary torsionless grading monoid Γ, $\bar{R}$ be the integral closure of R, H be the set of nonzero homogeneous elements of R, $C(f)$ be the fractional ideal of R generated by the homogeneous components of $f \in R_H$, and $N(H) = \{f \in R \mid C(f)_v = R\}$. Let R_H be a UFD. We say that a nonzero prime ideal Q of R is an *upper to zero* in R if $Q = fR_H \cap R$ for some $f \in R$ and that R is a *graded UMT-domain* if each upper to zero in R is a maximal t-ideal. In this paper, we study several ring-theoretic properties of graded UMT-domains. Among other things, we prove that if R has a unit of nonzero degree, then R is a graded UMT-domain if and only if every prime ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R, if and only if $\bar{R}_{H\setminus Q}$ is a graded-Prüfer domain for all homogeneous maximal t-ideals Q of R, if and only if $\bar{R}_{N(H)}$ is a Prüfer domain, if and only if R is a UMT-domain.

0. Introduction

Prüfer v-multiplication domains (PvMD) are one of the most important research topics in "Multiplicative Ideal Theory" because many essential non-Noetherian integral domains (e.g., Krull domains, Prüfer domains, GCD domains) are PvMDs and an integral domain D is a PvMD if and only if $D[X]$, the polynomial ring over D, is a PvMD. It is known that D is a PvMD if and only if D is an integrally closed UMT-domain; hence UMT-domains can be considered as non-integrally closed PvMDs. UMT-domains were introduced by Houston and Zafrullah [34] and studied in greater detail by Fontana, Gabelli, and Houston [26] and Chang and Fontana [17]. In this paper, we study UMT-domain properties of graded integral domains.

This section consists of three subsections. In Section 0.1, we review the definitions related to the t-operation and in Section 0.2, we review those of

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graded integral domains; so the reader who is familiar with these two notions can skip to Section 0.3 where we give the motivation and results of this paper.

0.1. The t-operation

Let D be an integral domain with quotient field K. An *overring* of D means a subring of K containing D. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. For $I \in \mathbf{F}(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \bigcup\{J_v \mid J \in \mathbf{F}(D)$ is finitely generated and $J \subseteq I\}$. An $I \in \mathbf{F}(D)$ is called a *t-ideal* (resp., *v-ideal*) if $I_t = I$ (resp., $I_v = I$). A t-ideal (resp., v-ideal) is a *maximal t-ideal* (resp., *maximal v-ideal*) if it is maximal among proper integral t-ideals (resp., v-ideals). Let t-Max(D) (resp., v-Max(D)) be the set of maximal t-ideals (resp., v-ideals) of D. It may happen that v-Max$(D) = \emptyset$ even though D is not a field as in the case of a rank-one nondiscrete valuation domain D. However, it is well known that t-Max$(D) \neq \emptyset$ if D is not a field; each maximal t-ideal is a prime ideal; each proper t-ideal is contained in a maximal t-ideal; each prime ideal minimal over a t-ideal is a t-ideal; and $D = \bigcap_{P \in t\text{-Max}(D)} D_P$. We mean by t-dim$(D) = 1$ that D is not a field and each prime t-ideal of D is a maximal t-ideal of D. Clearly, if dim$(D) = 1$ (i.e., D is one-dimensional), then t-dim$(D) = 1$.

An $I \in \mathbf{F}(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$, and D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. Let $T(D)$ (resp., $Prin(D)$) be the group of t-invertible fractional t-ideals (resp., nonzero principal fractional ideals) of D under the t-multiplication $I * J = (IJ)_t$. It is obvious that $Prin(D) \subseteq T(D)$. The *t-class group* of D is the abelian group $Cl(D) = T(D)/Prin(D)$. It is clear that if D is a Krull domain (resp., Prüfer domain), then $Cl(D)$ is the divisor class (resp., ideal class) group of D. Let $\{D_\alpha\}$ be a set of integral domains such that $D = \bigcap_\alpha D_\alpha$. We say that the intersection $D = \bigcap_\alpha D_\alpha$ is *locally finite* if each nonzero nonunit of D is a unit of D_α for all but a finite number of D_α.

Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D, $D[\{X_\alpha\}]$ be the polynomial ring over D, and $c_D(f)$ (simply $c(f)$) be the fractional ideal of D generated by the coefficients of a polynomial $f \in K[\{X_\alpha\}]$. It is known that if I is a nonzero fractional ideal of D, then $(ID[\{X_\alpha\}])^{-1} = I^{-1}D[\{X_\alpha\}]$, $(ID[\{X_\alpha\}])_v = I_vD[\{X_\alpha\}]$, and $(ID[\{X_\alpha\}])_t = I_tD[\{X_\alpha\}]$ [32, Lemma 4.1 and Proposition 4.3]; so I is a (prime) t-ideal of D if and only if $ID[\{X_\alpha\}]$ is a (prime) t-ideal of $D[\{X_\alpha\}]$.

0.2. Graded integral domains

Let Γ be a (nonzero) torsionless grading monoid, that is, Γ is a torsionless commutative cancellative monoid (written additively), and $\langle\Gamma\rangle = \{a - b \mid a, b \in \Gamma\}$ be the quotient group of Γ; so $\langle\Gamma\rangle$ is a torsionfree abelian group. It is well known that a cancellative monoid Γ is torsionless if and only if Γ can be given a total order compatible with the monoid operation [39, page 123]. By a ($\Gamma-$)

graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, we mean an integral domain graded by Γ. That is, each nonzero $x \in R_\alpha$ has degree α, i.e., $\deg(x) = \alpha$, and $\deg(0) = 0$. Thus, each nonzero $f \in R$ can be written uniquely as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$ with $\deg(x_{\alpha_i}) = \alpha_i$ and $\alpha_1 < \cdots < \alpha_n$. Since R is an integral domain, we may assume that $R_\alpha \neq \{0\}$ for all $\alpha \in \Gamma$.

A nonzero $x \in R_\alpha$ for every $\alpha \in \Gamma$ is said to be *homogeneous*. Let H be the saturated multiplicative set of nonzero homogeneous elements of R, i.e., $H = \bigcup_{\alpha \in \Gamma}(R_\alpha \setminus \{0\})$. Then R_H, called the *homogeneous quotient field* of R, is a graded integral domain whose nonzero homogeneous elements are units. Hence, R_H is a completely integrally closed GCD-domain [1, Proposition 2.1] and R_H is a $\langle\Gamma\rangle$-graded integral domain. We say that an overring T of R is a *homogeneous overring* of R if $T = \bigoplus_{\alpha \in \langle\Gamma\rangle}(T \cap (R_H)_\alpha)$; so T is a $\langle\Gamma\rangle$-graded integral domain such that $R \subseteq T \subseteq R_H$. Clearly, if $\Lambda = \{\alpha \in \langle\Gamma\rangle \mid T \cap (R_H)_\alpha \neq \{0\}\}$, then Λ is a torsionless grading monoid such that $\Gamma \subseteq \Lambda \subseteq \langle\Gamma\rangle$ and $T = \bigoplus_{\alpha \in \Lambda}(T \cap (R_H)_\alpha)$. The integral closure of R is a homogeneous overring of R by Lemma 1.6. Also, R_S is a homogeneous overring of R for a multiplicative set S of nonzero homogeneous elements of R (with $\deg(\frac{a}{b}) = \deg(a) - \deg(b)$ for $a \in H$ and $b \in S$).

For a fractional ideal A of R with $A \subseteq R_H$, let A^* be the fractional ideal of R generated by homogeneous elements in A. It is easy to see that $A^* \subseteq A$; and if A is a prime ideal, then A^* is a prime ideal. The A is said to be *homogeneous* if $A^* = A$. A homogeneous ideal (resp., homogeneous t-ideal) of R is called a *homogeneous maximal ideal* (resp., *homogeneous maximal t-ideal*) if it is maximal among proper homogeneous ideals (resp., homogeneous t-ideals) of R. It is known that a homogeneous maximal ideal need not be a maximal ideal, while a homogeneous maximal t-ideal is a maximal t-ideal [8, Lemma 2.1]. Also, it is easy to see that each proper homogeneous ideal (resp., homogeneous t-ideal) of R is contained in a homogeneous maximal ideal (resp., homogeneous maximal t-ideal) of R.

For $f \in R_H$, let $C_R(f)$ denote the fractional ideal of R generated by the homogeneous components of f. For a fractional ideal I of R with $I \subseteq R_H$, let $C_R(I) = \sum_{f \in I} C_R(f)$. It is clear that both $C_R(f)$ and $C_R(I)$ are homogeneous fractional ideals of R. If there is no confusion, we write $C(f)$ and $C(I)$ instead of $C_R(f)$ and $C_R(I)$. Let $N(H) = \{f \in R \mid C(f)_v = R\}$ and $S(H) = \{f \in R \mid C(f) = R\}$. It is well known that if $f, g \in R_H$, then $C(f)^{n+1}C(g) = C(f)^n C(fg)$ for some integer $n \geq 1$ [39]; so $N(H)$ and $S(H)$ are saturated multiplicative subsets of R and $S(H) \subseteq N(H)$. Let Ω be the set of maximal t-ideals Q of R with $Q \cap H \neq \emptyset$, i.e., $\Omega = \{Q \in t\text{-Max}(R) \mid Q \text{ is homogeneous}\}$ [8, Lemma 2.1]. As in [9], we say that R satisfies property (#) if $C(I)_t = R$ implies $I \cap N(H) \neq \emptyset$ for all nonzero ideals I of R; equivalently, $\text{Max}(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$ [9, Proposition 1.4]. It is known that R satisfies property (#) if R is one of the following integral domains: (i) R contains a unit of nonzero degree, (ii) $R = D[\Gamma]$ is the monoid domain of Γ over an integral domain D, (iii)

R contains a homogeneous prime element of nonzero degree, (iv) $R = D[\{X_\alpha\}]$ is the polynomial ring over D, or (v) the intersection $\bigcap_{Q\in\Omega} R_Q$ is locally finite [9, Example 1.6 and Lemma 2.2].

We say that R is a *graded-Prüfer domain* if each nonzero finitely generated homogeneous ideal of R is invertible. Clearly, invertible ideals are t-invertible, and hence a graded-Prüfer domain is a PvMD [1, Theorem 6.4] but need not be a Prüfer domain [9, Example 3.6]. The reader can refer to [10] or [42] for more on graded-Prüfer domains.

0.3. Motivation and results

Let X be an indeterminate over D and $D[X]$ be the polynomial ring over D. A nonzero prime ideal Q of $D[X]$ is called an *upper to zero* in $D[X]$ if $Q \cap D = (0)$. We say that D is a *UMT-domain* if each upper to zero in $D[X]$ is a maximal t-ideal of $D[X]$. (UMT stands for Upper to zero is a Maximal T-ideal.) A *quasi-Prüfer domain* is a UMT-domain in which every maximal ideal is a t-ideal; equivalently, its integral closure is a Prüfer domain [25, Chapter VI]. The most important properties of UMT-domains are that (i) D is a UMT-domain if and only if every prime ideal of $D[X]_{N_v}$, where $N_v = \{f \in D[X] \mid c(f)_v = D\}$, is extended from D and (ii) D is an integrally closed UMT-domain if and only if D is a PvMD [34, Theorem 3.1 and Proposition 3.2]. A subring $D[X^2, X^3] = D + X^2D[X]$ of $D[X]$ over a PvMD D is an easy example of a non-integrally closed UMT-domain. In many cases, UMT-domains are used like: $D[X]$ (or $D[X]_{N_v}$) has a ring-theoretic property (P) if and only if D is a UMT-domain with property (P). For example, t-$\dim(D[X]) = 1$ if and only if D is a UMT-domain with t-$\dim(D) = 1$; and $D[X]_{N_v}$ is a pseudo-valuation domain if and only if D is a pseudo-valuation UMT-domain [13, Lemma 3.7]. (A quasi-local domain D with maximal ideal M is a *pseudo-valuation domain* if and only if D has a unique valuation overring with maximal ideal M [31, Theorem 2.7].) For more results on UMT-domains, see, for example, [22, 23, 41, 44] including a survey article [33].

Clearly, Q is an upper to zero in $D[X]$ if and only if $Q = fK[X] \cap D[X]$ for some prime element $0 \neq f \in K[X]$, if and only if either $Q = XD[X]$ or $Q = fK[X, X^{-1}] \cap D[X]$ for some prime element $0 \neq f \in K[X]$. Note that $D[X] = \bigoplus_{n\geq 0} DX^n$ is an $\mathbb{N}_0$-graded integral domain, where $\mathbb{N}_0$ is the additive monoid of nonnegative integers, and if H is the set of nonzero homogeneous elements of $D[X]$, then $D[X]_H = K[X, X^{-1}]$ and $K[X, X^{-1}]$ is a unique factorization domain (UFD). In [19, Section 2], the notion of "upper to zero" was generalized to graded integral domains as follows: *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a* (*nontrivial*) *graded integral domain graded by an arbitrary torsionless grading monoid* Γ *and H be the set of nonzero homogeneous elements of R. Assume that R_H is a UFD. Then a nonzero prime ideal Q of R is called an upper to zero in R if $Q = fR_H \cap R$ for some $f \in R_H$.* Thus, Q is an upper to zero in $D[X]$ as the original definition if and only if either $Q = XD[X]$ or Q is an upper to zero in

$D[X]$ as a prime ideal of the $\mathbb{N}_0$-graded integral domain $D[X] = \bigoplus_{n\geq 0} DX^n$. As a graded integral domain analog, in [19, Theorem 2.5], it was shown that if $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a graded integral domain with a unit of nonzero degree such that R_H is a UFD, then R is a PvMD if and only if R is integrally closed and each upper to zero in R is a maximal t-ideal. In this paper, we further study some ring-theoretic properties of graded integral domains R such that R_H is a UFD and each upper to zero in R is a maximal t-ideal.

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a Γ-graded integral domain. In Section 1, we introduce the notion of graded UMT-domains, and we then study general properties of both UMT-domains and graded UMT-domains. For example, we prove that UMT-domains are graded UMT-domains, and R is a graded UMT-domain if and only if Q is homogeneous for all nonzero prime ideals Q of R with $C(Q)_t \subsetneq R$, if and only if $C(Q)_t = R$ for every upper to zero Q in R. In Section 2, we show that if R satisfies property (#), then R is a graded UMT-domain if and only if every prime ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R, and R is a weakly Krull domain if and only if $R_{N(H)}$ is a weakly Krull domain. We study in Section 3 graded UMT-domains with a unit of nonzero degree. Among other things, we prove that if R has a unit of nonzero degree, then R is a graded UMT-domain if and only if R is a UMT-domain, if and only if the integral closure of $R_{H\setminus Q}$ is a graded-Prüfer domain for all homogeneous maximal t-ideals Q of R, if and only if the integral closure of $R_{N(H)}$ is a Prüfer domain. Finally, in Section 4, we use the $D + XK[X]$ construction to give several counterexamples of the results in Sections 2 and 3. Assume that $D \subsetneq K$, and let $R = D + XK[X] (:= \{f \in K[X] \mid f(0) \in D\})$. Then R is an $\mathbb{N}_0$-graded integral domain such that $R_H = K[X, X^{-1}]$ is a UFD. We show that R is a graded UMT-domain, and R is a UMT-domain if and only if D is a UMT-domain. Thus, if D is not a UMT-domain, then $R = D + XK[X]$ is a graded UMT-domain but not a UMT-domain. We also give examples which show that the results of Section 3 do not hold without assuming that R has a unit of nonzero degree.

1. UMT-domains and graded UMT-domains

Let Γ be a nonzero torsionless grading monoid, $\langle\Gamma\rangle = \{a - b \mid a, b \in \Gamma\}$ be the quotient group of Γ, $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a nontrivial Γ-graded integral domain, and H be the set of nonzero homogeneous elements of R. Throughout this paper, R_H is always assumed to be a UFD.

We begin this section with examples of graded integral domains R such that R_H is a UFD.

Example 1.1. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then R_H is a UFD if one of the following conditions is satisfied.

(1) [7, Proposition 3.5] $\langle\Gamma\rangle$ satisfies the ascending chain condition on its cyclic subgroups.
(2) $R = D[\{X_\alpha\}]$ is the polynomial ring over an integral domain D.

(3) [38, Section A.I.4.] $\langle\Gamma\rangle = \mathbb{Z}$ is the additive group of integers.
(4) $R = D[\Gamma]$ is the monoid domain of Γ over D such that $\langle\Gamma\rangle$ satisfies the ascending chain condition on its cyclic subgroups.

Let $\bar{D}$ be the integral closure of an integral domain D. For easy reference, we recall from [37, Theorem 44] that (i) (Lying Over) if P is a prime ideal of D, then there is a prime ideal Q of $\bar{D}$ with $Q \cap D = P$; (ii) (Going Up) if $P_1 \subseteq P_2$ are prime ideals of D and Q_1 is a prime ideal of $\bar{D}$ with $Q_1 \cap D = P_1$, then there exists a prime ideal Q_2 of $\bar{D}$ such that $Q_1 \subseteq Q_2$ and $Q_2 \cap D = P_2$; and (iii) (Incomparable) if $P \subseteq Q$ are prime ideals of $\bar{D}$ with $P \cap D = Q \cap D$, then $P = Q$.

The next result appears in [26, Theorem 1.5], but we include it because our proof is easy and direct without using other results.

Theorem 1.2. *An integral domain D is a UMT-domain if and only if the integral closure of D_P is a Prüfer domain for all $P \in t$-Max(D).*

Proof. Let $\bar{D}$ be the integral closure of D. Hence, $\bar{D}_P$ is the integral closure of D_P for a prime ideal P of D.

($\Rightarrow$) Assume that $\bar{D}_P$ is not a Prüfer domain for some $P \in t$-Max(D), and let $T = \bar{D}_P$. Then there are some $0 \neq a, b \in T$ such that $(a, b)T$ is not invertible, and so if we let $f = a + bX$, then $fK[X] \cap T[X] = fc_T(f)^{-1}[X] \subseteq (c_T(f)c_T(f)^{-1})[X] \subseteq M[X]$ for some maximal ideal M of T (the first equality follows from [28, Corollary 34.9] because T is integrally closed). Thus, $fK[X] \cap D[X] = (fK[X] \cap T[X]) \cap D[X] \subseteq (M[X] \cap D_P[X]) \cap D[X] = P[X]$. Clearly, $fK[X] \cap D[X]$ is an upper to zero in $D[X]$, but $fK[X] \cap D[X]$ is not a maximal t-ideal, a contradiction.

($\Leftarrow$) Assume that D is not a UMT-domain. Then there are a maximal t-ideal P of D and an upper to zero Q in $D[X]$ such that $Q \subseteq P[X]$ (cf. [34, Proposition 1.1]). Since Q is an upper to zero in $D[X]$, there is an $f \in D[X]$ such that $Q = fK[X] \cap D[X]$. Note that $Q_f := fK[X] \cap \bar{D}_P$ is an upper to zero in $\bar{D}_P[X]$, $Q_f \cap D_P[X] = Q_{D \setminus P}$, and $\bar{D}_P[X]$ is integral over $D_P[X]$. Thus, there is a prime ideal M of $\bar{D}_P[X]$ such that $Q_f \subseteq M$ and $M \cap D_P[X] = PD_P[X]$. Clearly, $M = (M \cap \bar{D}_P)[X]$ because $(M \cap \bar{D}_P)[X] \cap D_P[X] = PD_P[X]$ and $(M \cap \bar{D}_P)[X] \subseteq M$. However, since $\bar{D}_P$ is a Prüfer domain, there is a $g \in Q_f$ such that $\bar{D}_P = c(g)\bar{D}_P \subseteq M \cap \bar{D}_P$, a contradiction. ☐

Bezout domains are Prüfer domains. Hence, if $\bar{D}_P$ is a Bezout domain for all $P \in t$-Max(D), then D is a UMT-domain by Theorem 1.2. In [13, Lemma 2.2], it was shown that D is a UMT-domain if and only if the integral closure of D_P is a Bezout domain for all $P \in t$-Max(D). Theorem 1.2 also shows that D_S is a UMT-domain for every multiplicative set S of a UMT-domain D.

Corollary 1.3 ([34, Proposition 3.2]). *D is a PvMD if and only if D is an integrally closed UMT-domain.*

Proof. It is well known that D is a PvMD if and only if D_P is a valuation domain for all $P \in t\text{-Max}(D)$ [30, Theorem 5] and $D = \bigcap_{P\in t\text{-Max}(D)} D_P$. Hence, the result follows directly from Theorem 1.2. □

Recall that D is an *S-domain* if $\text{ht}(PD[X]) = 1$ for every prime ideal P of D with $\text{ht}P = 1$ [37, p. 26]. It is easy to see that a UMT-domain is an S-domain; and if $t\text{-dim}(D) = 1$ (e.g., $\dim(D) = 1$), then D is an S-domain if and only if D is a UMT-domain (cf. [43, Theorem 8]). However, S-domains need not be UMT-domains. For example, if $D = \mathbb{R} + (X,Y)\mathbb{C}[\![X,Y]\!]$, where $\mathbb{C}[\![X,Y]\!]$ is the power series ring over the field $\mathbb{C}$ of complex numbers and $\mathbb{R}$ is the field of real numbers, then D is a 2-dimensional Noetherian domain [12, Theorem 4 and Corollary 9] whose maximal ideal is a t-ideal. Hence, D is an S-domain [37, Theorem 148] but not a UMT-domain [34, Theorem 3.7].

We next introduce the notion of graded UMT-domains.

Definition 1.4. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$, and assume that R_H is a UFD.

(1) A nonzero prime ideal Q of R is an *upper to zero in* R if $Q = fR_H \cap R$ for some $f \in R_H$. (In this case, f is a nonzero prime element of R_H and Q is a height-one prime t-ideal of R.)
(2) R is a *graded UMT-domain* if every upper to zero in R is a maximal t-ideal of R.

Recall that if Q is a maximal t-ideal of $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ with $Q \cap H \neq \emptyset$, then Q is homogeneous [8, Lemma 2.1]. We use this result without further citation.

Lemma 1.5. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded UMT-domain and Q be a nonzero prime ideal of R. Then Q is a maximal t-ideal of R if and only if either Q is an upper to zero in R or Q is a homogeneous maximal t-ideal.*

Proof. Let Q be a maximal t-ideal of R. If $Q \cap H \neq \emptyset$, then Q is homogeneous. Next, assume that $Q \cap H = \emptyset$. Then $Q = Q_H \cap R$, and hence Q contains an upper to zero in R. Thus, Q must be an upper to zero in R because R is a graded UMT-domain. The converse is clear. □

We say that $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a *gr-valuation ring* if $x \in R$ or $\frac{1}{x} \in R$ for all nonzero homogeneous elements $x \in R_H$. It is known that if R is a gr-valuation ring, then there is a valuation overring V of R such that $V \cap R_H = R$ [35, Theorem 2.3]. Hence, a gr-valuation ring is integrally closed.

Lemma 1.6. *Let $\bar{R}$ be the integral closure of $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$. Then $\bar{R}$ is a homogeneous overring of R.*

Proof. Let $\{V_\lambda\}$ be the set of all homogeneous gr-valuation overrings of R. Then $\bar{R} = \bigcap_\lambda V_\lambda$ [35, Theorem 2.10], and since each V_λ is a homogeneous overring of R, $\bar{R}$ is also a homogeneous overring of R. □

We next show that a UMT-domain is a graded UMT-domain, while a graded UMT-domain need not be a UMT-domain (see Example 4.3).

Proposition 1.7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a UMT-domain. Then R is a graded UMT-domain.*

Proof. Let Q' be a prime t-ideal of R such that $Q' \cap H = \emptyset$. Then Q'_H is a t-ideal of R_H [26, Proposition 1.4], and hence $\mathrm{ht}Q' = \mathrm{ht}(Q'_H) = 1$ because R_H is a UFD.

Let $U_f = fR_H \cap R$ be an upper to zero in R. If U_f is not a maximal t-ideal of R, there is a maximal t-ideal Q of R such that $U_f \subsetneq Q$, By the above paragraph, $Q \cap H \neq \emptyset$, and thus Q is homogeneous. Note that $U = fR_H \cap \bar{R}$ is a prime ideal of $\bar{R}$ and $U \cap R = U_f$; so there is a prime ideal M of $\bar{R}$ such that $U \subsetneq M$ and $M \cap R = Q$. However, note that $\bar{R}$ is a graded integral domain by Lemma 1.6; so M^* is a prime ideal of $\bar{R}$ and $M^* \cap R = Q$. Hence, $M^* = M$, and since $U = fC_{\bar{R}}(f)^{-1}$ [9, Lemma 1.2(4)], $C_{\bar{R}}(f)C_{\bar{R}}(f)^{-1} \subseteq M$. By Theorem 1.2, $\bar{R}_M = (\bar{R}_Q)_{M_Q}$ is a valuation domain, and hence $\bar{R}_M = (C_{\bar{R}}(f)_M)(C_{\bar{R}}(f)_M)^{-1} = (C_{\bar{R}}(f)_M)((C_{\bar{R}}(f)^{-1})_M) \subseteq M_M$, a contradiction. Thus, U_f is a maximal t-ideal of R. □

Let $D[X]$ be the polynomial ring over an integral domain D, and let Q be an upper to zero in $D[X]$. It is known that Q is a maximal t-ideal if and only if $c(Q)_t = D$, if and only if Q is t-invertible [34, Theorem 1.4] (see [27, Theorem 3.3] for the case of arbitrary sets of indeterminates). This was extended to graded integral domains $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ in [8, Corollary 2.2(2)] as follows: If Q is an upper to zero in R, then $C(Q)_t = R$ if and only if Q is t-invertible, if and only if Q is a maximal t-ideal. We next generalize [8, Corollary 2.2(2)] to prime t-ideals Q of R with $Q \cap H = \emptyset$.

Proposition 1.8. *Let Q be a prime t-ideal of $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ such that $Q \cap H = \emptyset$. Then the following statements are equivalent.*

(1) $C(Q)_t = R$.
(2) *Q is t-invertible.*
(3) *Q is a maximal t-ideal.*

In this case, ht$Q = 1$, and hence Q is an upper to zero in R.

Proof. (1) ⇒ (2) Since $C(Q)_t = R$, there are some $f_1, \dots, f_k \in Q$ such that $(C(f_1) + \cdots + C(f_k))_v = R$. Assume that $\mathrm{ht}Q \geq 2$. Since R_H is a UFD, there is a $g \in Q$ such that gR_H is a prime ideal and $f_1 \notin gR_H$. Clearly, $((f_1, \dots, f_k, g)R_H)_v = R_H$, and hence if $u \in (f_1, \dots, f_k, g)^{-1}$, then $u \in R_H$. Also, since $(C(f_1) + \cdots + C(f_k))_v = R$, $u \in R$. Thus, $R = (f_1, \dots, f_k, g)^{-1} = (f_1, \dots, f_k, g)_v \subseteq Q_t = Q \subsetneq R$, a contradiction. Hence, $\mathrm{ht}Q = 1$, and so Q is an upper to zero in R. Thus, Q is t-invertible [8, Corollary 2.2(2)].

(2) ⇒ (3) [34, Theorem 1.4].

(3) ⇒ (1) Note that $Q \subsetneq C(Q)_t \subseteq R$ and $C(Q)_t$ is a t-ideal. Hence, if Q is a maximal t-ideal, then $C(Q)_t = R$. □

Corollary 1.9. *Each homogeneous prime t-ideal of $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ has height-one if and only if t-dim$(R) = 1$. In this case, R is a graded UMT-domain.*

Proof. Assume that each homogeneous prime t-ideal of R has height-one, and let Q be a maximal t-ideal of R. If $Q \cap H \neq \emptyset$, then Q is homogeneous, and thus ht$Q = 1$. Next, if $Q \cap H = \emptyset$, then $C(Q)_t = R$ because each homogeneous maximal t-ideal has height-one. Thus, ht$Q = 1$ by Proposition 1.8. The converse is clear.

The "In this case" part follows because t-dim$(R) = 1$ implies that each prime t-ideal of R is a maximal t-ideal. □

Let $A \subseteq B$ be an extension of integral domains. As in [23], we say that B is *t-linked* over A if $I^{-1} = A$ for a nonzero finitely generated ideal I of A implies $(IB)^{-1} = B$. It is easy to see that B is t-linked over A if and only if $B = \bigcap_{P \in t\text{-Max}(A)} B_P$ [14, Lemma 3.2], if and only if either $Q \cap A = (0)$ or $Q \cap A \neq (0)$ and $(Q \cap A)_t \subsetneq A$ for all $Q \in t$-Max(B) [4, Propositions 2.1].

Corollary 1.10. *Let T be a homogeneous overing of $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, and assume that T is t-linked over R (e.g., $T = R_S$ for some multiplicative set $S \subseteq H$). If R is a graded UMT-domain, then T is a graded UMT-domain.*

Proof. Let U be an upper to zero in T. If U is not a maximal t-ideal, then $C_T(U)_t \subsetneq T$ by Proposition 1.8. Hence, there is a homogeneous maximal t-ideal Q of T such that $U \subsetneq Q$. Note that $U \cap R$ is an upper to zero in R, $Q \cap R$ is homogeneous, $(Q \cap R)_t \subsetneq R$ because T is t-linked over R, and $U \cap R \subseteq Q \cap R$. Thus, $U \cap R \subsetneq (Q \cap R)_t$, a contradiction because $U \cap R$ is a maximal t-ideal by assumption. Hence, U is a maximal t-ideal of T. □

Following [3], we say that a multiplicative subset S of D is a *t-splitting set* if for each $0 \neq d \in D$, $dD = (AB)_t$ for some integral ideals A and B of D, where $A_t \cap sD = sA_t$ (equivalently, $(A, s)_t = D$) for all $s \in S$ and $B_t \cap S \neq \emptyset$. It is known that S is a t-splitting set of D if and only if $dD_S \cap D$ is t-invertible for all $0 \neq d \in D$ [3, Corollary 2.3]. Also, D is a UMT-domain if and only if $D - \{0\}$ is a t-splitting set in $D[X]$ [16, Corollary 2.9].

Theorem 1.11. *The following statements are equivalent for $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$.*

(1) *R is a graded UMT-domain.*
(2) *Let Q be a nonzero prime ideal of R such that $C(Q)_t \subsetneq R$. Then Q is homogeneous.*
(3) *Let Q be a nonzero prime ideal of R such that $Q \subsetneq M$ for some homogeneous maximal t-ideal M of R. Then Q is homogeneous.*
(4) *$C(Q)_t = R$ for every upper to zero Q in R.*
(5) *If $I = fR_H \cap R$ for $0 \neq f \in R$, then $C(I)_t = R$.*
(6) *H is a t-splitting set of R.*
(7) *Every prime t-ideal of R disjoint from H is t-invertible.*
(8) *Every prime t-ideal of R disjoint from H is a maximal t-ideal.*

Proof. (1) ⇒ (2) Suppose that Q is not homogeneous. Clearly, there is an $f \in Q \setminus H$ such that $C(f) \nsubseteq Q$. Let P be a prime ideal of R such that P is minimal over fR and $P \subseteq Q$. If $P \cap H \neq \emptyset$, then $PR_{H \setminus P}$ must be a homogeneous maximal t-ideal of $R_{H\setminus P}$ (cf. [8, Lemma 2.1]); so P is homogeneous. Hence, $C(f) \subseteq P \subseteq Q$, a contradiction. Thus, $P \cap H = \emptyset$ and PR_H is a prime t-ideal because PR_H is minimal over fR_H, whence P is an upper to zero in R. Thus, $P = Q$ by (1), and so $C(Q)_t = R$ by Proposition 1.8, a contradiction. Thus, Q is homogeneous.

(2) $\Leftrightarrow$ (3) Clear.

(2) $\Rightarrow$ (4) Let Q be an upper to zero in R. Then Q is not homogeneous and $Q \subsetneq C(Q)$. However, if $C(Q)_t \subsetneq R$, then Q is homogeneous by (2), a contradiction. Thus, $C(Q)_t = R$.

(4) $\Rightarrow$ (1) Proposition 1.8.

(1) $\Rightarrow$ (5) Let $f = f_1^{e_1} \cdots f_n^{e_n}$ be the prime factorization of f in R_H, where $f_i \in R_H$ is a prime element. Then

$$\begin{aligned} I &= (f_1^{e_1} \cdots f_n^{e_n})R_H \cap R \\ &= (f_1^{e_1} R_H \cap \cdots \cap f_n^{e_n} R_H) \cap R \\ &= (f_1^{e_1} R_H \cap R) \cap \cdots \cap (f_n^{e_n} R_H \cap R) \\ &= ((f_1 R_H \cap R)^{e_1})_t \cap \cdots \cap ((f_n R_H \cap R)^{e_n})_t. \end{aligned}$$

(For the last equality, note that each $f_i R_H \cap R$ is a maximal t-ideal by (1) and $\sqrt{f_i^{e_i} R_H \cap R} = f_i R_H \cap R = \sqrt{((f_i R_H \cap R)^{e_i})_t}$; so $((f_i R_H \cap R)^{e_i})_t$ is primary. Clearly, $((f_i R_H \cap R)^{e_i})_t R_H = f_i^{e_i} R_H$, and thus $((f_i R_H \cap R)^{e_i})_t = f_i^{e_i} R_H \cap R$.) If $C(I)_t \subsetneq R$, then $I \subseteq C(I)_t \subseteq M$ for some homogeneous maximal t-ideal M of R. Since M is a prime ideal, $f_i R_H \cap R \subseteq M$ for some i, and hence $R = C(f_i R_H \cap R)_t \subseteq C(M)_t = M$ by the equivalence of (1) and (4) above, a contradiction. Thus, $C(I)_t = R$.

(5) $\Rightarrow$ (1) Let Q be an upper to zero in R. Then $Q = fR_H \cap R$ for some $f \in R$, and hence $C(Q)_t = R$ by (5). Thus, Q is a maximal t-ideal by Proposition 1.8.

(1) $\Rightarrow$ (6) Let Q be a prime t-ideal of R such that $Q \cap H = \emptyset$. Then Q_H is a prime ideal of R_H, and hence $fR_H \subseteq Q_H$ for some nonzero prime element f of R_H. Hence, $fR_H \cap R \subseteq Q$, and since $fR_H \cap R$ is a maximal t-ideal of R by (1), $Q = fR_H \cap R$ and $C(Q)_t = R$. Thus, H is a t-splitting set [8, Theorem 2.1].

(6) $\Rightarrow$ (4) Let Q be an upper to zero in R. Then Q is a prime t-ideal of R with $Q \cap H = \emptyset$, and thus $C(Q)_t = R$ [8, Theorem 2.1].

(6) $\Leftrightarrow$ (7) [8, Corollary 2.2].

(7) $\Leftrightarrow$ (8) Proposition 1.8. □

Let $D[X]$ be the polynomial ring over an integral domain D and $f \in D[X]$ be such that $c(f)_v = D$. If A is an ideal of $D[X]$ with $f \in A$, then A is t-invertible [34, Proposition 4.1] and $fD[X] = (Q_1^{e_1} \cdots Q_n^{e_n})_t$ for some uppers to zero Q_i in $D[X]$ and integers $e_i \geq 1$ [29, p. 144]. We end this section with an extension of these results to graded integral domains.

Proposition 1.12. *Let A be a nonzero ideal of $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ such that $C(A)_t = R$. If A contains a nonzero $f \in R$ with $C(f)_v = R$ (e.g., R satisfies* *property (#)), then* $A_t = (Q_1^{e_1} \cdots Q_n^{e_n})_t$ *for some* t*-invertible uppers to zero* Q_i *in* R *and integers* $e_i \geq 1$*. In particular,* A *is* t*-invertible.*

Proof. If $A_t = R$, then A is t-invertible; so assume that $A_t \subsetneq R$. Let Q be a maximal t-ideal of R with $A \subseteq Q$; then $f \in Q$. If $Q \cap H \neq \emptyset$, then Q is homogeneous, and hence $R = C(A)_t \subseteq Q_t = Q$, a contradiction. Hence, $Q \cap H = \emptyset$, and so Q contains an upper to zero U in R containing f. Clearly, $C(U)_t = R$; so by Proposition 1.8, U is a maximal t-ideal, and thus $Q = U$, i.e., Q is an upper to zero in R that is t-invertible. Hence, each prime t-ideal of R containing A is an upper to zero in R that is also t-invertible. Thus, $A_t = (Q_1^{e_1} \cdots Q_n^{e_n})_t$ for some uppers to zero Q_i in R and integers $e_i \geq 1$ (cf. the proof of [29, Theorem 1.3]) and A is t-invertible. □

Corollary 1.13. *Let* $f \in R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *be nonzero. If* $C(f)_v = R$*, then* $fR = (Q_1^{e_1} \cdots Q_n^{e_n})_t$ *for some uppers to zero* Q_i *in* R *and integers* $e_i \geq 1$*.*

Proof. Clearly, $C(fR)_t = R$ and $f \in fR$. Thus, the result is an immediate consequence of Proposition 1.12. □

A careful reading of the proof of Proposition 1.12 also shows:

Corollary 1.14. *Let* A *be a nonzero ideal of a graded UMT-domain* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *such that* $C(A)_t = R$*. Then* $A_t = (Q_1^{e_1} \cdots Q_n^{e_n})_t$ *for some uppers to zero* Q_i *in* R *and integers* $e_i \geq 1$*, and* A *is* t*-invertible.*

Let D be an integral domain, S be a t-splitting set of D, $\mathfrak{S} = \{A_1 \cdots A_n \mid A_i = d_i D_S \cap D$ for some $0 \neq d_i \in D\}$, and $D_{\mathfrak{S}} = \{x \in K \mid xA \subseteq D$ for some $A \in \mathfrak{S}\}$. Then $D_{\mathfrak{S}} = \bigcap\{D_P \mid P \in t\text{-Max}(D)$ and $P \cap S \neq \emptyset\}$ [3, Lemma 4.2 and Theorem 4.3]. The S is said to be *t-lcm* if $sD \cap dD$ is t-invertible for all $s \in S$ and $0 \neq d \in D$; and S is called a *t-complemented t-splitting set* if $D_{\mathfrak{S}} = D_T$ for some multiplicative set T of D and the saturation of T is the *t-complement of* S.

Corollary 1.15 (cf. [16, Proposition 3.7]). *Let* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *and* $N(H) = \{f \in R \mid C(f)_v = R\}$*. Then* $N(H)$ *is a* t*-lcm* t*-complemented* t*-splitting set of* R*.*

Proof. Let $0 \neq f \in R$ and $A = fR_{N(H)} \cap R$. For the t-splitting set property of $N(H)$, it suffices to show that A is t-invertible [3, Corollary 2.3]. Let Q be a maximal t-ideal of R. If $Q \cap N(H) = \emptyset$, then $A_Q = fR_Q$. Next, assume that $Q \cap N(H) \neq \emptyset$. Then $C(Q)_t = R$, and hence Q is an upper to zero in R and R_Q is a rank-one DVR by Proposition 1.8. Now, note that if Q' is an upper to zero in R containing A, then $f \in Q'_H$ and Q'_H is a height-one prime ideal of R_H; so there are only finitely many uppers to zero in R containing A, say $Q_1, \ldots, Q_n$. Hence, if $S = R \setminus \bigcup_{i=1}^n Q_i$, then R_S is a principal ideal domain, and thus $AR_S = gR_S$ for some $g \in A$. Let $I = (f, g)_v$. Then $IR_Q = fR_Q$ when $Q \cap N(H) = \emptyset$, and $IR_Q = gR_Q$ when $Q \cap N(H) \neq \emptyset$. Thus, $I = A$ [36, Proposition 2.8(3)]; so A is t-invertible [36, Corollary 2.7].

Next, note that every t-ideal of R intersecting $N(H)$ is t-invertible by Proposition 1.12. Thus, $N(H)$ is a t-lcm t-splitting set [16, Theorem 3.4]. Also, if $\mathfrak{S} = \{A_1 \cdots A_n \mid A_i = d_i R_{N(H)} \cap R \text{ for some } 0 \neq d_i \in R\}$, then $R_H \subseteq R_{\mathfrak{S}}$ because $aR_{N(H)} \cap R = aR$ for all $a \in H$. Hence, $R_{\mathfrak{S}}$ is t-linked over R_H [4, Proposition 2.3], and since R_H is a UFD, $R_{\mathfrak{S}} = (R_H)_T$ for some saturated multiplicative set T of R_H [24, Theorem 1.3]. Thus, if $N = T \cap R$, then $R_{\mathfrak{S}} = R_N$. □

An integral domain is called a *Mori domain* if it satisfies the ascending chain condition on its (integral) v-ideals. Clearly, Krull domains are Mori domains.

Corollary 1.16. *Let* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *and* $N(H) = \{f \in R \mid C(f)_v = R\}$. *Then R is a Mori domain* (*resp., UMT-domain*) *if and only if* $R_{N(H)}$ *is a Mori domain* (*resp., UMT-domain*).

Proof. By Corollary 1.15, $N(H)$ is a t-lcm t-complemented t-splitting set of R. Let N be the t-complement of $N(H)$; then $R_H \subseteq R_N$, and hence R_N is a UFD and $R = R_{N(H)} \cap R_N$. Thus, $R_{N(H)}$ is a Mori domain if and only if R is a Mori domain [40, Theorem 1]. The UMT-domain property follows directly from [16, Corollary 3.6] and Corollary 1.15. □

2. Graded integral domains with property (#)

Let Γ be a nonzero torsionless grading monoid, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a nontrivial Γ-graded integral domain, H be the set of nonzero homogeneous elements of R, and $N(H) = \{f \in R \mid C(f)_v = R\}$. Let Ω be the set of all homogeneous maximal t-ideals of R, i.e., $\Omega = \{Q \in t\text{-Max}(R) \mid Q \cap H \neq \emptyset\}$, and recall that R satisfies property (#) if and only if $\text{Max}(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$ [9, Proposition 1.4].

Lemma 2.1. *Let* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *be a graded integral domain with property* (#), *and let Q be an upper to zero in R.*

(1) *Q is a maximal t-ideal if and only if $C(g)_v = R$ for some $g \in Q$.*
(2) *If Q is a maximal t-ideal of R, then $Q = (f, g)_v$ for some $f, g \in R$.*

Proof. (1) Q is a maximal t-ideal if and only if $C(Q)_t = R$ by Proposition 1.8, if and only if $Q \cap N(H) \neq \emptyset$ by property (#).

(2) Since Q is an upper to zero in R, there is an $f \in R$ such that $Q = fR_H \cap R$. Also, there is a $g \in Q$ with $C(g)_v = R$ by (1). Clearly, $(f, g)_v \subseteq Q$. For the reverse containment, let $h \in Q$. Then $\alpha h \in fR$ for some $\alpha \in H$, and thus $h(\alpha, g) \subseteq (f, g)$. Hence, $h(\alpha, g)_v \subseteq (f, g)_v \subseteq Q$. If $\xi \in (\alpha, g)^{-1}$, then $\alpha \in H$ implies $\xi \in R_H$, and since $C(g)_v = R$, $\xi g \in R$ implies $\xi \in R$. Hence, $(\alpha, g)^{-1} = R$, and thus $h \in hR = h(\alpha, g)_v \subseteq (f, g)_v$. Thus, $Q \subseteq (f, g)_v$. □

We next give a characterization of graded UMT-domains $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ with property (#).

Theorem 2.2. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with property (#). Then the following statements are equivalent.*

(1) *R is a graded UMT-domain.*
(2) *If Q is an upper to zero in R, then there is an $f \in Q$ such that $C(f)_v = R$.*
(3) *Every prime ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R.*
(4) *$N(H)$ is a t-lcm t-complemented t-splitting set of R with t-complement H.*

Proof. (1) ⇔ (2) This follows directly from Lemma 2.1.

(1) ⇒ (3) Let Q' be a nonzero prime ideal of $R_{N(H)}$. Then $Q' = Q_{N(H)}$ for some prime ideal Q of R. Note that $Q \subseteq M$ for some homogeneous maximal t-ideal M of R because R satisfies property (#). Thus, Q is homogeneous by Theorem 1.11.

(3) ⇒ (1) Let Q be an upper to zero in R, and assume that Q is not a maximal t-ideal of R. Then $Q \cap N(H) = \emptyset$ by Lemma 2.1(1), and so $Q_{N(H)}$ is a proper ideal of $R_{N(H)}$. Hence, by (3), there is a homogeneous ideal P of R such that $Q_{N(H)} = PR_{N(H)}$. Thus, $P \subseteq PR_{N(H)} \cap R = Q_{N(H)} \cap R = Q$, and so $Q_H = R_H$, a contradiction. Thus, Q is a maximal t-ideal of R.

(1) ⇒ (4) By Corollary 1.15, $N(H)$ is a t-lcm t-complemented t-splitting set of R. Also, note that $\{Q \in t\text{-Max}(R) \mid Q \cap N(H) \neq \emptyset\}$ is the set of uppers to zero in R by property (#) and assumption; so $R_H = R_{\mathfrak{S}}$, where $\mathfrak{S} = \{A_1 \cdots A_n \mid A_i = d_i R_{N(H)} \cap R \text{ for some } 0 \neq d_i \in R\}$. Thus, H is the t-complement of $N(H)$.

(4) ⇒ (1) Let Q be an upper to zero in R. Then $Q \cap H = \emptyset$, and hence $Q \cap N(H) \neq \emptyset$ [3, Theorem 4.3] because H is the t-complement of $N(H)$. Thus, Q is a maximal t-ideal of R by Proposition 1.8. □

The next result is an immediate consequence of Corollary 1.9, but we use Theorem 2.2 to give another proof.

Corollary 2.3. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with property (#). Then t-$\dim(R) = 1$ if and only if $\dim(R_{N(H)}) = 1$. In this case, R is a graded UMT-domain.*

Proof. Assume t-$\dim(R) = 1$, and note that $\text{Max}(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$. Thus, $\dim(R_{N(H)}) = 1$. Conversely, suppose $\dim(R_{N(H)}) = 1$, and let Q be a maximal t-ideal of R. If $Q \cap H \neq \emptyset$, then Q is homogeneous, and thus $\text{ht}Q = \text{ht}(Q_{N(H)}) = 1$. Next, if $Q \cap H = \emptyset$, then $Q_H \subsetneq R_H$, and hence Q contains an upper to zero Q_0 in R. However, note that since R satisfies property (#), $\dim(R_{N(H)}) = 1$ implies $(Q_0)_{N(H)} = R_{N(H)}$. Thus, $Q_0 \cap N(H) \neq \emptyset$, and so Q_0 is a maximal t-ideal by Lemma 2.1. Hence, $Q = Q_0$ and $\text{ht}Q = 1$.

For "In this case", note that $\dim(R_{N(H)}) = 1$ implies that every prime ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R. Thus, R is a graded UMT-domain by Theorem 2.2. □

An integral domain D is called an *almost Dedekind domain* (resp., t-almost Dedekind domain) if D_P is a rank-one DVR for all maximal ideals (resp., maximal t-ideals) P of D. Clearly, Dedekind domains are almost Dedekind domains; Krull domains are t-almost Dedekind domains; and if D is an almost (resp., a t-almost) Dedekind domain, then $\dim(D) = 1$ (resp., t-$\dim(D) = 1$).

Corollary 2.4 (cf. [20, Corollary 9])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with property (#). Then R is a t-almost Dedekind domain if and only if $R_{N(H)}$ is an almost Dedekind domain.*

Proof. ($\Rightarrow$) By Corollary 2.3, $\dim(R_{N(H)}) = 1$. Note that $\mathrm{Max}(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$ and R_Q is a rank-one DVR for all $Q \in \Omega$. Thus, $R_{N(H)}$ is an almost Dedekind domain.

($\Leftarrow$) If $R_{N(H)}$ is an almost Dedekind domain, then $\dim(R_{N(H)}) = 1$, and thus t-$\dim(R) = 1$ by Corollary 2.3. Let Q be a maximal t-ideal of R. If $Q \cap H = \emptyset$, then $\mathrm{ht}(Q_H) = \mathrm{ht}Q = 1$, and since R_H is a UFD, R_Q is a rank-one DVR. Next, if $Q \cap H \neq \emptyset$, then Q is homogeneous, and hence $Q_{N(H)} \subsetneq R_{N(H)}$. Thus, R_Q is a rank-one DVR by assumption. □

An integral domain D is called a *weakly Krull domain* if (i) $D = \bigcap_{P\in X^1(D)} D_P$, where $X^1(D)$ is the set of height-one prime ideals of D, and (ii) the intersection $D = \bigcap_{P\in X^1(D)} D_P$ is locally finite. It is easy to see that if D is a weakly Krull domain, then t-$\dim(D) = 1$, i.e., $X^1(D) = t$-$\mathrm{Max}(D)$, and D_S is a weakly Krull domain for a multiplicative set S of D. Also, D is a Krull domain if and only if D is a weakly Krull domain and D_P is a rank-one DVR for all $P \in X^1(D)$.

Corollary 2.5. *The following statements are equivalent for $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$.*

(1) *R is a weakly Krull domain.*
(2) *R is a graded UMT-domain and $R_{N(H)}$ is a weakly Krull domain.*
(3) *$R_{N(H)}$ is a weakly Krull domain.*
(4) *$R_{N(H)}$ is an one-dimensional weakly Krull domain.*

Proof. Note that $R_{N(H)}$ is a weakly Krull domain in this corollary. Also, $Q_{N(H)}$ is a prime t-ideal of $R_{N(H)}$ for all $Q \in \Omega$ [9, Proposition 1.3]. Hence, the intersection $\bigcap_{Q\in\Omega} R_Q$ is locally finite, and thus R satisfies property (#) [9, Lemma 2.2].

(1) $\Rightarrow$ (2) If R is a weakly Krull domain, then t-$\dim(R) = 1$, and hence R is a graded UMT-domain by Corollary 2.3. Also, since $N(H)$ is a multiplicative subset of R, $R_{N(H)}$ is a weakly Krull domain.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) If $R_{N(H)}$ is a weakly Krull domain, then $\mathrm{ht}(Q_{N(H)}) = 1$ for all $Q \in \Omega$. Thus, $\dim(R_{N(H)}) = 1$ because R satisfies property (#).

(4) $\Rightarrow$ (1) By Corollary 2.3, t-$\dim(R) = 1$, and thus $R = \bigcap_{Q\in X^1(R)} R_Q$. Next, let $f \in R$ be a nonzero nonunit. Since $R_{N(H)}$ is a weakly Krull domain, f is contained in only finitely many homogeneous maximal t-ideals of R. Also,

since R_H is a UFD, f is contained in only finitely many uppers to zero in R. Therefore, R is a weakly Krull domain. □

It is clear that D is a Krull domain if and only if D is a t-almost Dedekind weakly Krull domain and that a Krull domain D is a Dedekind domain if and only if $\dim(D) = 1$. Hence, by Corollaries 2.4 and 2.5, we have:

Corollary 2.6 ([9, Corollary 2.4])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$. Then R is a Krull domain if and only if $R_{N(H)}$ is a Dedekind domain.*

An integral domain D is a *weakly factorial domain* if each nonzero nonunit of D can be written as a finite product of primary elements of D. (A nonzero element $x \in D$ is said to be *primary* if xD is a primary ideal.) Since a prime ideal is a primary ideal, prime elements are primary, and thus UFDs are weakly factorial domains. It is known that D is a weakly factorial domain if and only if D is a weakly Krull domain and $Cl(D) = \{0\}$ [6, Theorem]. Note that X is a prime element of the polynomial ring $D[X]$; so $D[X]$ is a weakly factorial domain if and only if $D[X, X^{-1}]$ is a weakly factorial domain. Thus, the next result is a generalization of [5, Theorem 17] that D is a weakly factorial GCD-domain if and only if $D[X]$ is a weakly factorial domain.

Corollary 2.7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then the following statements are equivalent.*

1. *R is a weakly factorial domain.*
2. *R is a weakly factorial GCD-domain.*
3. *R is a weakly factorial PvMD.*

Proof. (1) ⇒ (2) If R is a weakly factorial domain, then R is a weakly Krull domain and $Cl(R) = \{0\}$. Hence, each upper to zero Q in R is t-invertible by Corollary 2.5 and Proposition 1.8, and so Q is principal. Thus, every upper to zero in R contains a (nonzero) prime element, and hence R is a GCD-domain [19, Theorem 2.2].

(2) ⇒ (3) ⇒ (1) Clear. □

3. Graded integral domains with a unit of nonzero degree

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an integral domain graded by a nonzero torsionless grading monoid Γ, H be the set of nonzero homogeneous elements of R, $N(H) = \{f \in R \mid C(f)_v = R\}$, and $\bar{R}$ be the integral closure of R. Note that $\bar{R}$ is a graded integral domain by Lemma 1.6 such that $R \subseteq \bar{R} \subseteq R_H = \bar{R}_H$. In this section, we study a graded UMT-domain property of R with a unit of nonzero degree.

Lemma 3.1. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, and let Q be a nonzero homogeneous prime ideal of R. If Q is not a t-ideal, then there is an upper to zero U in R such that $U \subseteq Q$.*

Proof. Since Q is not a t-ideal, there are some $a_0, a_1, \dots, a_n \in Q \cap H$ such that $(a_0, a_1, \dots, a_n)_v \nsubseteq Q$. Let

$$f = a_0 + a_1 x^{k_1} + \cdots + a_n x^{k_n},$$

where $x \in R$ is a unit of nonzero degree and $k_i \geq 1$ is an integer such that $C(f) = (a_0, a_1, \dots, a_n)$, and let $U \subseteq Q$ be a prime ideal of R minimal over fR. Then U is a t-ideal. We claim that U is an upper to zero in R.

Let $S = H \setminus Q$. Then Q_S is a unique homogeneous maximal ideal of R_S, and so $(C(f)R_S)_t = R_S$ because $(C(f)R_S)_t = (C(f)_t R_S)_t \nsubseteq Q_S$. Also, note that U_S is a t-ideal of R_S; hence if $a \in U \cap H (\neq \emptyset)$, then $R_S = ((a, f)R_S)_v \subseteq (U_S)_t = U_S$, a contradiction. Thus, $U \cap H = \emptyset$, and so U_H is a prime t-ideal because U_H is minimal over fR_H. Since R_H is a UFD, $U_H = gR_H$ for some $g \in R$. Thus, $U = U_H \cap R = gR_H \cap R$ is an upper to zero in R. □

Proposition 3.2. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded UMT-domain with a unit of nonzero degree, T be a homogeneous overring of R, and Q be a homogeneous prime t-ideal of R. If M is a homogeneous prime ideal of T such that $M \cap R = Q$, then M is a t-ideal of T.*

Proof. If M is not a t-ideal of T, then there is an upper to zero U in T such that $U \subseteq M$ by Lemma 3.1. Clearly, $U \cap R$ is an upper to zero in R and $U \cap R \subsetneq M \cap R = Q$. Thus, $U \cap R$ is not a maximal t-ideal of R, a contradiction. □

Corollary 3.3. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded UMT-domain with a unit of nonzero degree. If Q is a homogeneous prime t-ideal of R, then $R_{H \setminus Q}$ is a graded UMT-domain with a unique homogeneous maximal ideal that is a t-ideal.*

Proof. Clearly, $R_{H \setminus Q}$ is a homogeneous t-linked overring of R, and hence $R_{H \setminus Q}$ is a graded UMT-domain by Corollary 1.10. Also, $Q_{H \setminus Q}$ is a unique homogeneous maximal ideal of $R_{H \setminus Q}$, and by Proposition 3.2, $Q_{H \setminus Q}$ is a t-ideal. □

Lemma 3.4. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then R is a graded-Prüfer domain if and only if R_Q is a valuation domain for all homogeneous maximal ideals Q of R.*

Proof. This follows from the following two observations: (i) R is a (graded) PvMD if and only if R_Q is a valuation domain for all homogeneous maximal t-ideals Q of R [18, Lemma 2.7] and (ii) R is a graded-Prüfer domain if and only if R is a graded PvMD whose homogeneous maximal ideals are t-ideals. □

We next give the main result of this section which provides characterizations of graded UMT-domains with a unit of nonzero degree.

Theorem 3.5. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then the following statements are equivalent.*

(1) *R is a graded UMT-domain.*

(2) *If Q is an upper to zero in R, then there is an $f \in Q$ such that $C(f)_v = R$.*
(3) *Every prime ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R.*
(4) *$\bar{R}_{H\setminus Q}$ is a graded-Prüfer domain for all homogeneous maximal t-ideals Q of R.*
(5) *R is a UMT-domain.*
(6) *$\bar{R}_{N(H)}$ is a Prüfer domain.*
(7) *$R_{N(H)}$ is a UMT-domain.*
(8) *$R_{N(H)}$ is a quasi-Prüfer domain.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) Since R has a unit of nonzero degree, R satisfies property (#). Thus, the results follow directly from Theorem 2.2.

(1) $\Rightarrow$ (4) Let Q be a homogeneous maximal t-ideal of R. Replacing R and Q with $R_{H\setminus Q}$ and $Q_{H\setminus Q}$ respectively, by Corollary 3.3, we may assume that R has a unique homogeneous maximal ideal Q and Q is a t-ideal.

Assume to the contrary that $\bar{R}$ is not a graded-Prüfer domain. Then there are some $a_0, a_1, \ldots, a_k \in H$ such that $I = (a_0, a_1, \ldots, a_k)\bar{R}$ is not invertible. Let $f = a_0 + a_1x^{m_1} + \cdots + a_kx^{m_k}$, where $x \in R$ is a unit of nonzero degree and $m_i \geq 1$ is an integer such that $C_{\bar{R}}(f) = I$. Then $fR_H \cap \bar{R} = fC_{\bar{R}}(f)^{-1}$ [9, Lemma 1.2(4)], and since I is not invertible and $C_{\bar{R}}(f)C_{\bar{R}}(f)^{-1}$ is homogeneous, we have $U = fC_{\bar{R}}(f)^{-1} \subseteq C_{\bar{R}}(f)C_{\bar{R}}(f)^{-1} \subseteq M$ for some homogeneous maximal ideal M of $\bar{R}$. Note that R_H is a UFD; so $f = f_1^{e_1} \cdots f_n^{e_n}$ for some prime elements $f_i \in R_H$ and integers $e_i \geq 1$. Thus,

$$\begin{aligned} fR_H \cap \bar{R} &= ((f_1R_H)^{e_1} \cdots (f_nR_H)^{e_n}) \cap \bar{R} \\ &= ((f_1R_H)^{e_1} \cap \cdots \cap (f_nR_H)^{e_n}) \cap \bar{R} \\ &= ((f_1R_H)^{e_1} \cap \bar{R}) \cap \cdots \cap ((f_nR_H)^{e_n} \cap \bar{R}) \\ &\supseteq (f_1R_H \cap \bar{R})^{e_1} \cap \cdots \cap (f_nR_H \cap \bar{R})^{e_n}. \end{aligned}$$

Thus, $M \supsetneq f_iR_H \cap \bar{R}$ for some i, and so

$$Q = M \cap R \supsetneq (f_iR_H \cap \bar{R}) \cap R = f_iR_H \cap R,$$

which is contrary to the fact that Q is a t-ideal. Therefore, $\bar{R}$ is a graded-Prüfer domain.

(4) $\Rightarrow$ (1) Assume that R is not a graded UMT-domain, and let $Q_f = fR_H \cap R$ be an upper to zero in R such that $Q_f \subseteq Q$ for some homogeneous maximal t-ideal Q of R (cf. Theorem 1.11). Let $T = \bar{R}_{H\setminus Q}$. Then by (4), T is a graded-Prüfer domain, and hence $U_f = fR_H \cap T = fC_T(f)^{-1} \nsubseteq M_0$ for all homogeneous maximal ideals M_0 of T. Note that $U_f \cap R_{H\setminus Q} = (Q_f)_{H\setminus Q}$, $(Q_f)_{H\setminus Q} \subsetneq Q_{H\setminus Q}$, and T is integral over $R_{H\setminus Q}$. Thus, there is a prime ideal M of T such that $U_f \subseteq M$ and $M \cap R_{H\setminus Q} = Q_{H\setminus Q}$. Since Q is homogeneous, $M^* \cap R_{H\setminus Q} = Q_{H\setminus Q}$. Thus, $M = M^*$ is homogeneous, a contradiction.

(1) $\Rightarrow$ (5) Let Q be a maximal t-ideal of R. If $Q \cap H \neq \emptyset$, then Q is homogeneous, and thus $\bar{R}_{H\setminus Q}$ is a graded-Prüfer domain by the equivalence of

(1) and (4). Note that if M is a prime ideal of $\bar{R}_{H\setminus Q}$ such that $M \cap R_{H\setminus Q} = Q_{H\setminus Q}$, then M is homogeneous because Q is homogeneous; hence $(\bar{R}_{H\setminus Q})_M$ is a valuation domain by Lemma 3.4. Clearly, $\bar{R}_{R\setminus Q} = (\bar{R}_{H\setminus Q})_{R\setminus Q}$. Thus, $\bar{R}_{R\setminus Q}$ is a Prüfer domain. Next, assume $Q \cap H = \emptyset$. Then $Q = Q_H \cap R$, and so if ht$Q \geq 2$, then there is an $0 \neq f \in R$ such that $fR_H \subseteq Q_H$ is a prime ideal of R_H. Hence, $fR_H \cap R \subsetneq Q_H \cap R = Q$, a contradiction. Thus, ht$Q = 1$ and so $R_Q = (R_H)_{Q_H}$ is a rank-one DVR. Therefore, by Theorem 1.2, R is a UMT-domain.

(5) $\Rightarrow$ (6) Let M be a prime ideal of $\bar{R}$ such that $M_{N(H)}$ is a maximal ideal of $\bar{R}_{N(H)}$. Then $(M \cap R) \cap N(H) = \emptyset$, and hence $M \cap R$ is a homogeneous maximal t-ideal of R. Since R is a UMT-domain, $\bar{R}_{M\cap R}$ is a Prüfer domain by Theorem 1.2. Note that $\bar{R}_{M\cap R} \subseteq \bar{R}_M = (\bar{R}_{N(H)})_{M_{N(H)}}$; so $(\bar{R}_{N(H)})_{M_{N(H)}}$ is a valuation domain. Thus, $\bar{R}_{N(H)}$ is a Prüfer domain.

(6) $\Rightarrow$ (4) Let M be a homogeneous prime ideal of $\bar{R}$ such that $M_{H\setminus Q}$ is a homogeneous maximal ideal of $\bar{R}_{H\setminus Q}$. Then $M \cap R \subseteq Q$, and so $M \cap N(H) = \emptyset$. Thus, $M_{N(H)}$ is a proper prime ideal of $\bar{R}_{N(H)}$, and so $\bar{R}_M = (\bar{R}_{N(H)})_{M_{N(H)}}$ is a valuation domain. Thus, by Lemma 3.4, $\bar{R}_{H\setminus Q}$ is a graded-Prüfer domain.

(6) $\Leftrightarrow$ (8) [25, Corollary 6.5.14].

(7) $\Leftrightarrow$ (8) This follows because each maximal ideal of $R_{N(H)}$ is a t-ideal [9, Propositions 1.3 and 1.4]. $\square$

Corollary 3.6 ([19, Theorem 2.5])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then R is an integrally closed graded UMT-domain if and only if R is a PvMD.*

Proof. R is an integrally closed graded UMT-domain if and only if R is an integrally closed UMT-domain (by Theorem 3.5), if and only if R is a PvMD (by Corollary 1.3). $\square$

Corollary 3.7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then $\bar{R}$ is a graded-Prüfer domain if and only if R is a graded UMT-domain whose homogeneous maximal ideals are t-ideals.*

Proof. ($\Rightarrow$) Clearly, $\bar{R}_{H\setminus Q}$ is a graded-Prüfer domain for all homogeneous maximal t-ideals Q of R. Thus, by Theorem 3.5, R is a graded UMT-domain. Next, let $f \in R$ be nonzero such that fR_H is a prime ideal. Note that $fR_H \cap \bar{R} = fC_{\bar{R}}(f)^{-1}$ [9, Lemma 1.2(4)]; so if $h \in R_H$ with $C_{\bar{R}}(h) = C_{\bar{R}}(f)^{-1}$ (such h exists because R has a unit of nonzero degree), then $fh \in fC_{\bar{R}}(f)^{-1}$ and $C_{\bar{R}}(fh) = \bar{R}$. Thus, $C(fC_{\bar{R}}(f)^{-1}) = \bar{R}$. Note also that $fR_H \cap R = fC_{\bar{R}}(f)^{-1} \cap R$ and $\bar{R}$ is integral over R. Hence, $C(fR_H \cap R) = R$. Thus, by Lemma 3.1, each homogeneous maximal ideal of R is a t-ideal.

($\Leftarrow$) Let M be a homogeneous maximal ideal of $\bar{R}$. Then $M \cap R$ is a homogeneous ideal of R; so $(M \cap R) \cap N(H) = \emptyset$ by assumption. Hence, $M \cap N(H) = \emptyset$, and thus $\bar{R}_M = (\bar{R}_{N(H)})_{M_{N(H)}}$ is a valuation domain by Theorem 3.5. Thus, by Lemma 3.4, $\bar{R}$ is a graded-Prüfer domain. $\square$

It is well known that each overring of a Prüfer domain is a Prüfer domain [28, Theorem 26.1]. The next result is the graded-Prüfer domain analog.

Lemma 3.8 ([10, Theorem 2.5(2)]). *Let T be a homogeneous overring of a graded-Prüfer domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$. Then T is a graded-Prüfer domain.*

Proof. Let A be a nonzero finitely generated homogeneous ideal of T. Since $R \subseteq T \subseteq R_H$, there are an $\alpha \in H$ and a finitely generated homogeneous ideal I of R such that $A = \frac{1}{\alpha}IT$. Since R is a graded-Prüfer domain, I is invertible, and thus $A = \frac{1}{\alpha}IT$ is invertible. Hence, T is a graded-Prüfer domain. □

Let D be a UMT-domain, and recall that if P is a nonzero prime ideal of D with $P_t \subsetneq D$, then P is a t-ideal [26, Corollary 1.6]. We next give the graded UMT-domain analog.

Corollary 3.9. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded UMT-domain with a unit of nonzero degree, and let M be a homogeneous maximal t-ideal of R. If $P \subseteq M$ is a nonzero prime ideal of R, then P is a homogeneous prime t-ideal.*

Proof. Since M is homogeneous, $C(P)_t \subseteq M_t = M \subsetneq R$. Thus, P is homogeneous by Theorem 1.11. Next, note that $\bar{R}_{H\setminus M}$ is a graded-Prüfer domain and $\bar{R}_{H\setminus P}$ is a homogeneous overring of $\bar{R}_{H\setminus M}$; so by Lemma 3.8, $\bar{R}_{H\setminus P}$ is a graded-Prüfer domain. Thus, by Corollary 3.7, $PR_{H\setminus P}$ is a prime t-ideal, and hence P is a prime t-ideal of R. □

We next give another characterization of graded UMT-domains.

Corollary 3.10. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then the following statements are equivalent.*

(1) *R is a graded UMT-domain.*
(2) *Let Q be a nonzero prime ideal of R with $C(Q)_t \subsetneq R$. Then Q is a homogeneous prime t-ideal.*
(3) *Let Q be a nonzero prime ideal of R such that $Q \subseteq M$ for some homogeneous maximal t-ideal M of R. Then Q is a homogeneous prime t-ideal.*

Proof. (1) ⇒ (2) Let Q be a nonzero prime ideal of R with $C(Q)_t \subsetneq R$. Clearly, there is a homogeneous maximal t-ideal M of R such that $Q \subseteq M$. Hence, by Corollary 3.9, Q is a homogeneous prime t-ideal.

(2) ⇔ (3) Clear.

(3) ⇒ (1) This follows from Theorem 1.11. □

An integral domain D is called a *generalized Krull domain* if (i) $D = \bigcap_{P\in X^1(D)} D_P$, (ii) the intersection $D = \bigcap_{P\in X^1(D)} D_P$ is locally finite, and (iii) D_P is a (rank-one) valuation domain for all $P \in X^1(D)$. Clearly, D is a generalized Krull domain if and only if D is a weakly Krull domain and D_P is a valuation domain for all $P \in X^1(D)$, if and only if D is a weakly Krull PvMD; and a generalized Krull domain D is a Krull domain, if and only if D_P is a DVR for all $P \in X^1(D)$.

Corollary 3.11. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then the following statements are equivalent.*

(1) *R is an integrally closed weakly Krull domain.*
(2) *R is a generalized Krull domain.*
(3) *$R_{N(H)}$ is a generalized Krull domain.*
(4) *$R_{N(H)}$ is an one-dimensional generalized Krull domain*

Proof. (1) $\Rightarrow$ (2) It suffices to show that R_Q is a valuation domain for all $Q \in X^1(R)$. Let Q be a height-one prime ideal of R. If $Q \cap H = \emptyset$, then Q_H is a height-one prime ideal of R_H, and since R_H is a UFD, $R_Q = (R_H)_{Q_H}$ is a valuation domain. If $Q \cap H \neq \emptyset$, then Q is homogeneous, and hence $Q_{N(H)}$ is a proper prime ideal of $R_{N(H)}$. Note that R is a graded UMT-domain by Corollary 2.5 and $R_{N(H)}$ is integrally closed; hence $R_{N(H)}$ is a Prüfer domain by Theorem 3.5. Thus, $R_Q = (R_{N(H)})_{Q_{N(H)}}$ is a valuation domain.

(2) $\Rightarrow$ (3) [28, Corollary 43.6].

(3) $\Rightarrow$ (4) $\Rightarrow$ (1) This follows from Corollary 2.5 because $R = R_H \cap R_{N(H)}$, R_H is integrally closed, and a generalized Krull domain is a weakly Krull domain. □

Let $\bar{D}$ be the integral closure of an integral domain D, $\{X_\alpha\}$ be a nonempty set of indeterminates over D, and $N_v = \{f \in D[\{X_\alpha\}] \mid c(f)_v = D\}$. It is known that D is a UMT-domain if and only if $D[\{X_\alpha\}]$ is a UMT-domain, if and only if $D[\{X_\alpha\}]_{N_v}$ is a UMT-domain, if and only if $\bar{D}[\{X_\alpha\}]_{N_v}$ is a Prüfer domain [26, Theorems 2.4 and 2.5], if and only if every prime ideal of $D[\{X_\alpha\}]_{N_v}$ is extended from D (cf. [34, Theorem 3.1]). We next recover this result as a corollary of Theorem 3.5, and for this we first need a simple lemma.

Lemma 3.12. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a set $\{p_\beta\}$ of nonzero homogeneous prime elements such that* (i) *$ht(p_\beta R) = 1$ for each β and* (ii) *$\bigcap_{n=1}^{\infty} p_{\beta_n} R = (0)$ for any sequence $\{p_{\beta_n}\}$ of nonassociate members of $\{p_\beta\}$, and let S be the saturated multiplicative set of R generated by $\{p_\beta\}$.*

(1) *R_S is a homogeneous overring of R.*
(2) *R is a graded UMT-domain if and only if R_S is a graded UMT-domain.*
(3) *R is a UMT-domain if and only if R_S is a UMT-domain.*

Proof. (1) Clear.

(2) It is clear that each upper to zero in R is not comparable with $p_\beta R$ under inclusion for all β. Also, Q is an upper to zero in R if and only if Q_S is an upper to zero in R_S. Note that $t\text{-Max}(R_S) = \{Q_S \mid Q \in t\text{-Max}(R)$ and $Q \neq p_\beta R$ for all $\beta\}$ [2, Proposition 2.6 and Corollary 3.5]. Thus, each upper to zero in R is a maximal t-ideal if and only if each upper to zero in R_S is a maximal t-ideal.

(3) Clearly, $R_{p_\beta R}$ is a rank-one DVR for all β. Also, if Q is a prime ideal of R with $Q \cap S = \emptyset$, then $(\overline{R_S})_{Q_S} = \bar{R}_Q$. Thus, the result follows from Theorem 1.2 and [2, Proposition 2.6 and Corollary 3.5]. □

Corollary 3.13. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a nonzero homogeneous prime element p such that $ht(pR) = 1$ and $\deg(p) \neq 0$. Then R is a graded UMT-domain if and only if R is a UMT-domain.*

Proof. Clearly, $\{p\}$ satisfies the conditions (i) and (ii) of Lemma 3.12. Also, if $S = \{up^n \mid u$ is a unit of R and $n \geq 0\}$, then R_S has a unit of nonzero degree. Thus, R is a graded UMT-domain if and only if R_S is a graded UMT-domain, if and only if R_S is a UMT-domain, if and only if R is a UMT-domain by Lemma 3.12 and Theorem 3.5. □

For each α, let $\mathbb{Z}_\alpha = \mathbb{Z}$ be the additive group of integers; so if $G = \bigoplus_\alpha \mathbb{Z}_\alpha$, then G is a torsionfree abelian group and the group ring $D[G]$ of G over D is isomorphic to $D[\{X_\alpha, X_\alpha^{-1}\}]$. Thus, if $R = D[\{X_\alpha, X_\alpha^{-1}\}]$, then R has a unit of nonzero degree and $R_{N(H)} = D[\{X_\alpha\}]_{N_v}$ [9, Proposition 3.1] and every homogeneous ideal of R has the form IR for an ideal I of D.

Corollary 3.14. *Let D be an integral domain, $\{X_\alpha\}$ be a nonempty set of indeterminates over D, and $N_v = \{f \in D[\{X_\alpha\}] \mid c(f)_v = D\}$. Then the following statements are equivalent.*

(1) *D is a UMT-domain.*
(2) *$D[\{X_\alpha\}]$ is a UMT-domain.*
(3) *$D[\{X_\alpha\}]$ is a graded UMT-domain.*
(4) *$D[\{X_\alpha, X_\alpha^{-1}\}]$ is a UMT-domain.*
(5) *$D[\{X_\alpha, X_\alpha^{-1}\}]$ is a graded UMT-domain.*
(6) *$\bar{D}[\{X_\alpha\}]_{N_v}$ is a Prüfer domain.*
(7) *$D[\{X_\alpha\}]_{N_v}$ is a UMT-domain.*
(8) *$D[\{X_\alpha\}]_{N_v}$ is a quasi-Prüfer domain.*
(9) *Every prime ideal of $D[\{X_\alpha\}]_{N_v}$ is extended from D.*

Proof. (1) ⇔ (5) Let $R = D[\{X_\alpha, X_\alpha^{-1}\}]$. Then $R_{N(H)} = D[\{X_\alpha\}]_{N_v}$ and $\{PR \mid P \in t\text{-Max}(D)\}$ is the set of homogeneous maximal t-ideals of R. Note that $\bar{R}_{H\setminus PR} = \bar{D}_P[\{X_\alpha, X_\alpha^{-1}\}]$; and $\bar{D}_P[\{X_\alpha, X_\alpha^{-1}\}]$ is a graded-Prüfer domain if and only if $\bar{D}_P$ is a Prüfer domain for all $P \in t\text{-Max}(D)$ (cf. [9, Example 3.6]). Thus, the result follows from Theorems 1.2 and 3.5.

(2) ⇔ (3) This follows from Corollary 3.13 because each X_β is a height-one homogeneous prime element of nonzero degree.

(3) ⇔ (5) Clearly, $\{X_\alpha\}$ is a set of nonzero homogeneous prime elements of $D[\{X_\alpha\}]$ satisfying the two conditions of Lemma 3.12. Also, if S is the multiplicative set of $D[\{X_\alpha\}]$ generated by $\{X_\alpha\}$, then $D[\{X_\alpha\}]_S = D[\{X_\alpha, X_\alpha^{-1}\}]$. Thus, the result is an immediate consequence of Lemma 3.12(2).

(4) ⇔ (5) ⇔ (6) ⇔ (7) ⇔ (8) ⇔ (9) Theorem 3.5. □

4. Counterexamples via the $D + XK[X]$ construction

In this section we use the $D + XK[X]$ construction to show that a graded UMT-domain need not be a UMT-domain in general. For this, let D be an integral domain with quotient field K and $D \subsetneq K$, X be an indeterminate over D, $K[X]$ be the polynomial ring over K, and $R = D + XK[X]$ be a subring of $K[X]$, i.e., $R = \{f \in K[X] \mid f(0) \in D\}$; so $D[X] \subsetneq R \subsetneq K[X]$ and R is an $\mathbb{N}_0$-graded integral domain with $\deg(aX^n) = n$ for $0 \neq a \in K$ and integer $n \geq 0$ ($a \in D$ when $n = 0$). Let H be the set of nonzero homogeneous elements of R and $N(H) = \{f \in R \mid C(f)_v = R\}$; then $N(H) = \{f \in R \mid f(0)$ is a unit of $R\}$ [15, Lemma 6] and $R_H = K[X, X^{-1}]$.

Lemma 4.1. *If Q is an upper to zero in $R = D + XK[X]$, then $Q = fR$ for some $f \in R$ with $f(0) = 1$, and hence Q is a maximal t-ideal of R.*

Proof. Note that $R_H = K[X, X^{-1}]$; so $Q = fK[X, X^{-1}] \cap R$ for some $f \in K[X, X^{-1}]$. Since X is a unit of $K[X, X^{-1}]$ and K is the quotient field of D, we may assume that $f \in R$ with $f(0) = 1$. Hence, if $g \in K[X, X^{-1}]$ is such that $fg \in R$, then $g \in K[X]$, and since $f(0) = 1$, we have $g(0) \in D$; so $g \in R$. Thus, $Q = fR$. □

It is known that $R = D + XK[X]$ is a PvMD if and only if D is a PvMD [21, Theorem 4.43]. We next give a UMT-domain analog.

Proposition 4.2. *Let $R = D + XK[X]$.*

(1) *R is a graded UMT-domain.*
(2) *R is a UMT-domain if and only if D is a UMT-domain.*

Proof. (1) Lemma 4.1.

(2) Note that $K[X]$ is a UMT-domain and $XK[X]$ is a maximal t-ideal of $K[X]$. Thus, R is a UMT-domain if and only if D is a UMT-domain [26, Proposition 3.5]. □

We end this paper with some counterexamples.

Example 4.3. Let $R = D + XK[X]$. Then R is a graded UMT-domain.

(1) Counterexample to Proposition 1.7, Theorem 3.5, Corollary 3.6, and Corollary 3.7: Let $\mathbb{R}$ be the field of real numbers, $\bar{\mathbb{Q}}$ be the algebraic closure of the field $\mathbb{Q}$ of rational numbers in $\mathbb{R}$, $\mathbb{R}[\![y]\!]$ be the power series ring over $\mathbb{R}$, and $D = \bar{\mathbb{Q}} + y\mathbb{R}[\![y]\!]$. Then D is an integrally closed one-dimensional local integral domain that is not a valuation domain [11, Theorem 2.1] (hence D is not a UMT-domain). Hence, R satisfies property (#) [15, Corollary 9], R is an integrally closed graded UMT-domain, but R is not a UMT-domain (so not a PvMD). (i) Thus, the converse of Proposition 1.7 does not hold in general. (ii) Moreover, this shows that Theorem 3.5 is not true if $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ does not contain a unit of nonzero degree. (iii) This also shows that Corollary 3.6 is not true in general. (iv) Finally, $R = D + XK[X]$ is an integrally closed domain but not a graded-Prüfer domain, while $R = D + XK[X]$ has a unique homogeneous maximal t-ideal (which must be a unique homogeneous maximal ideal). Thus, Corollary 3.7 does not hold in general.

(2) Let D be an integral domain with a prime ideal P such that $P \subsetneq P_t \subsetneq D$. (For example, let $D = \mathbb{R} + (X, Y, Z)\mathbb{C}[\![X, Y, Z]\!]$, where $\mathbb{C}$ is the field of complex numbers and $\mathbb{C}[\![X, Y, Z]\!]$ is the power series ring, and let $P = (X, Y)\mathbb{C}[\![X, Y, Z]\!]$. Then P is a prime ideal of D such that $P \subsetneq P_t = (X, Y, Z)\mathbb{C}[\![X, Y, Z]\!] \subsetneq D$.) Then $PR = P + XK[X] \subsetneq P_t + XK[X] = (P + XK[X])_t \subsetneq R = D + XK[X]$, and hence PR is a prime ideal of R contained in a homogeneous maximal t-ideal but PR is not a t-ideal. Thus, Corollary 3.9 does not hold if $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ does not contain a unit of nonzero degree.

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