J. Korean Math. Soc. **54** (2017), No. 6, pp. 1699–1731
https://doi.org/10.4134/JKMS.j160606
pISSN: 0304-9914 / eISSN: 2234-3008

CLOSURE OPERATIONS AND THE DESCENDING CHAIN CONDITION

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Abstract. In this note, we define and compare some closures which behave somewhat like the radical closure. Using these closures as a starting point allows us to classify all semiprime closures on the nodal curve. Several examples provided show how these closures can differ significantly in the non-Noetherian setting.

1. Introduction

The importance of the radical can be seen in Hilbert's Nullstellensatz which illustrates the one to one correspondence between radical ideals in a polynomial ring over and algebraically closed field with affine varieties. The radical is a type of closure operation which the current author defined to be bounded in [9].

Let $(R, \mathfrak{m})$ be a local ring. The author defined a closure operation c on the ideals of R to be bounded if there exists an $\mathfrak{m}$-primary ideal J such that for all $\mathfrak{m}$-primary ideals $\mathfrak{q} \subseteq J$, $\mathfrak{q}^c = J$. For the rings considered in [9] and [10] the bounded closures c also satisfy the property that a descending chain of closures of $\mathfrak{m}$-primary ideals stabilizes. The radical is in fact more specialized than a bounded closure. For Λ an indexing set, let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a collection of ideals with the same radical. If $I_{\lambda_1} \supseteq I_{\lambda_2} \supseteq \cdots \supseteq I_{\lambda_n} \supseteq \cdots$ is a descending chain of ideals from this set, then since they all have the same radical, they satisfy the descending chain condition on their closures since in fact their radicals are all equal.

We broaden the notion of bounded closures in two ways. Firstly, we suggest the notion of I-bounded closures for a radical ideal I which is defined similarly to the $(\mathfrak{m})$-bounded we defined in [9]. Secondly, we will impose the condition that the closures of a descending chain of ideals with radical I stabilize for some or all radical ideals I. So although we previously used the term bounded, maybe a more appropriate name should be I-DCC closures where I is a radical

Received September 14, 2016; Accepted July 3, 2017.
2010 *Mathematics Subject Classification.* 13A15, 13C05, 13E99.
Key words and phrases. closure operation, semiprime operation, prime operation.

ideal. The radical is a closure that behaves quite differently than other familiar closures such as the integral closure or tight closure. In fact, in a recent paper [2], Epstein showed that the radical is almost never standard which is a property that some common closure operations share on nice rings. In general, closures which are I-DCC will not typically be standard either; however, for the nodal curve, Morre and the author [5] have determined all the standard closure operations and among the list are some I-DCC closures which are standard. For a nice survey on closure operations see [1].

Although we define the new notions of I-DCC closures and I-bounded closures for radical ideals I, we, for the most part, focus on closures that are additionally semiprime. The notion of prime operations dates back to work by Krull [3], [4] on closure operations on the set of fractional ideals of a domain. In this work, he used the prime symbol to denote a closure operation satisfying certain properties. Sakuma [8] produced further work on prime operations restricted to the set of ideals of a ring. Petro deleted a condition from those required for prime operations and called these more generalized closures semiprime [6]. The author further studies both prime and semiprime closures in [9] and [10] in the setting of one dimensional domains.

An outline of the paper follows. In Section 2, we recall the definitions of both closure operations and semiprime operations and introduce the new notions of I-bounded closures and I-DCC closures for radical ideals I. We also include a few examples to illustrate the differences between these two notions. In Section 3, we recall the lattice structure on the set of ideals of the nodal curve $k[[x,y]]/(xy)$ and show that (x,y)-DCC and (x,y)-bounded are equivalent notions in this ring. We also prove several Lemmas to determine if a semiprime closure c is (x,y)-bounded and classify the semiprime operations c which are not (x,y)-bounded. In Section 4, we classify the (x,y)-bounded semiprime operations. In Section 5, we give some examples of closures c which are P-bounded but not Q-bounded for primes $Q \subseteq P$. For semiprime closures this seems to only occur in the non-Noetherian setting.

2. Closure operations

Let R be commutative ring with ideals $I, J \subseteq R$. Let $I \mapsto I^c$ be an operation on the ideals of R. We will denote the operation by c. The operation may satisfy the following properties:

(a) c is expansive if $I \subseteq I^c$ for all $I \subseteq R$.
(b) c is order preserving if when $I \subseteq J$, then $I^c \subseteq J^c$.
(c) c is idempotent if $(I^c)^c = I^c$ for all $I \subseteq R$.
(d) $I^c J^c \subseteq (IJ)^c$ for all $I, J \subseteq R$.

Definition 2.1. We say c is a *closure operation* if c satisfies (a)-(c). If a closure operation c further satisfies (d), we call c a *semiprime operation.*

We will say an ideal $I \subseteq R$ is c-closed if $I^c = I$ for the closure operation or semiprime operation c. If $I^c \neq I$ we will say that I is not c-closed. Here are a couple of properties of closure operations that we will make use of.

Lemma 2.2. *Let R be a commutative ring and c a closure operation defined on the ideals of R. Suppose $\mathfrak{a}, \mathfrak{b} \subseteq I$ are ideals of R.*

(1) *If $(\mathfrak{a} \cap \mathfrak{b})^c = I$, then $\mathfrak{a}^c = I = \mathfrak{b}^c$.*
(2) *If $\mathfrak{a}^c = I$, then $(\mathfrak{a} + \mathfrak{b})^c = I$.*

Proof. These are clear by the order preservation property of closure operations. □

Lemma 2.3. *Let R be a commutative ring and c a closure operation defined on the ideals of R. Suppose $\mathfrak{a}, \mathfrak{b}$ are ideals of R which are incomparable. Suppose $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ such that $I^c = \mathfrak{a}$. Then $\mathfrak{a} + \mathfrak{b} \subseteq \mathfrak{b}^c$.*

Proof. Since $I \subseteq \mathfrak{b}$ then by the order preservation property $\mathfrak{a} = I^c \subseteq \mathfrak{b}^c$. However, $\mathfrak{b} \subseteq \mathfrak{b}^c$ by the expansive property of closure operations so $\mathfrak{a} + \mathfrak{b} \subseteq \mathfrak{b}^c$. □

The notion of bounded closure operation, was essential in the classification of semiprime operations on both $k[[t]]$ and $k[[t^2, t^3]]$ in [9].

Definition 2.4. A closure operation c is *bounded* if for each maximal ideal $\mathfrak{m}$ of R, there exists an $\mathfrak{m}$-primary ideal J such that for all $\mathfrak{m}$-primary ideals $I \subseteq J$, $I^c = J$.

In a local one-dimensional domain, if c is a bounded closure, then since there exists an $\mathfrak{m}$-primary ideal J such that for all $\mathfrak{m}$-primary ideals $I \subseteq J$, then $I^c = J$. Since $(0) \subseteq I$ for all $I \subseteq R$, $(0)^c \subseteq J$. So there are only two possibilities of what the closure of (0) may be. Either $(0)^c = (0)$ or $(0)^c = J$.

However if R is a local commutative ring which is not a domain or which has dimension greater than 1 and c is a bounded closure operation on the ideals of R, what can we say about the closures of the possibly many ideals which are not $\mathfrak{m}$-primary? One possible way to extend the definition of bounded closures is to define a similar notion in terms of P-primary ideals for a prime P. For example, we could define a closure operation c on the set of ideals of R to be *weakly-P-bounded* if there exists a P-primary ideal Q such that for all P-primary ideals $\mathfrak{q} \subseteq Q$, $\mathfrak{q}^c = Q^c$. Similarly we could say a closure c is *P-bounded* if there exists a P-primary ideal Q such that for all P-primary ideals $\mathfrak{q} \subseteq Q$, $\mathfrak{q}^c = Q$. However, the following example, seems to indicate that this notion is not quite strong enough.

Example 2.5. Let $R = k[[x, y]]/(xy)$. The primes of R are given by the following partially ordered set.

$$\begin{tikzcd} & (x,y) \arrow[dl, no head] \arrow[dr, no head] & \\ (x) & & (y) \end{tikzcd}$$

Suppose $I \to I^c$ is given by $\mathfrak{A}^c = (x, y)$ for all (x, y)-primary ideals $\mathfrak{A}$ and $(x^n)^c = (x^n)$ for all $n \in \mathbb{N}$ and $(y^m)^c = (y^m)$ for all $m \in \mathbb{N}$. Note that $J \subseteq J^c$ for all ideals J in R. Also, for $J \subseteq I$ ideals, $J^c \subseteq I^c$. Lastly, $I^c = (I^c)^c$ for all ideals I of R.

The only (x)-primary ideal is (x) itself. This is because $(x^n) = (x^n, y) \cap (x)$ is a primary decomposition of (x^n). Similarly, (y) is the only (y)-primary ideal. This closure is (x, y)-bounded, (x)-bounded and (y)-bounded by our "definitions" above.

However, the ideals (x^n) for $n > 1$ and (y^m) for $m > 1$ don't satisfy the descending chain condition on the closures like their primary counterparts.

Instead we will extend the notion of boundedness by an examination of the closures of ideals with a common radical.

Definition 2.6. Let $(R, \mathfrak{m})$ be a local ring and c be a closure operation defined on R. Suppose I is a radical ideal. We say that c is *weakly-I-bounded* if there exists some ideal J with $\sqrt{J} = I$ such that for all ideals $\mathfrak{A} \subseteq J$ with $\sqrt{\mathfrak{A}} = I$, $\mathfrak{A}^c = J^c$. We say c is *I-bounded* if there exists some ideal J with $\sqrt{J} = I$ such that for all ideals $\mathfrak{A} \subseteq J$ with $\sqrt{\mathfrak{A}} = I$, $\mathfrak{A}^c = J$.

With our new definition, we will see that the closure c in Example 2.5 is actually now neither (x)-bounded nor (y)-bounded. Let $\mathfrak{A}$ be a radical ideal, $\mathcal{I}$ be a set of ideals and $\mathcal{I}_{\mathfrak{A}}$ be the set of all ideals with radical $\mathfrak{A}$.

The following two examples show that even with this new definition of a closure being I-bounded for radical ideals we may still have a descending chain of ideals all with radical I and the closures in the chain may not stabilize:

Example 2.7. Let $R = k[[x, y]]$. Note $I = (xy)$ is a radical ideal. Define a closure c on the R such that for ideals J with radical (xy), if $J \subseteq (x^2y)$, then $J^c = (x^2y)$, if $J \supseteq (x^2y)$, then $J^c = J$ and if J is incomparable to (x^2y), then $J^c = (x^2y) + J$. Consider the descending chain of ideals

$$(xy) \supseteq (x^2y, xy^2) \supseteq (x^2y, xy^3) \supseteq \cdots (x^2y, xy^r) \supseteq \cdots .$$

Then c is an (xy)-bounded closure. However, it is not an (xy)-DCC closure.

Example 2.8. Let $S = k[x_1, x_2, \ldots, x_n, \ldots]$ and $\mathfrak{m} = (x_1, x_2, \ldots, x_n, \ldots)$. Set $R = S_{\mathfrak{m}}$. Define the closure c on the ideals of R satisfying $J^c = \mathfrak{m}^2$ for all $J \subseteq \mathfrak{m}^2$ and $J^c = J$ for all other J. c is an $\mathfrak{m}$-bounded closure. However, c

does not satisfy the descending chain condition on the closures of $\mathfrak{m}$-primary ideals. Denote the ideal $P_n = (x_n, x_{n+1}, \ldots)$. Then

$$\mathfrak{m} \supseteq (x_1)^2 + P_2 \supseteq (x_1, x_2)^2 + P_3 \supseteq \cdots \supseteq (x_1, \ldots, x_{n-1})^2 + P_n \supseteq \cdots$$

is a descending chain of $\mathfrak{m}$-primary ideals whose closures do not stabilize.

Both examples indicate that I-bounded may not be the best generalization for closures which behave like the radical. Of course, the radical ideal that we picked in the first example is not prime and the ring in the second example is not Noetherian. It may be the case that a weakly I-bounded closure satisfies the descending chain condition on descending chains of closures of ideals with radical P, a prime ideal in a Noetherian ring. In fact, we make the following definition.

Definition 2.9. Let R be a commutative ring and c a closure operation defined on the ideals of R. Let I be a radical ideal of R. We will say that c is an *I-DCC closure* if for any descending chain of ideals

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_n \supseteq \cdots$$

with $\sqrt{J_n} = I$, then

$$J_1^c \supseteq J_2^c \supseteq \cdots \supseteq J_n^c \supseteq \cdots$$

stabilizes. In other words there exists an n such that for all $m \geq n$ $J_m^c = J_n^c$.

Proposition 2.10. *Let R be a commutative ring and c a closure operation defined on the ideals of R. Let I be a radical ideal of R. If c is I-DCC and $\sqrt{J^c} = I$ for all J with $\sqrt{J} = I$, then c is I-bounded.*

Proof. First we claim that there are elements $\mathfrak{a}$ of the set of ideals with radical I which are c-closed and minimal with respect to inclusion. Suppose there is a chain of ideals with radical I:

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_n \supseteq \cdots$$

such that for each $n \in \mathbb{N}$, J_n is c-closed. Since c is I-DCC then for some $n \in \mathbb{N}$, for every $m \geq n$, $J_m = J_m^c = J_n^c = J_n$. In particular, if chains C_i stabilize at the c-closed ideals $\mathfrak{a}_i$, then $\{\mathfrak{a}_i \mid i \in I\}$ cannot contain an infinite descending chain. So in particular the set of c-closed ideals with radical I has minimal elements. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are minimal c-closed elements with radical I. Note that

$$\mathfrak{a} \cap \mathfrak{b} \subseteq (\mathfrak{a} \cap \mathfrak{b})^c = \mathfrak{a}^c \cap \mathfrak{b}^c = \mathfrak{a} \cap \mathfrak{b}.$$

However since $\mathfrak{a}$ and $\mathfrak{b}$ are minimal c-closed elements with radical I and $\mathfrak{a} \cap \mathfrak{b}$ is c-closed, has radical I and is properly contained in $\mathfrak{a}$ and $\mathfrak{b}$ implying there is only one minimal element. Thus for any minimal c-closed element $\mathfrak{a}$ and all ideals $J \subseteq \mathfrak{a}$ with $\sqrt{J} = I$ then $J^c = \mathfrak{a}$. This implies that c is I-bounded. $\square$

One particular descending chain of ideals that we will consider is a descending chain of powers of an ideal I. If c is a semiprime operation and for some ideal I and all natural numbers n, $(I^n)^c = I^k$ for some $k \leq n$, then we can precisely describe what c behaves like for all natural numbers.

Proposition 2.11. *Let R be a commutative ring and c a semiprime operation defined on the ideals of R. Let I be an ideal of R. Suppose for all natural numbers n $(I^n)^c = I^k$ for some $k \leq n$ for some semiprime operation c. Then either c is the identity on I^n or c is defined as follows*:

$$(I^n)^c = \begin{cases} I^n \text{ for all } 1 \leq n \leq m \\ I^m \text{ for all } n > m. \end{cases}$$

Proof. We know that for all natural numbers n, $(I^n)^c = I^k$ for some $k \leq n$. Suppose for some particular n, $(I^n)^c = I^m$ for $m < n$. Note that for all $m \leq r \leq n$,

$$I^m = (I^n)^c \subseteq (I^r)^c \subseteq (I^m)^c = I^m,$$

where the first and second containments follow from the extensive property of closure operations and the last equality follows from idempotence. Thus $(I^r)^c = I^m$ for all $m \leq r \leq n$. We use induction to show that $(I^r)^c = I^m$ for all $r \geq n$. Suppose that $(I^r)^c = I^m$ for some $r > n$. Then

$$I^{m+1} = (I^r)^c I \subseteq (I^{r+1})^c \subseteq (I^m)^c = I^m.$$

Since $(I^{m+1})^c = I^m$, then $(I^{r+1})^c = I^m$ concluding our induction argument. Now suppose that $(I^j)^c = I^i$ for some $i < j \leq m$. Then $I^{m-j+i} = (I^j)^c I^{m-j} \subseteq (I^m)^c = I^m$ is a contradiction since $I^{m-j+i} \not\subseteq I^m$. Thus $(I^j)^c = I^j$ for all $1 \leq j \leq m$. If there exists no such m with $(I^n)^c = I^m$ for $m < n$, then $(I^n)^c = I^n$ for all $n \geq 1$ indicating that c is the identity on I^n. □

For the ring $R = k[[x, y]]/(xy)$, the ideals (x) and (y) are both radical ideals. The only ideals in R which have radical (x) are of the form $(x^r) = (x)^r$ and the only ideals which have radical (y) are of the form $(y^s) = (y)^s$. The following is a corollary of Proposition 2.11:

Corollary 2.12. *Let $R = k[[x, y]]/(xy)$ and c is a semiprime operation.*

(1) *If c is (x)-bounded, then there exists an natural number m such that*

$$(x^r)^c = \begin{cases} (x^r) \text{ for } 1 \leq r \leq m \\ (x^m) \text{ for } r > m. \end{cases}$$

(2) *If c is (y)-bounded, then there exists an natural number n such that*

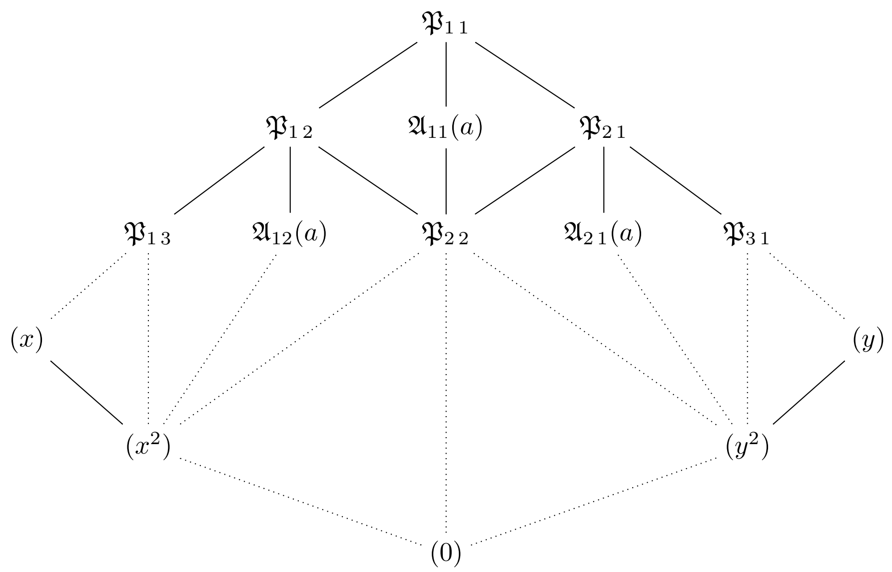
$$(y^s)^c = \begin{cases} (y^s) \text{ for } 1 \leq s \leq n \\ (y^n) \text{ for } s > n. \end{cases}$$

3. Classifying closures on the nodal curve

Using these new definitions, we would like to classify the semiprime operations on the ring $R = k[[x,y]]/(xy)$. In [7, Theorem 1], Ran showed that all the (x,y)-primary ideals of $R = k[[x,y]]/(xy)$ are either of the form (x^i, y^j) which we will denote $\mathfrak{P}_{i\,j}$ or $(x^i + ay^j)$ which we will denote $\mathfrak{A}_{i\,j}(a)$ with $a \neq 0$. (Note, this notation differs from Ran's notation as he was grouping the ideals in terms of colength m.) Clearly $\mathfrak{P}_{i+1\,j+1} \subseteq \mathfrak{A}_{i\,j}(a) \subseteq \mathfrak{P}_{i\,j}$ and $\mathfrak{A}_{i\,j}(a) + \mathfrak{A}_{i\,j}(b) = \mathfrak{P}_{i\,j}$ and $\mathfrak{A}_{i\,j}(a) \bigcap \mathfrak{A}_{i\,j}(b) = \mathfrak{P}_{i+1\,j+1}$ for $a \neq b$, both not 0. Also,

$$\bigcap_{i\geq 1} \mathfrak{P}_{i\,j} = (y^j), \quad \bigcap_{j \geq 1} \mathfrak{P}_{i\,j} = (x^i).$$

Hence, a portion of the lattice of ideals is:



where $a \neq 0$ and the node at $\mathfrak{A}_{i\,j}(a) = (x^i + ay^j)$ has $k\backslash\{0\} := k^\times$ incomparable ideals.

We will show that a semiprime operation c is weakly (x,y)-bounded if and only if c is weakly (x)-bounded and weakly (y)-bounded. For the nodal curve we will see that the weak (x,y)-boundedness of c is equivalent to c being (x,y)-DCC.

Proposition 3.1. *Let $R = k[[x,y]]/(xy)$. Suppose a closure operation c defined on the ideals of R is weakly (x,y)-bounded. Then c is (x,y)-DCC.*

Proof. Since c is (x,y)-bounded there exists an (x,y)-primary ideal J such that for all (x,y)-primary ideals $I \subseteq J$, $I^c = J^c$. There are three possibilities for J^c.

Either $J^c = R$, $J^c = \mathfrak{P}_{i\,j}$ or $J^c = \mathfrak{A}_{i\,j}(a)$ for some $i, j \in \mathbb{N}$ and some $a \in k^\times$. In either of these cases if we have a chain of (x, y)-primary ideals

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_n \supseteq \cdots$$

such that $J_n^c = J = J^c$ for some $n \geq 1$, then clearly the chain terminates since for $m \geq n$ $J_m^c \subseteq J^c$ which implies $J_m^c = J = J^c$. Suppose

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_n \supseteq \cdots$$

is a chain of (x, y)-primary ideals in R such that J_i are not comparable to J for $i \geq n$. We need to show that the chain of closures

$$J_1^c \supseteq J_2^c \supseteq \cdots \supseteq J_n^c \supseteq \cdots$$

terminate. First consider the chain

$$J \cap J_1 \supseteq J \cap J_2 \supseteq \cdots \supseteq J \cap J_n \supseteq \cdots .$$

All the ideals in this chain are contained in J so the chain of closures terminates at J^c. Now since $J \cap J_i \subseteq J_i$ for all i, then by Lemma 2.2 $J^c = (J \cap J_i)^c \subseteq J_i^c$. So in fact by Lemma 2.3 the chain of closures

$$J_1^c \supseteq J_2^c \supseteq \cdots \supseteq J_n^c \supseteq \cdots$$

is

$$(J + J_1)^c \supseteq (J + J_2)^c \supseteq \cdots \supseteq (J + J_n)^c \supseteq \cdots$$

all of which are comparable to J. Since any J^c listed above is of finite length then the chain must terminate. □

Hence, all semiprime operations which are weakly (x, y)-bounded will be (x, y)-DCC. So this is why we focus on the weakly (x, y)-bounded closures from this point on.

Lemma 3.2. *Let $R = k[[x, y]]/(xy)$. Suppose a closure operation c defined on the ideals of R is weakly (x)-bounded. Then there exists an $n \in \mathbb{N}$ such that for every $r \geq 1$ and $m \geq n$, $\mathfrak{P}_{m\,r}^c = \mathfrak{P}_{n\,r}^c$.*

Proof. Suppose c is weakly (x)-bounded. Then there exists an $n \geq 1$ such that for all $m \geq n$, $(x^n)^c = (x^m)^c$. Consider the chain

$$\mathfrak{P}_{n\,r} = (x^n) + (y^r) \subseteq (x^n)^c + (y^r) = (x^m)^c + (y^r) \subseteq \mathfrak{P}_{m\,r}^c \subseteq \mathfrak{P}_{n\,r}^c.$$

This implies that for every $r \geq 1$ and $m \geq n$, $\mathfrak{P}_{m\,r}^c = \mathfrak{P}_{n\,r}^c$. □

Lemma 3.3. *Let $R = k[[x, y]]/(xy)$. Suppose a closure operation c defined on the ideals of R is weakly (y)-bounded. Then there exists an $n \in \mathbb{N}$ such that for every $r \geq 1$ and $m \geq n$, $\mathfrak{P}_{r\,m}^c = \mathfrak{P}_{r\,n}^c$.*

Proof. Exchanging the roles of (x) and (y) in the proof of Lemma 3.2 we obtain for every $r \geq 1$ and $m \geq n$, $\mathfrak{P}_{r\,m}^c = \mathfrak{P}_{r\,n}^c$. □

Proposition 3.4. *Let $R = k[[x,y]]/(xy)$. Suppose a closure operation c defined on the ideals of R is both weakly-(x)-bounded and weakly-(y)-bounded. Then c is weakly-(x,y)-bounded and if the there is an (x,y)-primary ideal J such that $J = I^c$ for all $I \subseteq J$, then c is (x,y)-bounded.*

Proof. Since c is weakly-(x)-bounded by Lemma 3.2 there exists an $n \in \mathbb{N}$ such that for all $m \geq n$ $\mathfrak{P}^c_{m\,r} = \mathfrak{P}^c_{n\,r}$. Since c is weakly-(y)-bounded by Lemma 3.3 there exists an $s \in \mathbb{N}$ such that for all $t \geq s$ $\mathfrak{P}^c_{r\,t} = \mathfrak{P}^c_{r\,s}$. Putting these together we see for all $m \geq n$ and all $t \geq s$ $\mathfrak{P}^c_{m\,t} = \mathfrak{P}^c_{n\,s}$. Note that for any ideal I such that $\mathfrak{P}_{m\,t} \subseteq I \subseteq \mathfrak{P}_{n\,s}$, $I^c = \mathfrak{P}^c_{n\,s}$. Thus c is weakly (x,y)-bounded. If $\mathfrak{P}^c_{n\,s}$ is an (x,y)-primary ideal, then c is (x,y)-bounded. □

Corollary 3.5. *Let $R = k[[x,y]]/(xy)$. Suppose a closure operation c defined on the ideals of R is both (x)-bounded and (y)-bounded. Then c is weakly-(x,y)-bounded and (x,y)-bounded if there exists some (x,y)-primary ideal J such that for all (x,y)-primary $I \subseteq J$ $I^c = J$.*

Proof. Since I-bounded closures are weakly-I-bounded for all radical ideals I, the proof follows. □

We saw above that Example 2.5 exhibits a closure operation c which is (x,y)-bounded on R but not (x)-bounded or (y)-bounded. However, if c is a bounded semiprime operation or even a weakly bounded semiprime operation, then c is both weakly (x)-bounded and weakly (y)-bounded.

Proposition 3.6. *Let $R = k[[x,y]]/(xy)$. Suppose a closure operation c is a weakly-(x,y)-bounded semiprime operation. Then c is both weakly-(x)-bounded and weakly-(y)-bounded.*

Proof. Since c is weakly-(x,y)-bounded then there exists an (x,y)-primary ideal J such that for all (x,y)-primary ideals $I \subseteq J$, $I^c = J^c$. J is either of the form $\mathfrak{A}_{i\,j}(a)$ or $\mathfrak{P}_{i\,j}$. Note that $(x^r)\mathfrak{A}_{i\,j} = (x^r)\mathfrak{P}_{i\,j} = (x^r)J = (x^{r+i})$ for all $r \in \mathbb{N}$. For either such J for $m > i$ and $n > j$ and any nonzero $b \in k$,

$$(x^{r+i}) = (x^r)J \subseteq (x^r)J^c \subseteq ((x^r)J)^c = (x^{r+m})^c \subseteq (x^{r+i})^c$$

which implies that $(x^{r+i})^c = (x^{r+m})^c$ for all $m \geq i$. Thus, c is a weakly-(x)-bounded operation.

We can use a similar argument to see that c is a weakly-(y)-bounded operation. □

We have seen that a semiprime operation on $R = k[[x,y]]/(xy)$ is weakly (x,y)-bounded if and only if it is both weakly (x)-bounded and weakly (y)-bounded. Next we provide several lemmas which we can use to determine if a semiprime operation on R is weakly P-bounded for one of the primes $P \subseteq R$ since the only nonzero radical ideals of R are prime.

Lemma 3.7. *Let $R = k[[x,y]]/(xy)$. Let c be a semiprime operation defined on the ideals of R and J be a radical ideal such that $\sqrt{J^n} = J$ for all $n \geq 1$. If*

$(J^n)^c = (J^{n+1})^c$ for some n, then c is a weakly-J-bounded semiprime operation. If $(J^n)^c$ is an ideal with radical J, then c is J-bounded.

Proof. Suppose $(J^n)^c = (J^{n+1})^c = K$ for some ideal $K \supseteq J^n$ for some n. We need to show that any ideal $L \subseteq K$ satisfying $\sqrt{L} = \sqrt{J}$ has the property $L^c = K$. Note that J is either (1) an (x, y)-primary ideal, (2) (x^i) for some $i \geq 1$ or (3) (y^j) for some $j \geq 1$. (1) If J is (x, y)-primary, then $J = \mathfrak{A}_{i\,j}(a)$ or $J = \mathfrak{P}_{i\,j}$ for some $i, j \geq 1$ and $a \in k \setminus \{0\}$. Since $\mathfrak{A}_{i\,j}(a)J = J^2$ for some $a \in k \setminus \{0\}$ then

$$\begin{aligned} J^{n+2} &\subseteq J^{n+1} = \mathfrak{A}_{i\,j}(a)J^n \\ &\subseteq \mathfrak{A}_{i\,j}(a)(J^n)^c = \mathfrak{A}_{i\,j}(a)(J^{n+1})^c \\ &\subseteq (\mathfrak{A}_{i\,j}(a)J^{n+1})^c = (J^{n+2})^c. \end{aligned}$$

Applying c everywhere, we see that $(J^{n+2})^c = (J^{n+1})^c = K$. By a similar argument, we see that $(J^{n+m})^c = K$ for all $m \geq 0$. If $J^n \subseteq L \subseteq K$, then applying c we see that $(J^n)^c = L^c = K$. Note that as J is an (x, y)-primary, for any (x, y)-primary ideal $L \subseteq J^n$ there exist positive integers $r \geq s \geq n$ with $J^r \subseteq L \subseteq J^s$. As $(J^r)^c = K$ for all $r \geq n$, then $L^c = K$. Clearly, if K is (x, y)-primary, then c is (x, y)-bounded.

(2) Replacing (x, y)-primary with any ideal with radical (x) and $\mathfrak{A}_{i\,j}(a)$ with (x^i) in (1) above we see that c is weakly (x)-bounded and (x)-bounded if $K = (x^m)$ for some $m \geq 1$.

(3) Replacing (x, y)-primary with any ideal with radical (y) and $\mathfrak{A}_{i\,j}(a)$ with (y^j) in (1) above we see that c is weakly (y)-bounded and (y)-bounded if $K = (y^n)$ for some $n \geq 1$. □

Lemma 3.8. *Let $R = k[[x, y]]/(xy)$. Suppose a semiprime operation c defined on the ideals of R has the property that $\mathfrak{P}^c_{i+1\,j} = \mathfrak{P}^c_{i\,j}$ for some $i, j \in \mathbb{N}$. Then c is a weakly-(x)-bounded semiprime operation.*

Proof. If $\mathfrak{P}^c_{i+1\,j} = \mathfrak{P}^c_{i\,j}$ for some $i \geq 1$, then for all $r \geq 1$

$$(x^{i+r}) = (x^r)\mathfrak{P}_{i\,j} \subseteq (x^r)\mathfrak{P}^c_{i\,j} = (x^r)\mathfrak{P}^c_{i+1\,j} \subseteq (x^{i+r+1})^c \subseteq (x^{i+r})^c$$

implying that $(x^{i+r})^c = (x^{r+i+1})^c$ for all $r \geq 1$ or $(x^{i+1})^c = (x^{i+r})^c$ for all $r \geq 1$. In particular $(x^{i+1})^c = (x^{2i+2})^c = ((x^{i+1})^2)^c$. By Lemma 3.7 we conclude that c is weakly-(x)-bounded. □

Lemma 3.9. *Let $R = k[[x, y]]/(xy)$. Suppose a semiprime operation c defined on the ideals of R has the property that $\mathfrak{P}^c_{i\,j+1} = \mathfrak{P}^c_{i\,j}$ for some $i, j \in \mathbb{N}$. Then c is a weakly-(y)-bounded semiprime operation.*

Proof. Reversing the roles of (x) and (y) in the proof of Lemma 3.8 we obtain the result. □

The preceding two lemmas give us criteria to determine if a semiprime operation is weakly-(x)-bounded or weakly-(y)-bounded. We now prove several lemmas that will give us criteria for a semiprime operation to be weakly-(x, y)-bounded

Lemma 3.10. *Let $R = k[[x, y]]/(xy)$. Then if c is a semiprime operation defined on the ideals of R satisfying $\mathfrak{P}^c_{i\,j} = \mathfrak{P}^c_{i+1\,j+1}$ for some $i, j \in \mathbb{N}$, then c is a weakly-(x, y)-bounded semiprime operation. If in addition $\mathfrak{P}^c_{i\,j}$ is (x, y)-primary, then c is (x, y)-bounded.*

Proof. Since $\mathfrak{P}^c_{i\,j} = \mathfrak{P}^c_{i+1\,j+1}$ implies also that

$$\mathfrak{P}^c_{i+1\,j} = \mathfrak{P}^c_{i\,j} = \mathfrak{P}^c_{i\,j+1}$$

then by Lemmas 3.8 and 3.9, c is both weakly (x)-bounded and weakly (y)-bounded. Now by Proposition 3.4 c is weakly (x, y)-bounded. □

Lemma 3.11. *Let $R = k[[x, y]]/(xy)$. Suppose a semiprime operation c defined on the ideals of R has the property that $\mathfrak{P}^c_{i+1\,j} = \mathfrak{P}_{i\,j} = \mathfrak{P}^c_{i\,j+1}$ for some integers i and j. Then c is an (x, y)-bounded semiprime operation.*

Proof. Since $\mathfrak{P}^c_{i+1\,j} = \mathfrak{P}_{i\,j} = \mathfrak{P}^c_{i\,j+1}$ then by Lemmas 3.8 and 3.9, c is both weakly (x)-bounded and weakly (y)-bounded. Now by Proposition 3.4, c is (x, y)-bounded. □

Lemma 3.12. *Let $R = k[[x, y]]/(xy)$. Suppose a semiprime operation c defined on the ideals of R has the property that $(x^m)^c = \mathfrak{P}^c_{m\,n}$ for some integers m and n. Then c is a weakly (y)-bounded semiprime operation. If $(x^m)^c = \mathfrak{P}^c_{(m-1)\,n}$ for some integers m and n, then c is a weakly (x, y)-bounded semiprime operation.*

Proof. If $(x^m)^c = \mathfrak{P}_{m\,n}$ for some integers m and n, then for all $r \geq n$, $\mathfrak{P}_{m\,n} = (x^m)^c + (y^r) \subseteq \mathfrak{P}^c_{m\,r} \subseteq \mathfrak{P}^c_{m\,n}$ which implies that $\mathfrak{P}^c_{m\,n} = \mathfrak{P}^c_{m\,r}$ for all $r \geq n$. Thus $(y^{n+1}) = (y)\mathfrak{P}_{m\,n} \subseteq (y)\mathfrak{P}^c_{m\,n} = (y)\mathfrak{P}^c_{m\,r} \subseteq (y^{r+1})^c \subseteq (y^{n+1})^c$ for all $r \geq n$ implying that c is weakly (y)-bounded.

If $(x^m)^c = \mathfrak{P}^c_{(m-1)\,n}$ for some integers m and n, then $\mathfrak{P}^c_{(m-1)\,n} = (x^m)^c + (y^n)^c \subseteq \mathfrak{P}^c_{m\,n} \subseteq \mathfrak{P}^c_{(m-1)\,n}$ implies that c is weakly (x)-bounded by Lemma 3.8. As c is both weakly (x)-bounded and weakly (y)-bounded then c is also (x, y)-bounded by Proposition 3.4. □

Lemma 3.13. *Let $R = k[[x, y]]/(xy)$. Suppose a semiprime operation c defined on the ideals of R has the property that $(y^n)^c = \mathfrak{P}^c_{m\,n}$ for some integers m and n. Then c is a weakly (x)-bounded semiprime operation. If $(y^n)^c = \mathfrak{P}^c_{m\,(n-1)}$ for some integers m and n, then c is a weakly (x, y)-bounded semiprime operation.*

Proof. Exchanging the roles of (x) and (y) in the proof of Lemma 3.12 we obtain the result. □

Lemma 3.14. *Let $R = k[[x, y]]/(xy)$. Suppose a semiprime operation c is defined on the ideals of R has the property that $\mathfrak{P}^c_{i\,j} = \mathfrak{A}_{i-1\,j-1}(a)$ for some*

$i, j \in \mathbb{N}$ and some nonzero $a \in k$. Then c is an (x, y)-bounded semiprime operation.

Proof. Consider the chain of containments

$$\mathfrak{P}_{(2i-1)\,(2j-1)} = \mathfrak{P}_{i\,j}\mathfrak{A}_{i-1\,j-1}(a) = \mathfrak{P}_{i\,j}\mathfrak{P}^c_{i\,j} \subseteq (\mathfrak{P}^2_{i\,j})^c = \mathfrak{P}^c_{2i\,2j} \subseteq \mathfrak{P}^c_{(2i-1)\,(2j-1)}.$$

Applying c to the chain, we see that $\mathfrak{P}^c_{2i\,2j} = \mathfrak{P}^c_{(2i-1)\,(2j-1)}$. By Lemma 3.10, we see that as c is an (x, y)-bounded semiprime operation. □

In fact, even if c is a semiprime operation which is (x)-bounded or (y)-bounded we can specifically determine which monomial (x, y)-primary ideals are c-closed.

Proposition 3.15. *Let $R = k[[x, y]]/(xy)$. Suppose c is a semiprime operation defined on the ideals of R.*

1. *If c is (x)-bounded but not weakly (y)-bounded, then there exists a natural number m such that $\mathfrak{P}^c_{r\,t} = \mathfrak{P}_{r\,t}$ for all $1 \le r \le m-1$ and all $t \in \mathbb{N}$.*
2. *If c is (y)-bounded but not weakly (x)-bounded, then there exists a natural number n such that $\mathfrak{P}^c_{t\,s} = \mathfrak{P}_{t\,s}$ for all $1 \le s \le n-1$ and all $t \in \mathbb{N}$.*
3. *If c is both (x)-bounded and (y)-bounded, then there exist natural numbers m and n such that $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,s}$ for all $1 \le r \le m-1$ and all $1 \le s \le n-1$.*

Proof. (1) Suppose c is (x)-bounded then by Corollary 2.12

$$(x^r)^c = \begin{cases} (x^r) \text{ for } 1 \le r \le m \\ (x^m) \text{ for } r > m. \end{cases}$$

Note that either $(y^s)^c = (y^t)$ for some $t \le s$, $(y^s)^c = \mathfrak{P}_{i\,t}$ for some i and $t \le s$ or $(y^s)^c = \mathfrak{A}_{i\,t}(a)$ for some i, $a \in k^\times$ and $t < s$.

First suppose that $(y^s)^c = (y^t)$. Then for $1 \le r \le m$,

$$\mathfrak{P}_{r\,t} = (x^r) + (y^t) = (x^r)^c + (y^s)^c \subseteq \mathfrak{P}^c_{r\,s} \subseteq \mathfrak{P}^c_{r\,t}$$

or $\mathfrak{P}^c_{r\,t} = \mathfrak{P}^c_{r\,s}$. Note that since, c is not weakly (y)-bounded then Lemma 3.9 $t = s$.

Suppose now that $(y^s)^c = \mathfrak{P}_{i\,t}$, $\mathfrak{P}_{i\,t} \subseteq (y^s)^c \subseteq \mathfrak{P}^c_{i\,s} \subseteq \mathfrak{P}^c_{i\,t}$ or $\mathfrak{P}^c_{r\,t} = \mathfrak{P}^c_{r\,s}$. Again Lemma 3.9 implies that $t = s$.

Note that $(y^s)^c \ne \mathfrak{A}_{i\,t}(a)$ for $t < s$. Otherwise $(y^s)^c = \mathfrak{A}_{i\,t}(a) \supseteq \mathfrak{P}_{i+1\,t+1} \supseteq (y^{t+1}) \supseteq (y^s)$ implying $\mathfrak{A}_{i\,t}(a)^c = \mathfrak{P}^c_{i+1\,t+1}$. However, by Lemma 3.14 this would imply that c is weakly (x, y)-bounded and hence weakly (y)-bounded which is a contradiction.

Thus $(y^s)^c = (y^s)$ for all s or $(y^s)^c = \mathfrak{P}_{i\,s}$ for all s and some i. Consider $\mathfrak{P}^c_{r\,s}$ for $1 \le r < m$. Suppose $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{i\,s}$ for some $i \le r \le m-1$ then

$$(x^{i+1}) = (x)\mathfrak{P}_{i\,s} = (x)\mathfrak{P}^c_{r\,s} \subseteq (x^{r+1})^c \subseteq (x^{i+1})^c = (x^{i+1})$$

or $(x^{r+1})^c = (x^{i+1})$ which implies that $r = i$. Thus for all $1 \le r \le m-1$ $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,s}$.

(2) The proof is the same as (1) exchanging the roles of (x) and (y) and replacing Lemma 3.9 with Lemma 3.8.

(3) Suppose c is both (x)-bounded and (y)-bounded, then there exist natural numbers m and n such that

$$(x^r)^c = \begin{cases} (x^r) \text{ if } 1 \le r \le m \\ (x^m) \text{ if } r > m \end{cases} \qquad \text{and} \qquad (y^s)^c = \begin{cases} (y^s) \text{ if } 1 \le s \le n \\ (y^s)^c = (y^n) \text{ if } s > n. \end{cases}$$

If $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{i\,j}$ for some $1 \le i \le r < m$ and $1 \le j \le s < n$,

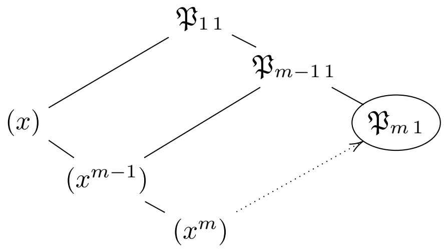
$$(x^{i+1}) = (x)\mathfrak{P}^c_{r\,s} \subseteq (x^{r+1})^c \subseteq (x^{i+1})^c = (x^{i+1})$$

and

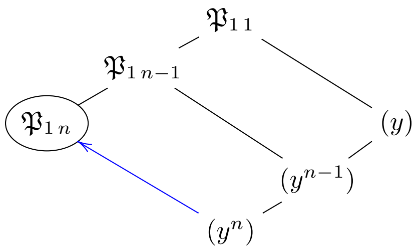
$$(y^{j+1}) = (y)\mathfrak{P}^c_{r\,s} \subseteq (y^{s+1})^c \subseteq (y^{j+1})^c = (y^{j+1})$$

imply that $i = r$ and $j = s$. □

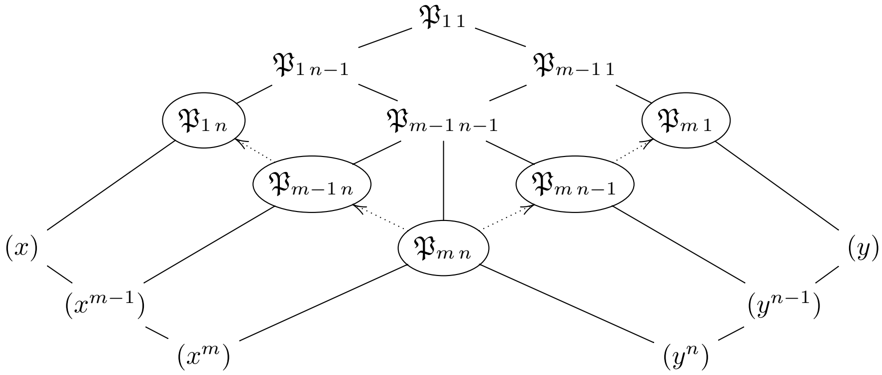
We illustrate what we have obtained in this proposition through some pictures:



If c is (x)-bounded but not (y)-bounded, then the ideals contained in the lattice of ideals of R inside the upper parallelogram are all c-closed. We cannot determine the closures of the ideals between (x^m) and $\mathfrak{P}_{m\,1}$ which are indicated by the dotted arrow and the circled ideals. However, by Lemma 3.2 we do know all the ideals lying below the chain of ideals between (x^m) and $\mathfrak{P}_{m\,1}$ are not c-closed.



If c is (y)-bounded but not (x)-bounded, then the ideals contained in the lattice of ideals of R inside the upper parallelogram are all c-closed. We cannot determine the closures of the ideals between (y^n) and $\mathfrak{P}_{1\,n}$ which are indicated by the dotted arrow and the circled ideals. However, by Lemma 3.3 we do know all the ideals lying below the chain of ideals between (y^n) and $\mathfrak{P}_{1\,n}$ are not c-closed.



If c is both (x)- and (y)-bounded, then the ideals contained in the lattice of ideals of R inside the top parallelogram are all c-closed. We cannot determine the closures of the ideals between (x^m) and $\mathfrak{P}_{m\,1}$ and the ideals between (y^n) and $\mathfrak{P}_{1\,n}$ which are indicated by the dotted arrow and the circled ideals. However, by Lemmas 3.2 and 3.3 we do know all the ideals lying below the chain of ideals between (x^m) and $\mathfrak{P}_{m\,1}$, lying below the chain of ideals between (y^n) and $\mathfrak{P}_{1\,n}$ and inside $\mathfrak{P}_{m\,n}$ are not c-closed.

We can also determine the closures of the ideals in the "border" regions of the above diagrams. In particular, the following proposition determines the closures of the chain of ideals between (y^r) and $\mathfrak{P}_{n\,r}$ for each r.

Proposition 3.16. *Let $R = k[[x,y]]/(xy)$. Suppose a semiprime operation c defined on the ideals of R is (x)-bounded with* $(x^r)^c = \begin{cases} (x^r) \text{ for } 1 \le r \le n \\ (x^r)^c = (x^n) \text{ for } r > n \end{cases}$ *but not weakly (y)-bounded. Then precisely one of the following holds*:

1. $\mathfrak{P}^c_{m\,r} = \mathfrak{P}_{n\,r}$ *for every $r \ge 1$ and $m \ge n$.*
2. $\mathfrak{P}^c_{m\,r} = \mathfrak{P}_{n\,r}$ *for every $r > i$ and $m \ge n$, and $\mathfrak{P}^c_{m\,r} = \mathfrak{P}_{n-1\,r}$ for every $1 \le r \le i$ and $m \ge n-1$.*

Proof. Since $(x^r)^c = \begin{cases} (x^r) \text{ for } 1 \le r \le n \\ (x^n) \text{ for } r > n \end{cases}$, we know by Lemma 3.2 that $\mathfrak{P}^c_{m\,r} = \mathfrak{P}^c_{n\,r}$ for every $r \ge 1$ and $m \ge n$. To prove this proposition we need only determine $\mathfrak{P}^c_{n\,r}$ for each r. Suppose for some $r \ge 1$, $\mathfrak{P}^c_{n\,r} \supsetneq \mathfrak{P}_{n\,r}$, then since c is not (y)-bounded then $\mathfrak{P}^c_{n\,r} \ne \mathfrak{A}_{n-1\,r-1}(a)$ for any a by Lemma 3.14 and $\mathfrak{P}^c_{n\,r} \ne \mathfrak{P}_{n\,r-1}$ by Lemma 3.9. The only possibility is that $\mathfrak{P}^c_{n\,r} = \mathfrak{P}_{n-1\,r}$ since $\mathfrak{P}_{n-1\,r}$ is c-closed by Proposition 3.15(1). We need to determine $\mathfrak{P}^c_{n\,j}$ for $1 \le j < r$. Since $\mathfrak{P}_{n\,j} \supseteq \mathfrak{P}_{n\,r}$ then $\mathfrak{P}^c_{n\,j} \supseteq \mathfrak{P}_{n-1\,r}$. But $\mathfrak{P}_{n-1\,r} + \mathfrak{P}_{n\,j} = \mathfrak{P}_{n-1\,j}$. Thus $\mathfrak{P}^c_{n\,j} \supseteq \mathfrak{P}_{n-1\,j}$ and since $\mathfrak{P}_{n-1\,j}$ is c-closed, then $\mathfrak{P}^c_{n\,j} = \mathfrak{P}_{n-1\,j}$ for all $1 \le j < r$. Thus either $\mathfrak{P}^c_{m\,r} = \mathfrak{P}_{n\,r}$ for every $r \ge 1$ and $m \ge n$ or $\mathfrak{P}^c_{m\,r} = \mathfrak{P}_{n\,r}$

for every $r < i$ and $m \geq n$, and $\mathfrak{P}_{m\,r}^c = \mathfrak{P}_{n-1\,r}$ for every $1 \leq r \leq i$ and $m \geq n-1$. $\square$

The next proposition determines the closures of the chain of ideals between (x^r) and $\mathfrak{P}_{r\,n}$ for each r.

Proposition 3.17. *Let $R = k[[x,y]]/(xy)$. Suppose a semiprime operation c defined on the ideals of R is (y)-bounded with* $(y^r)^c = \begin{cases} (y^r) \text{ for } 1 \leq r \leq n \\ (y^r)^c = (y^n) \text{ for } r > n \end{cases}$ *but not weakly (x)-bounded. Then precisely one of the following holds*:

(1) $\mathfrak{P}_{r\,m}^c = \mathfrak{P}_{r\,n}$ *for every $r \geq 1$ and $m \geq n$.*

(2) $\mathfrak{P}_{r\,m}^c = \mathfrak{P}_{r\,n}$ *for every $r > i$ and $m \geq n$, and $\mathfrak{P}_{r\,m}^c = \mathfrak{P}_{r\,n-1}$ for every $1 \leq r \leq i$ and $m \geq n-1$.*

Proof. The proof is the same as Proposition 3.16 replacing the roles of x and y. Hence, instead of using Lemma 3.9 we use Lemma 3.8 and we use Proposition 3.15(2) instead of Proposition 3.15(1). $\square$

Lemma 3.18. *Let $R = k[[x,y]]/(xy)$ and c be a semiprime operation.*

(1) *If c is not weakly (x)-bounded, then either*:
   (a) $(x^r)^c = (x^r)$ *for all natural numbers r,*
   (b) *there exists a natural number n such that either*:
      (i) $(x^r)^c = \mathfrak{P}_{r\,n}$ *for all natural numbers r or*
      (ii) *there exists $j \in \mathbb{N} \cup \{\infty\}$ such that*

$$(x^r)^c = \begin{cases} \mathfrak{P}_{r\,n-1} \text{ for } 1 \leq r \leq j \\ \mathfrak{P}_{r\,n} \text{ for all } r > j. \end{cases}$$

(2) *If c is not weakly (y)-bounded, then either*:
   (a) $(y^s)^c = (y^s)$ *for all natural numbers s,*
   (b) *there exists a natural number m such that either*:
      (i) $(y^s)^c = \mathfrak{P}_{m\,s}$ *for all natural numbers s or*
      (ii) *there exists $i \in \mathbb{N} \cup \{\infty\}$ such that*

$$(y^s)^c = \begin{cases} \mathfrak{P}_{m-1\,s} \text{ for } 1 \leq s \leq i \\ \mathfrak{P}_{m\,s} \text{ for all } s > i. \end{cases}$$

Proof. We will exhibit a proof for (1) and note that the same proof works for (2) replacing the roles of x and y.

Suppose first that $(x^r)^c = (x^t)$ for some $t < r$. This implies that $(x^i)^c = (x^t)$ for all $t \leq i \leq r$. Consider the chain

$$(x^{t+1}) = (x)(x^i)^c \subseteq (x^{i+1})^c \subseteq (x^{t+1})^c$$

and the fact that $(x^{t+1})^c = (x^t)$. We can now conclude that $(x^t) = (x^{s+1})^c$. Thus $(x^t) = (x^i)^c$ for all $i \geq t$ by induction. However, this contradicts the fact that c was not weakly (x)-bounded. Thus $(x^r)^c = (x^r)$ for all natural numbers r.

Now suppose that $(x^r)^c = \mathfrak{P}_{t\,n}$ for some natural number n and some $t < r$. Then

$$\mathfrak{P}_{t\,n} \subseteq (x^i)^c \subseteq \mathfrak{P}^c_{i\,n} \subseteq \mathfrak{P}^c_{t\,n}$$

for all $t \leq i \leq r$ which implies in particular that $\mathfrak{P}^c_{t\,n} = \mathfrak{P}^c_{t+1\,n}$. Using Lemma 3.8, we conclude that c is weakly (x)-bounded. However, this contradicts our assumption that c was not weakly (x)-bounded so $(x^r)^c = \mathfrak{P}_{r\,n}$ for some natural number n.

For all $t < r$, we observe that $\mathfrak{P}_{t\,n} = (x^t) + \mathfrak{P}_{r\,n} \subseteq (x^t)^c \subseteq \mathfrak{P}^c_{t\,n}$ which implies that $\mathfrak{P}^c_{t\,n} = (x^t)^c$ for $t < r$. Again by induction we can conclude that $(x^r)^c = \mathfrak{P}^c_{r\,n}$ for all $r \geq t$. Observe that the following chain

$$\mathfrak{P}^c_{r\,m} = (x^r)^c \subseteq \mathfrak{P}^c_{r+1\,n} \subseteq \mathfrak{P}^c_{r\,n}$$

implies that $\mathfrak{P}^c_{r\,n} = \mathfrak{P}^c_{r+1\,n}$. Lemma 3.9 implies that c is weakly (y)-bounded. Since c is not weakly (x)-bounded then Lemma 3.12 implies $(y^n)^c \neq \mathfrak{P}^c_{i\,n}$ for any natural numbers i or n. Thus for every natural number n, $(y^n)^c = (y^i)$ for some $1 \leq i \leq n$ which implies that c is (y)-bounded by Corollary 2.12. We can now conclude that $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,s}$ for all natural numbers r and $1 \leq s \leq n-1$ by Proposition 3.15. Finally, we conclude that either $(x^r)^c = \mathfrak{P}_{r\,n}$ for all natural numbers r or that there exists $j \in \mathbb{N} \cup \{\infty\}$ such that $(x^r)^c = \begin{cases} \mathfrak{P}_{r\,n-1} \text{ for } 1 \leq r \leq j \\ \mathfrak{P}_{r\,n} \text{ for } r > j \end{cases}$ by Proposition 3.17. □

Lemma 3.19. *Let $R = k[[x,y]]/(xy)$ and c be a semiprime operation.*

1. *If $\mathfrak{A}_{i\,j}(a)^c = \mathfrak{P}_{i\,j}$ for some $a \in k \setminus \{0\}$ and $i, j \in \mathbb{N}$, then $\mathfrak{A}_{m\,n}(b)^c = \mathfrak{P}^c_{m\,n}$ for all $b \in k \setminus \{0\}$ and all $m \geq i+1$ and $n \geq j+1$.*
2. *If $\mathfrak{A}_{i\,j}(a)^c = \mathfrak{A}_{i\,j}(a)$ for some $a \in k \setminus \{0\}$ and $i, j \in \mathbb{N}$, then $\mathfrak{A}_{m\,n}(b)^c = \mathfrak{A}_{m\,n}(b)$ for all $b \in k \setminus \{0\}$ and all $m \leq i-1$ and $n \leq j-1$.*

Proof. (1) Since $\mathfrak{A}_{i\,j}(a)\mathfrak{A}_{r\,s}(ba^{-1}) = \mathfrak{A}_{(i+r)\,(j+s)}(b)$ and

$$\begin{aligned}\mathfrak{P}_{(i+r)\,(j+s)} &\subseteq \mathfrak{P}_{i\,j}\mathfrak{A}_{r\,s}(ba^{-1})^c \subseteq \mathfrak{A}_{i\,j}(a)^c\mathfrak{A}_{r\,s}(ba^{-1})^c \\ &\subseteq \mathfrak{A}_{(i+r)\,(j+s)}(b)^c \subseteq \mathfrak{P}^c_{(i+r)\,(j+s)}\end{aligned}$$

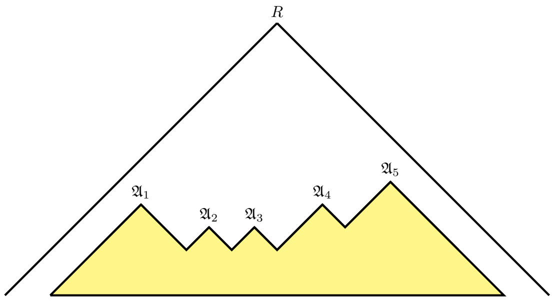
for all $r, s \geq 1$, we see that $\mathfrak{A}_{(i+r)\,(j+s)}(b)^c = \mathfrak{P}^c_{(i+r)\,(j+s)}$ for all $r, s \geq 1$.

(2) Note that $\mathfrak{A}_{r\,s}(b)\mathfrak{A}_{i-r\,j-s}(b^{-1}a) = \mathfrak{A}_{i\,j}(a)$ and $\mathfrak{A}_{r\,s}(b)^c\mathfrak{A}_{i-r\,j-s}(b^{-1}a)^c \subseteq \mathfrak{A}_{i\,j}(a)^c = \mathfrak{A}_{i\,j}(a)$. If $\mathfrak{P}_{r\,s}(b) \subseteq \mathfrak{A}_{r\,s}(b)^c$ or $\mathfrak{P}_{i-r\,j-s}(b^{-1}a) \subseteq \mathfrak{A}_{i-r\,j-s}(b^{-1}(a)^c$ then $\mathfrak{P}_{i\,j} \subseteq \mathfrak{A}_{r\,s}(b)^c\mathfrak{A}_{i-r\,j-s}(b^{-1}a)^c \subseteq \mathfrak{A}_{i\,j}(a)$ gives a contradiction. Thus for $r < i$ and $s < j$ $\mathfrak{A}_{r\,s}(b)^c = \mathfrak{A}_{r\,s}(b)$. □

To illustrate the previous lemma in the unbounded case, observe the following diagram where each arrow indicates the closure of the ideal at the base of the arrow and the boxed ideals represent $|k^{\times}|$ ideals:

$$
\begin{tikzcd}
 & & \mathfrak{P}_{m\,n} \arrow[loop above] & & \\
 & & \mathfrak{A}_{m\,n}(a) \arrow[u] & & \\
 & & \mathfrak{P}_{(m+1)\,(n+1)} \arrow[loop above] & & \\
 & \mathfrak{P}_{(m+1)\,(n+2)} \arrow[loop above] & \boxed{\mathfrak{A}_{(m+1)\,(n+1)}(b)} \arrow[u] & \mathfrak{P}_{(m+2)\,(n+1)} \arrow[loop above] & \\
\mathfrak{P}_{(m+1)\,(n+3)} \arrow[loop above] & \boxed{\mathfrak{A}_{(m+1)\,(n+2)}(b)} \arrow[u] & \mathfrak{P}_{(m+2)\,(n+2)} \arrow[loop above] & \boxed{\mathfrak{A}_{(m+2)\,(n+1)}(b)} \arrow[u] & \mathfrak{P}_{(m+3)\,(n+1)} \arrow[loop above]
\end{tikzcd}
$$

There may be several incomparable ideals $\mathfrak{A}_{r_i\,s_i}(a_i)$ satisfying the property $\mathfrak{A}_{r_i\,s_i}(a_i)^c = \mathfrak{P}_{r_i\,s_i}$ and every ideal $\mathfrak{A}_{m\,n}(b)$ containing each $\mathfrak{A}_{r_i\,s_i}(a_i)$ is c-closed. The shaded region in the following picture illustrates a possible region of the lattice of ideals where the ideals $\mathfrak{A}_i = \mathfrak{A}_{r_i\,s_i}(a_i)$ are not c-closed.



As we can see from the illustration and Lemma 3.19, once the closure of an ideal $\mathfrak{A}_{i\,j}(a)$ is $\mathfrak{P}_{i\,j}$, all ideals $\mathfrak{A}_{r\,s}(b)$ will have closure containing $\mathfrak{P}_{r\,s}$ for $r > i$ and $s > j$. We state it this way because of the possibility that the closure may be (x, y)-bounded, (x)-bounded or (y)-bounded. To describe the ideals $\mathfrak{A}_{i\,j}(a)$ such that $\mathfrak{P}_{i\,j} \subseteq \mathfrak{A}_{i\,j}(a)^c$ we will use a subset

$$T = \{(r, s, a) \in \mathbb{N}^2 \times k^{\times} \mid \mathfrak{P}_{r\,s} \subseteq \mathfrak{A}_{r\,s}(a)^c\}.$$

As observed in the illustration, the set $S = \{\mathfrak{A}_{i\,j}(a) \mid (i, j, a) \in T\}$ will have maximal elements. The set $T \subseteq \mathbb{N}^2 \times k^{\times}$ also satisfies the property that for all $(r, s, a) \notin T$ then $(m, n, b) \notin T$ for all $m < r$, $n < s$ and all $b \in k^{\times}$ which is the second statement proved in Lemma 3.19.

Proposition 3.20. *Let $R = k[[x, y]]/(xy)$ and c be a semiprime operation. Let $T \subseteq \mathbb{N}^2 \times k^{\times}$ satisfy the property that for all $(r, s, a) \notin T$. Then $(m, n, b) \notin T$*

for all $m < r$, $n < s$ *and all* $b \in k^{\times}$. *Let* $m, n \in \mathbb{N}$ *and* $i, j \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. *If* c *is not weakly* (x, y)*-bounded, then* c *is either the identity or one of the following*:

(1) $c = c(\infty, \infty, T)$ *where* $I^c = I$ *for all monomial ideals* I *of* R *and*

$$\mathfrak{A}_{r\,s}(a)^c = \begin{cases} \mathfrak{A}_{r\,s}(a) \text{ for all } (r, s, a) \notin T \\ \mathfrak{P}_{r\,s} \text{ for all } (r, s, a) \in T. \end{cases}$$

(2) $c = c(m, \infty(j), T)$ *where* $(0)^c = (0)$, $(y^s)^c = (y^s)$ *for all* $s \in \mathbb{N}$,

$$(x^r)^c = \begin{cases} (x^r) \text{ for } r \leq m \\ (x^m) \text{ for } r > m, \end{cases}$$

$$\mathfrak{P}^c_{r\,s} = \begin{cases} \mathfrak{P}_{r\,s} \text{ for } 1 \leq r \leq m-1 \text{ and all } s \\ \mathfrak{P}_{(m-1)\,s} \text{ for } r > m-1 \text{ and all } 1 \leq s \leq j \text{ if } j \geq 1 \\ \mathfrak{P}_{m\,s} \text{ for } r \geq m \text{ and all } s > j, \end{cases}$$

$$\mathfrak{A}_{r\,s}(a)^c = \begin{cases} \mathfrak{A}_{r\,s}(a) \text{ for all } (r, s, a) \notin T \\ \mathfrak{P}_{r\,s} \text{ for all } 1 \leq r \leq m-2 \text{ and all } s \\ \qquad \text{or } r = m-1 \text{ and } s \geq j \text{ such that } (r, s, a) \in T \\ \mathfrak{P}_{m-1\,s} \text{ for } r \geq m-1 \text{ and } 1 \leq s < j \text{ if } j \geq 1 \\ \mathfrak{P}_{m\,s} \text{ for } r \geq m \text{ and } s > j. \end{cases}$$

(3) $c = c(m, \infty(j)', T)$ *where* $(0)^c = (x^m)$,

$$(y^s)^c = \begin{cases} \mathfrak{P}_{m\,s} \text{ for all } s > j \\ \mathfrak{P}_{(m-1)\,s} \text{ for all } 1 \leq s \leq j, \end{cases} \qquad (x^r)^c = \begin{cases} (x^r) \text{ for } r \leq m \\ (x^m) \text{ for } r > m, \end{cases}$$

$$\mathfrak{P}^c_{r\,s} = \begin{cases} \mathfrak{P}_{r\,s} \text{ for } 1 \leq r \leq m-1 \text{ and all } s \\ \mathfrak{P}_{(m-1)\,s} \text{ for } r > m-1 \text{ and all } 1 \leq s \leq j \text{ if } j \geq 1 \\ \mathfrak{P}_{m\,s} \text{ for } r \geq m \text{ and all } s > j, \end{cases}$$

$$\mathfrak{A}_{r\,s}(a)^c = \begin{cases} \mathfrak{A}_{r\,s}(a) \text{ for all } (r, s, a) \notin T \\ \mathfrak{P}_{r\,s} \text{ for all } 1 \leq r \leq m-2 \text{ and all } s \\ \qquad \text{or } r = m-1 \text{ and } s \geq j \text{ such that } (r, s, a) \in T \\ \mathfrak{P}_{m-1\,s} \text{ for } r \geq m-1 \text{ and } 1 \leq s < j \text{ if } j \geq 1 \\ \mathfrak{P}_{m\,s} \text{ for } r \geq m \text{ and } s > j. \end{cases}$$

(4) $c = c(\infty(i), n, T)$ *where* $(0)^c = (0)$, $(x^r)^c = (x^r)$ *for all* $r \in \mathbb{N}$,

$$(y^s)^c = \begin{cases} (y^s) \text{ for } y \leq n \\ (y^n) \text{ for } s > n, \end{cases}$$

$$\mathfrak{P}^c_{r\,s} = \begin{cases} \mathfrak{P}_{r\,s} \text{ for all } r \text{ and } 1 \leq s \leq m-1 \\ \mathfrak{P}_{r\,(n-1)} \text{ for } 1 \leq r \leq i \text{ and all } s > n-1 \text{ if } i \geq 1 \\ \mathfrak{P}_{r\,n} \text{ for } r > i \text{ and all } s \geq n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(a)^c = \begin{cases} \mathfrak{A}_{r\,s}(a) \text{ for all } (r,s,a) \notin T \\ \mathfrak{P}_{r\,s} \text{ for all } r \text{ and all } 1 \le s \le n-2 \\ \qquad \text{or } r \ge i \text{ and } s = n-1 \text{ such that } (r,s,a) \in T \\ \mathfrak{P}_{r\,(n-1)} \text{ for } 1 \le r < i \text{ and } s \ge n-1 \text{ if } i \ge 1 \\ \mathfrak{P}_{r\,n} \text{ for } r > i \text{ and } s \ge n. \end{cases}$$

(5) $c = c(\infty(i)', n, T)$ *where* $(0)^c = (y^n)$,

$$(x^r)^c = \begin{cases} \mathfrak{P}_{r\,n} \text{ for all } r > i \\ \mathfrak{P}_{r\,(n-1)} \text{ for all } 1 \le r \le i, \end{cases} \qquad (y^s)^c = \begin{cases} (y^s) \text{ for } y \le n \\ (y^n) \text{ for } s > n, \end{cases}$$

$$\mathfrak{P}_{r\,s}^c = \begin{cases} \mathfrak{P}_{r\,s} \text{ for all } r \text{ and } 1 \le s \le m-1 \\ \mathfrak{P}_{r\,(n-1)} \text{ for } 1 \le r \le i \text{ and all } s > n-1 \text{ if } i \ge 1 \\ \mathfrak{P}_{r\,n} \text{ for } r > i \text{ and all } s \ge n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(a)^c = \begin{cases} \mathfrak{A}_{r\,s}(a) \text{ for all } (r,s,a) \notin T \\ \mathfrak{P}_{r\,s} \text{ for all } r \text{ and all } 1 \le s \le n-2 \\ \qquad \text{or } r \ge i \text{ and } s = n-1 \text{ such that } (r,s,a) \in T \\ \mathfrak{P}_{r\,(n-1)} \text{ for } 1 \le r < i \text{ and } s \ge n-1 \text{ if } i \ge 1 \\ \mathfrak{P}_{r\,n} \text{ for } r > i \text{ and } s \ge n. \end{cases}$$

Proof. Suppose c is a semiprime operation which is not weakly (x, y)-bounded. By Lemma 3.11 it cannot be the case that $\mathfrak{P}_{i\,j}^c = \mathfrak{P}_{(i-1)\,j}$ for some i and j and $\mathfrak{P}_{r\,s}^c = \mathfrak{P}_{r\,(s-1)}$ for some r and s. So either one of the equalities holds or neither hold.

If neither hold, then Lemma 3.14 tells us that $\mathfrak{P}_{i\,j}^c \neq \mathfrak{A}_{(i-1)\,(j-1)}(a)$ for any $a \in k^\times$ so $\mathfrak{P}_{i\,j}^c = \mathfrak{P}_{i\,j}$ for all $i, j \in \mathbb{N}$. Now $(x^i) \subseteq \mathfrak{P}_{i\,j}$ for all $j \in \mathbb{N}$ implies that

$$(x^i)^c \subseteq \bigcap_{j \ge 1} \mathfrak{P}_{i\,j}^c = \bigcap_{j \ge 1} \mathfrak{P}_{i\,j} = (x^i).$$

Thus $(x^i)^c = (x^i)$ for all $i \in \mathbb{N}$. Similarly $(y^j)^c = (y^j)$ for all $j \in \mathbb{N}$. So all monomial ideals are c-closed. Using Lemma 3.19 we obtain the set T which describes which principal (x, y)-primary ideals which are not closed. Thus we have obtained the closure c described in (1).

Suppose now that $\mathfrak{P}_{i\,j}^c = \mathfrak{P}_{(i-1)\,j}$ some i. By Lemma 3.8, c is weakly (x)-bounded. Since c is not weakly (x, y)-bounded then c cannot be weakly (y)-bounded by Proposition 3.5. Also c must be (x)-bounded, for if $(x^i)^c = J$ for some (x, y)-primary ideal, then by Lemma 3.12 c is weakly (y)- bounded which cannot be the case. Now by Corollary 2.12 there exists an $m \in \mathbb{N}$ such that $(x^r)^c = \begin{cases} (x^r) \text{ for } 1 \le r \le m \\ (x^m) \text{ for } r > m. \end{cases}$ By Lemma 3.15 $\mathfrak{P}_{r\,s}^c = \mathfrak{P}_{r\,s}$ for all $1 \le r \le m-1$ and all $s \in \mathbb{N}$. By Lemma 3.2, $\mathfrak{P}_{r\,s}^c = \mathfrak{P}_{m\,s}^c$ for all $r \ge m$ and all

$s \in \mathbb{N}$. Once we know the closures of $\mathfrak{P}_{m\,s}$ for $s \in \mathbb{N}$ and the closures of (y^s) we will have determined c.

By Lemma 3.18(2), the closures of the ideals (y^s) may be: (a) $(y^s)^c = (y^s)$ for all s, (b) $(y^s)^c = \mathfrak{P}_{m\,s}$ for all s or (c) $(y^s)^c = \begin{cases} \mathfrak{P}_{m-1\,s} \text{ for } 1 \leq s \leq j \\ \mathfrak{P}_{m\,s} \text{ for } s > j. \end{cases}$

For (b) and (c) described above we immediately see that we obtain the closures described in (3) above because the closures of the ideals (y^s) describe which ideals among the set $\mathfrak{P}_{m\,s}$ must be closed and Lemma 3.19 describes which principal (x, y)-primary ideals are closed. In both cases, $(0)^c = (x^m) = \bigcap_{s>>0} (y^s)^c = \bigcap_{s>>0} \mathfrak{P}_{m\,s}$. If the (y^s) are closed, then using Proposition 3.16, we can determine the closures of $\mathfrak{P}_{m\,s}$ and we obtain the closures described in (2) above after applying Lemma 3.19 to describe the closures of the principal (x, y)-primary ideals. Again in both cases $(0)^c = (0) = \bigcap_{s>>0} (y^s)$.

If $\mathfrak{P}^c_{i\,j} = \mathfrak{P}_{i\,(j-1)}$ some j following similar reasoning as above but exchanging the roles of x and y and using Lemma 3.9 in place of Lemma 3.8 and Lemma 3.13 in place of Lemma 3.12 and Lemma 3.3 in place of Lemma 3.2 and Proposition 3.17 in place of Proposition 3.16 we obtain the closures described in (4) and (5). $\square$

4. Bounded closures on the nodal curve

Although we have proved some lemmas about bounded closures on the nodal curve, we have yet to classify them. We prove a few more lemmas before we embark on this goal.

Lemma 4.1. *Let $R = k[[x, y]]/(xy)$. Suppose a semiprime operation c defined on the ideals of R satisfies $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m\,n}$ for every $r \geq m$ and $s \geq n$. Then $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,s}$ for every $r \leq m - 1$ and $s \leq n - 1$ and one of the following holds*:

1. *$\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,n}$ for all $1 \leq r \leq m$ and all $s \geq n$ and $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m\,s}$ for all $r \geq m$ and all $1 \leq s \leq n$.*
2. *$\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,n}$ for all $1 \leq r \leq m$ and all $s \geq n$, $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m-1\,s}$ for all $r \geq m - 1$ and all $1 \leq s \leq j$ and $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m\,s}$ for all $r \geq m$ and all $j < s \leq n$.*
3. *$\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,n-1}$ for all $1 \leq r \leq i$ and all $s \geq n - 1$, $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,n}$ for all $i < r \leq m$ and all $s \geq n$ and $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m\,s}$ for all $r \geq m$ and all $1 \leq s \leq n$.*
4. *$\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,n-1}$ for all $1 \leq r \leq i$ and all $s \geq n - 1$, $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,n}$ for all $i < r \leq m$ and all $s \geq n$, $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m-1\,s}$ for all $r \geq m - 1$ and all $1 \leq s \leq j$ and $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m\,s}$ for all $r \geq m$ and all $j < s \leq n$.*

Proof. Suppose first that $\mathfrak{P}^c_{r\,s} \neq \mathfrak{P}_{r\,s}$ for some $1 \leq r < m$ and $1 \leq s < n$. Then either $\mathfrak{P}^c_{r\,s} \supseteq \mathfrak{P}_{r-1\,s}$, $\mathfrak{P}^c_{r\,s} \supseteq \mathfrak{P}_{r\,s-1}$ or $\mathfrak{P}^c_{r\,s} \supseteq \mathfrak{A}_{r-1\,s-1}(a)$ for some $a \in k^\times$. Since c is semiprime $\mathfrak{P}^c_{r\,s}\mathfrak{P}_{m-r\,n-s} \subseteq \mathfrak{P}^c_{m\,n} = \mathfrak{P}_{m\,n}$ implying either

$\mathfrak{P}_{m-1\,n} \subseteq \mathfrak{P}_{m\,n}$, $\mathfrak{P}_{m\,n-1} \subseteq \mathfrak{P}_{m\,n}$ or $\mathfrak{P}_{m-1\,n-1} \subseteq \mathfrak{P}_{m\,n}$ each leading to a contradiction. Thus $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,s}$ for all $1 \leq m-1$ and $1 \leq s \leq n-1$.

For $1 \leq s \leq n-1$, $\mathfrak{P}_{m\,n} \subseteq \mathfrak{P}_{m\,s}$. Hence $\mathfrak{P}_{m\,n} \subseteq \mathfrak{P}^c_{m\,s}$ for $1 \leq s \leq n-1$. For $r > m$ our assumption implies that $\mathfrak{P}^c_{r\,n} = \mathfrak{P}_{m\,n}$. Since $\mathfrak{P}_{r\,n} \subseteq \mathfrak{P}_{r\,s}$ for $1 \leq s \leq n-1$ and $r > m$ we see that $\mathfrak{P}_{m\,n} \subseteq \mathfrak{P}^c_{r\,s}$. Now $\mathfrak{P}_{m\,s} = \mathfrak{P}_{r\,s} + \mathfrak{P}_{m\,n} \subseteq \mathfrak{P}^c_{r\,s} \subseteq \mathfrak{P}^c_{m\,s}$ for $1 \leq s \leq n-1$ and $r > m$ which implies that $\mathfrak{P}^c_{r\,s} = \mathfrak{P}^c_{m\,s}$ for $1 \leq s \leq n-1$ and $r > m$. Similarly $\mathfrak{P}^c_{r\,s} = \mathfrak{P}^c_{r\,n}$ for $1 \leq r \leq m-1$ and $s > n$. Thus we need only determine $\mathfrak{P}^c_{r\,n}$ for $1 \leq r \leq n-1$ and $\mathfrak{P}^c_{m\,s}$ for $1 \leq s \leq n-1$.

If $\mathfrak{P}^c_{r\,n} \neq \mathfrak{P}_{r\,n}$ for some $1 \leq r \leq n-1$, then $\mathfrak{P}^c_{r\,n} = \mathfrak{P}_{r\,n-1}$. It cannot be the case that $\mathfrak{P}^c_{r\,n} = \mathfrak{P}_{r-1\,n}$ or $\mathfrak{P}^c_{r\,n} = \mathfrak{A}_{r-1\,n-1}(a)$ for some $a \in k^\times$. This is because $\mathfrak{P}^c_{r\,n} \subseteq \mathfrak{P}^c_{r\,n-1} = \mathfrak{P}_{r\,n-1}$ and both $\mathfrak{P}_{r-1\,n}$ and $\mathfrak{A}_{r-1\,n-1}(a)$ are incomparable with $\mathfrak{P}_{r\,n-1}$. If $\mathfrak{P}^c_{r\,n} = \mathfrak{P}_{r\,n-1}$, then for all $1 \leq i \leq r$,

$$\mathfrak{P}_{i\,n-1} = \mathfrak{P}^c_{i\,n-1} \supseteq \mathfrak{P}^c_{i\,n} \supseteq \mathfrak{P}^c_{r\,n} + \mathfrak{P}_{i\,n} = \mathfrak{P}_{i\,n-1}.$$

Therefore $\mathfrak{P}^c_{i\,n} = \mathfrak{P}_{i\,n-1}$ for all $1 \leq i \leq r$. So there will be some maximal r with $1 \leq r < n$ with $\mathfrak{P}^c_{i\,n} = \mathfrak{P}_{i\,n-1}$ for all $1 \leq i \leq r$. For all $m \geq i > r$, $\mathfrak{P}^c_{i\,n} = \mathfrak{P}_{i\,n}$.

Similarly, if $\mathfrak{P}^c_{m\,s} \neq \mathfrak{P}_{m\,s}$, then $\mathfrak{P}^c_{m\,s}$ must be $\mathfrak{P}_{m-1\,s}$ and this would imply that $\mathfrak{P}^c_{m-1\,j} = \mathfrak{P}_{m-1\,j}$ for all $1 \leq j \leq s$.

So either (1) $\mathfrak{P}^c_{r\,n} = \mathfrak{P}_{r\,n}$ for all $1 \leq r \leq m$ and $\mathfrak{P}^c_{m\,s} = \mathfrak{P}_{m\,s}$ for all $1 \leq s \leq n$ or (2) $\mathfrak{P}^c_{r\,n} = \mathfrak{P}_{r\,n}$ for all $1 \leq r \leq m$ and $\mathfrak{P}^c_{m\,s} = \mathfrak{P}_{m-1\,s}$ for all $1 \leq s \leq j$ and $\mathfrak{P}^c_{m\,s} = \mathfrak{P}_{m\,s}$ for all $j < s \leq n$ or (3) $\mathfrak{P}^c_{r\,n} = \mathfrak{P}_{r\,n-1}$ for all $1 \leq r \leq i$, $\mathfrak{P}^c_{r\,n} = \mathfrak{P}_{r\,n}$ for all $i < r \leq m$ and $\mathfrak{P}^c_{m\,s} = \mathfrak{P}_{m\,s}$ for all $1 \leq s \leq n$ or (4) $\mathfrak{P}^c_{r\,n} = \mathfrak{P}_{r\,n-1}$ for all $1 \leq r \leq i$, $\mathfrak{P}^c_{r\,n} = \mathfrak{P}_{r\,n}$ for all $i < r \leq m$, $\mathfrak{P}^c_{m\,s} = \mathfrak{P}_{m-1\,s}$ for all $1 \leq s \leq j$ and $\mathfrak{P}^c_{m\,s} = \mathfrak{P}_{m\,s}$ for all $j < s \leq n$. □

Lemma 4.2. *Let $R = k[[x,y]]/(xy)$ and c be a semiprime operation defined on R which satisfies $\mathfrak{P}^c_{r\,s} = \mathfrak{A}_{m-1\,n-1}(a)$ for some $a \in k^\times$ and for all $r \geq m$ and $s \geq n$. Then $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,s}$ for all $1 \leq r \leq m-1$ and $1 \leq s \leq n-1$; $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,n-1}$ for all $1 \leq r \leq m-1$ and $s > n-1$; and $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m-1\,s}$ for all $r > m-1$ and $1 \leq s \leq n-1$.*

Proof. Using Lemma 3.19, we see that $\mathfrak{A}_{r\,s}(b)^c = \mathfrak{A}_{r\,s}(b)$ for all $1 \leq r < m-1$, $1 \leq s < n-1$ and all $b \in k^\times$. Thus $\mathfrak{P}^c_{r\,s} \subseteq \mathfrak{A}_{r-1\,s-1}(a)^c \cap \mathfrak{A}_{r-1\,s-1}(b)^c = \mathfrak{P}_{r\,s}$ for all $1 \leq r \leq m-1$, $1 \leq s \leq n-1$ implying that $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,s}$ for all $1 \leq r < m-1$, $1 \leq s < n-1$.

Since $\mathfrak{A}_{m-1\,n-1}(a) = \mathfrak{P}^c_{m\,s} \subseteq \mathfrak{P}^c_{r\,s}$ for all $1 \leq r < m$ and $s \geq n$ the chain

$$\mathfrak{P}_{r\,n-1} = \mathfrak{A}_{m-1\,n-1}(a) + \mathfrak{P}_{r\,s} \subseteq \mathfrak{P}^c_{r\,n-1}$$

implies $\mathfrak{P}^c_{r\,s} = \mathfrak{P}^c_{r\,n-1}$ for $1 \leq r < m$ and $s \geq n$. Similarly $\mathfrak{P}^c_{r\,s} = \mathfrak{P}^c_{m-1\,s}$ for $1 \leq s < n$ and $r \geq m$. Note that we have already shown above that both $\mathfrak{P}^c_{r\,n-1} = \mathfrak{P}_{r\,n-1}$ for $1 \leq r \leq m-1$ and $\mathfrak{P}^c_{m-1\,s} = \mathfrak{P}_{m-1\,s}$ for $1 \leq s \leq n-1$ which concludes the lemma. □

Proposition 4.3. *Let $R = k[[x,y]]/(xy)$. Suppose a semiprime operation c is defined on the ideals of R.*

(1) *If c has the property that $(0)^c = (y^n)$ for some $n \geq 1$, then*

$$(y^j)^c = \begin{cases} (y^j) \textit{ for } 1 \leq j \leq n \\ (y^r) \textit{ for } j > n. \end{cases}$$

(2) *If c has the property that $(0)^c = (x^m)$ for some $m \geq 1$, then*

$$(x^i)^c = \begin{cases} (x^i) \textit{ for } 1 \leq i \leq m \\ (x^m) \textit{ for } i > m. \end{cases}$$

(3) *If c has the property that $(0)^c = (0)$ and c is weakly (x, y)-bounded, then c is both (x)-bounded and (y)-bounded and hence there exists natural number m and n such that*

$$(x^i)^c = \begin{cases} (x^i) \textit{ for } 1 \leq i \leq m \\ (x^m) \textit{ for } i > m \end{cases}$$

and

$$(y^j)^c = \begin{cases} (y^j) \textit{ for } 1 \leq j \leq n \\ (y^r) \textit{ for } j > n. \end{cases}$$

Proof. (1) Since $(0)^c = (y^n)$ then for all $j \geq n$ $(y^j)^c = (y^n)$. Suppose $(y^j)^c \neq (y^j)$ some $1 \leq j < n$. Then either $(y^i) \subseteq (y^j)^c$ for $1 \leq i < j$ or $\mathfrak{P}_{m\,j} \subseteq (y^j)^c$ for some natural number m. Both lead to contradictions from the observations below:

$$(y^{n-j+i}) = (y^{n-j})(y^i) \subseteq (y^{n-j})(y^j)^c \subseteq (y^n)^c = (y^n)$$

which cannot be the case for $i < j$.

For k such that $kj > n$,

$$\mathfrak{P}_{km\,kj} = \mathfrak{P}_{m\,j}^k \subseteq ((y^j)^k)^c \subseteq (y^n)^c = (y^n)$$

which also cannot be the case. Hence,

$$(y^j)^c = \begin{cases} (y^j) \text{ for } 1 \leq j \leq n \\ (y^n) \text{ for } j > n. \end{cases}$$

The proof of (2) is identical to the above exchanging the roles of y and x.

For (3) observe that $(0) = (x^m)(y^n)$ for all natural numbers m and n. If $(x^i)^c \neq (x^j)$ for some $j \leq i$ or $(y^i)^c \neq (y^j)$ for some $j \leq i$, then $(x^i)^c$ or $(y^j)^c$ is either R, the ring itself, an (x, y)-primary ideal $J = \mathfrak{P}_{r\,s}$ or $J = \mathfrak{A}_{r\,s}(a)$ for some natural numbers r and s and $a \in k^\times$. Since c is semiprime $(x^m)^c(y^n)^c \subseteq (0)^c$ and all above possibilities lead to a contradiction. Thus for all natural numbers i, $(x^i)^c = (x^j)$ and $(y^i)^c = (y^k)$ for some j, k natural numbers. Corollary 2.12 implies that c is (x)-bounded and (y)-bounded and gives the classification of the closures of all the ideals (x^i) and (y^j) for all natural numbers i and j. $\square$

Lemma 4.4. *Let $R = k[[x, y]]/(xy)$ and c be a weakly (x, y)-bounded semiprime operation defined on the ideals of R.*

(1) *If* $(0)^c = (x^m)$ *for some natural number* m, *then* $(y^s)^c = \mathfrak{P}^c_{m\,s}$ *for all natural numbers* s *and there exists a natural number* n *such that one of the following holds*:

(a) $(y^j)^c = R$ *for all natural numbers* j. (*In the case that* $m = 1$ *only.*)

(b) $(y^j)^c = \begin{cases} \mathfrak{P}_{m\,j} \textit{ for } 1 \le j \le n \\ \mathfrak{P}_{m\,n} \textit{ for } j > n. \end{cases}$

(c) $(y^j)^c = \begin{cases} \mathfrak{P}_{m-1\,j} \textit{ for } 1 \le j < v \textit{ for some natural number } 1 \le v \le n \\ \mathfrak{P}_{m\,j} \textit{ for } v \le j \le n \\ \mathfrak{P}_{m\,n} \textit{ for } j > n. \end{cases}$

(d) $(y^j)^c = \begin{cases} \mathfrak{P}_{m-1\,j} \textit{ for } 1 \le j \le n \\ \mathfrak{P}_{m-1\,n} \textit{ for } j > n. \end{cases}$

(e) $(y^j)^c = \begin{cases} \mathfrak{P}_{m-1\,j} \textit{ for } 1 \le j \le n \\ \mathfrak{A}_{m-1\,n}(a) \textit{ for } j > n \textit{ and some} a \in k^\times. \end{cases}$

(2) *If* $(0)^c = (y^n)$ *for some natural number* n, *then* $(x^r)^c = \mathfrak{P}^c_{r\,n}$ *for all natural numbers* r *and there exists a natural number* m *such that one of the following holds*:

(a) $(x^i)^c = R$ *for all natural numbers* i. (*In the case that* $n = 1$ *only.*)

(b) $(x^i)^c = \begin{cases} \mathfrak{P}_{i\,n} \textit{ for } 1 \le i \le m \\ \mathfrak{P}_{m\,n} \textit{ for } i > m. \end{cases}$

(c) $(x^i)^c = \begin{cases} \mathfrak{P}_{i\,n-1} \textit{ for } 1 \le i < u \textit{ for some natural number } u \le r \\ \mathfrak{P}_{i\,n} \textit{ for } u \le i \le m \\ \mathfrak{P}_{m\,n} \textit{ for } i > m. \end{cases}$

(d) $(x^i)^c = \begin{cases} \mathfrak{P}_{i\,n-1} \textit{ for } 1 \le i \le m \\ \mathfrak{P}_{m\,n-1} \textit{ for } i > m. \end{cases}$

(e) $(x^i)^c = \begin{cases} \mathfrak{P}_{i\,n-1} \textit{ for } 1 \le i \le m \\ \mathfrak{A}_{m\,n-1}(a) \textit{ for } i > m \textit{ and some } a \in k^\times. \end{cases}$

Proof. (1) For all natural numbers s, $\mathfrak{P}_{m\,s} = (0)^c + (y^s) \subseteq (y^s)^c \subseteq \mathfrak{P}^c_{m\,s}$. Thus $(y^s)^c = \mathfrak{P}^c_{m\,s}$. Since c is weakly (x, y)-bounded then c is weakly (y)-bounded by Proposition 3.6. Thus there exists a natural number n such that for all $s \ge n$ $(y^s)^c = (y^n)^c = \mathfrak{P}^c_{m\,n}$. Assume n is the smallest such natural number such that $(y^n)^c = (y^s)^c$ for all $s \ge n$.

By Lemma 4.3, $(x^r)^c = \begin{cases} (x^r) \text{ for } 1 \le r \le m \\ (x^m) \text{ for } r > m. \end{cases}$ First note that for $1 \le r < m$, $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,j}$ for some $1 \le j \le s$. If $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{i\,j}$ or $\mathfrak{P}^c_{r\,s} = \mathfrak{A}_{i\,j}(a)$ for some $1 \le i < r$ and some $1 \le j < s$, then $(x^{i+1}) = (x)\mathfrak{P}^c_{r\,s} \subseteq (x^{r+1})^c \subseteq (x^{i+1})^c$ implies that $(x^{i+1})^c = (x^{r+1})^c$ which implies that $i = r$ since $i, r < m$.

Note that $\mathfrak{P}^c_{r\,1} = \mathfrak{P}_{r\,1}$ for all $1 \le r < m$. Suppose that for $1 \le r < m$, $\mathfrak{P}^c_{r\,i} = \mathfrak{P}_{r\,j}$ for some $1 \le j < i < n$. Then $(y^n)^c = (y^{n-i})(y^i)^c = (y^{n-s})\mathfrak{P}_{r\,j} =$

$(y^{n-i+j}) \subseteq (y^n)^c$ which implies that $(y^s)^c = (y^n)^c = (y^{n-s+j})^c$ for all $s > n-s+j$ contradicting that n was the smallest such natural number. Thus for all $1 \leq r < m$ and $1 \leq s < n$ $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{r\,s}$. In particular $\mathfrak{P}^c_{m-1\,s} = \mathfrak{P}_{m-1\,s}$ for all $1 \leq s < n$.

Since $\mathfrak{P}^c_{m\,s} = (y^s)^c$ for all $1 \leq s \leq n$, we need to determine the closures of $\mathfrak{P}^c_{m\,s}$ for $1 \leq s \leq n$. For $s = n$ it must be the case that $\mathfrak{P}^c_{m\,n} = \mathfrak{P}_{m\,n}$, $\mathfrak{P}^c_{m\,n} = \mathfrak{P}_{m-1\,n}$ or $\mathfrak{P}^c_{m\,n} = \mathfrak{A}_{m-1\,n-1}(a)$ for some $a \in k^\times$ since $\mathfrak{P}_{m-1\,n-1}$ is closed and $\mathfrak{P}^c_{m\,n} \subseteq \mathfrak{P}_{m-1\,n-1}$. If $\mathfrak{P}^c_{n\,n}$ is either $\mathfrak{P}_{m-1\,n}$ or $\mathfrak{A}_{m-1\,n-1}(a)$, then since $\mathfrak{P}^c_{m\,n} \subseteq \mathfrak{P}^c_{m\,s}$ for all $1 \leq s \leq n$ we see that $\mathfrak{P}^c_{m\,s} = \mathfrak{P}_{m-1\,s}$. This covers cases (1)(d) and (1)(e) above. If $\mathfrak{P}_{m\,n}$ is closed, then we obtain either (1)(b) or (1)(c) above holds depending on whether $\mathfrak{P}_{m\,j} = \mathfrak{P}_{m-1\,j}$ for some $1 \leq j < n$. If $\mathfrak{P}_{m\,j} = \mathfrak{P}_{m-1\,j}$ for some $1 \leq j < n$, then for all $1 \leq s \leq j < n$ $\mathfrak{P}_{m\,s} = \mathfrak{P}_{m-1\,s}$ similar to how we obtained (1)(d) and (1)(e).

In the case that $m = 1$ then $(y^s)^c = \mathfrak{P}^c_{1\,s}$ for all s. Either $\mathfrak{P}_{1\,s}$ is closed for all $1 \leq s \leq n$ or $\mathfrak{P}^c_{1\,s} = R$ for all s which gives us (1)(a).

The proof of (2) is identical to that of (1) exchanging the roles of x and y. ☐

Lemma 4.5. *Let $R = k[[x,y]]/(xy)$. Suppose c is an (x,y)-bounded semiprime operation defined on the ideals of R and $(0)^c = (0)$. Then there exist natural numbers m and n such that*

$$(x^r)^c = \begin{cases} (x^r) \text{ for } 1 \leq r \leq m \\ (x^m) \text{ for } r > m \end{cases}$$

and

$$(y^s)^c = \begin{cases} (y^s) \text{ for } 1 \leq s \leq n \\ (y^s)^c = (y^n) \text{ for } s > n. \end{cases}$$

Also, precisely one of the following holds:

(1) $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m\,n}$ *for every* $r \geq m$ *and* $s \geq n$.
(2) $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m-1\,n}$ *for every* $r \geq m-1$ *and* $s \geq n$.
(3) $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m\,n-1}$ *for every* $r \geq m$ *and* $s \geq n-1$.
(4) $\mathfrak{P}^c_{r\,s} = \mathfrak{P}_{m-1\,n-1}$ *for every* $r \geq m-1$ *and* $s \geq n-1$.
(5) $\mathfrak{P}^c_{r\,s} = \mathfrak{A}_{m-1\,n-1}(a)$ *for some* $a \in k^\times$ *and for every* $r \geq m$ *and* $s \geq n$.

Proof. Since $(0)^c = (0)$, Lemma 4.3 implies that c is (x)-bounded and (y)-bounded and there exist natural numbers m and n such that

$$(x^i)^c = \begin{cases} (x^i) \text{ for } 1 \leq i \leq m \\ (x^m) \text{ for } i > m \end{cases}$$

and

$$(y^j)^c = \begin{cases} (y^j) \text{ for } 1 \leq j \leq n \\ (y^r) \text{ for } j > n. \end{cases}$$

For $r \geq m$ and $s \geq n$, $\mathfrak{P}_{m\,n} = (x^r)^c + (y^s)^c \subseteq \mathfrak{P}^c_{r\,s} \subseteq \mathfrak{P}^c_{m\,n}$. Thus, for $r \geq m$ and $s \geq n$ $\mathfrak{P}^c_{r\,s} = \mathfrak{P}^c_{m\,n}$. We need to determine $\mathfrak{P}^c_{m\,n}$. By Proposition 3.15

$\mathfrak{P}^c_{m-1\,n-1} = \mathfrak{P}_{m-1\,n-1}$, implying either $\mathfrak{P}^c_{m\,n} = \mathfrak{P}_{m\,n}$, $\mathfrak{P}^c_{m\,n} = \mathfrak{P}_{m-1\,n}$, $\mathfrak{P}^c_{m\,n} = \mathfrak{P}_{m\,n-1}$ $\mathfrak{P}^c_{m\,n} = \mathfrak{A}_{m-1\,n-1}(a)$ for some $a \in k^\times$ or $\mathfrak{P}^c_{m\,n} = \mathfrak{P}_{m-1\,n-1}$. □

At this point we would like to give a classification of the semiprime operations which are weakly-(x, y)-bounded. We will assume that the ideals $(x^0) = (y^0) = \mathfrak{P}_{0\,s} = \mathfrak{P}_{r\,0} = \mathfrak{A}_{0\,s}(a) = \mathfrak{A}_{r\,0}(a) = R$ for all non-negative integers r and s and $a \in k^\times$.

Proposition 4.6. *Let $R = k[[x, y]]/(xy)$ and c be a semiprime operation which is weakly (x, y)-bounded. Set $T \subseteq \mathbb{N}^2 \times k^\times$ satisfying the property that if $(r, s, a) \notin T$, then $(m, n, b) \notin T$ for $m < r$, $n < s$ and all $b \in k^\times$ and let m, n, i and j be non-negative integers with $0 \le i \le m$ and $1 \le j \le n$. Then c is one of the following*:

(1) *For $i < m$ and $j < n$: $c = c(m(i), n(j), T)$ where $(0)^c = (0)$,*

$$(x^r)^c = \begin{cases} (x^r) \text{ for } 0 \le r \le m \\ (x^m) \text{ for } r > m, \end{cases} \qquad (y^s)^c = \begin{cases} (y^s) \text{ for } 0 \le s \le n \\ (y^n) \text{ for } s > n, \end{cases}$$

$$\mathfrak{P}^c_{r\,s} = \begin{cases} \mathfrak{P}_{r\,s} \text{ for all } 0 \le r \le m-1 \text{ and all } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,n-1} \text{ for } 0 \le r \le i \text{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \text{ for } i < r \le m \text{ and } s \ge n \\ \mathfrak{P}_{m-1\,s} \text{ for } r \ge m-1 \text{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \text{ for } r \ge m \text{ and } j < s \le n \\ \mathfrak{P}_{m\,n} \text{ for } r > m \text{ and } s > n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \text{ for } (r, s, b) \notin T \\ \mathfrak{P}_{r\,s} \text{ for } (r, s, b) \in T \text{ and } 0 \le r \le m-1 \text{ and } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,n-1} \text{ for } 0 \le r \le i \text{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \text{ for } i < r \le m \text{ and } s \ge n \\ \mathfrak{P}_{m-1\,s} \text{ for } r \ge m-1 \text{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \text{ for } r \ge m \text{ and } j < s \le n \\ \mathfrak{P}_{m\,n} \text{ for } r \ge m \text{ and } s \ge n. \end{cases}$$

(2) *For $i < m$ and $j < n$: $c = c(m(i)', n(j), T)$ where $(0)^c = (y^n)$,*

$$(x^r)^c = \begin{cases} \mathfrak{P}_{r\,n-1} \text{ for } 0 \le r \le i \\ \mathfrak{P}_{r\,n} \text{ for } i < r \le m \\ \mathfrak{P}_{m\,n} \text{ for } r > m, \end{cases} \qquad (y^s)^c = \begin{cases} (y^s) \text{ for } 0 \le s \le n \\ (y^n) \text{ for } s > n, \end{cases}$$

$$\mathfrak{P}^c_{r\,s} = \begin{cases} \mathfrak{P}_{r\,s} \textit{ for all } 0 \le r \le m-1 \textit{ and all } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,n-1} \textit{ for } 0 \le r \le i \textit{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \textit{ for } i < r \le m \textit{ and } s \ge n \\ \mathfrak{P}_{m-1\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } r \ge m \textit{ and } j < s \le n \\ \mathfrak{P}_{m\,n} \textit{ for } r > m \textit{ and } s > n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \textit{ for } (r,s,b) \notin T \\ \mathfrak{P}_{r\,s} \textit{ for } (r,s,b) \in T \textit{ and } 0 \le r \le m-1 \textit{ and } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,n-1} \textit{ for } 0 \le r \le i \textit{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \textit{ for } i < r \le m \textit{ and } s \ge n \\ \mathfrak{P}_{m-1\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } r \ge m \textit{ and } j < s \le n \\ \mathfrak{P}_{m\,n} \textit{ for } r \ge m \textit{ and } s \ge n. \end{cases}$$

(3) *For* $i < m$ *and* $j < n$*:* $c = c(m(i), n(j)', T)$ *where* $(0)^c = (x^m)$,

$$(x^r)^c = \begin{cases} (x^r) \textit{ for } 0 \le r \le m \\ (x^m) \textit{ for } r > m, \end{cases} \qquad (y^s)^c = \begin{cases} \mathfrak{P}_{m-1\,s} \textit{ for } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } j < s \le n \\ \mathfrak{P}_{m\,n} \textit{ for } s > n, \end{cases}$$

$$\mathfrak{P}^c_{r\,s} = \begin{cases} \mathfrak{P}_{r\,s} \textit{ for all } 0 \le r \le m-1 \textit{ and all } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,n-1} \textit{ for } 0 \le r \le i \textit{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \textit{ for } i < r \le m \textit{ and } s \ge n \\ \mathfrak{P}_{m-1\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } r \ge m \textit{ and } j < s \le n \\ \mathfrak{P}_{m\,n} \textit{ for } r > m \textit{ and } s > n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \textit{ for } (r,s,b) \notin T \\ \mathfrak{P}_{r\,s} \textit{ for } (r,s,b) \in T \textit{ and } 0 \le r \le m-1 \textit{ and } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,n-1} \textit{ for } 0 \le r \le i \textit{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \textit{ for } i < r \le m \textit{ and } s \ge n \\ \mathfrak{P}_{m-1\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } r \ge m \textit{ and } j < s \le n \\ \mathfrak{P}_{m\,n} \textit{ for } r \ge m \textit{ and } s \ge n. \end{cases}$$

(4) *For* $i < m$ *and* $j < n$*:* $c = c(m(i)', n(j)', T)$ *where* $(0)^c = \mathfrak{P}_{m\,n}$,

$$(x^r)^c = \begin{cases} \mathfrak{P}_{r\,n-1} \textit{ for } 0 \le r \le i \\ \mathfrak{P}_{r\,n} \textit{ for } i < r \le m \\ \mathfrak{P}_{m\,n} \textit{ for } r > m, \end{cases} \qquad (y^s)^c = \begin{cases} \mathfrak{P}_{m-1\,s} \textit{ for } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } j < s \le n \\ \mathfrak{P}_{m\,n} \textit{ for } s > n, \end{cases}$$

$$\mathfrak{P}^c_{r\,s} = \begin{cases} \mathfrak{P}_{r\,s} \textit{ for all } 0 \le r \le m-1 \textit{ and all } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,n-1} \textit{ for } 0 \le r \le i \textit{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \textit{ for } i < r \le m \textit{ and } s \ge n \\ \mathfrak{P}_{m-1\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } r \ge m \textit{ and } j < s \le n \\ \mathfrak{P}_{m\,n} \textit{ for } r > m \textit{ and } s > n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \textit{ for } (r,s,b) \notin T \\ \mathfrak{P}_{r\,s} \textit{ for } (r,s,b) \in T \textit{ and } 0 \le r \le m-1 \textit{ and } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,n-1} \textit{ for } 0 \le r \le i \textit{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \textit{ for } i < r \le m \textit{ and } s \ge n \\ \mathfrak{P}_{m-1\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } r \ge m \textit{ and } j < s \le n \\ \mathfrak{P}_{m\,n} \textit{ for } r \ge m \textit{ and } s \ge n. \end{cases}$$

(5) *For* $j < n$: $c = c(m(m), n(j), T)$ *where* $(0)^c = (0)$,

$$(x^r)^c = \begin{cases} (x^r) \textit{ for } 0 \le r \le m \\ (x^m) \textit{ for } r > m, \end{cases} \qquad (y^s)^c = \begin{cases} (y^s) \textit{ for } 0 \le s \le n \\ (y^n) \textit{ for } s > n, \end{cases}$$

$$\mathfrak{P}^c_{r\,s} = \begin{cases} \mathfrak{P}_{r\,s} \textit{ for all } 0 \le r \le m-1 \textit{ and all } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,(n-1)} \textit{ for } 0 \le r \le m \textit{ and } s \ge n-1 \\ \mathfrak{P}_{(m-1)\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } r \ge m \textit{ and } j < s \le n-1 \\ \mathfrak{P}_{m\,(n-1)} \textit{ for } r \ge m \textit{ and } s \ge n-1, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \textit{ for } (r,s,b) \notin T \\ \mathfrak{P}_{r\,s} \textit{ for } (r,s,b) \in T, 0 \le r \le m-1 \textit{ and } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,(n-1)} \textit{ for } 1 \le r \le m \textit{ and } s \ge n-1 \\ \mathfrak{P}_{(m-1)\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } r \ge m \textit{ and } j < s \le n-1 \\ \mathfrak{P}_{m\,(n-1)} \textit{ for } r \ge m \textit{ and } s \ge n-1. \end{cases}$$

(6) *For* $j < n$: $c = c(m(m)', n(j), T)$ *where* $(0)^c = (y^n)$,

$$(x^r)^c = \begin{cases} \mathfrak{P}_{r\,n-1} \textit{ for } 0 \le r \le m \\ \mathfrak{P}_{m\,n-1} \textit{ for } r > m, \end{cases} \qquad (y^s)^c = \begin{cases} (y^s) \textit{ for } 0 \le s \le n \\ (y^n) \textit{ for } s > n, \end{cases}$$

$$\mathfrak{P}^c_{r\,s} = \begin{cases} \mathfrak{P}_{r\,s} \textit{ for all } 0 \le r \le m-1 \textit{ and all } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,(n-1)} \textit{ for } 0 \le r \le m \textit{ and } s \ge n-1 \\ \mathfrak{P}_{(m-1)\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } r \ge m \textit{ and } j < s \le n-1 \\ \mathfrak{P}_{m\,(n-1)} \textit{ for } r \ge m \textit{ and } s \ge n-1, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \textit{ for } (r,s,b) \notin T \\ \mathfrak{P}_{r\,s} \textit{ for } (r,s,b) \in T, 0 \le r \le m-1 \textit{ and } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,(n-1)} \textit{ for } 0 \le r \le m \textit{ and } s \ge n-1 \\ \mathfrak{P}_{(m-1)\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le j \\ \mathfrak{P}_{m\,s} \textit{ for } r \ge m \textit{ and } j < s \le n-1 \\ \mathfrak{P}_{m\,(n-1)} \textit{ for } r \ge m \textit{ and } s \ge n-1. \end{cases}$$

(7) *For $i < m$: $c = c(m(i), n(n), T)$ where $(0)^c = (0)$,*

$$(x^r)^c = \begin{cases} (x^r) \textit{ for } 0 \le r \le m \\ (x^m) \textit{ for } r > m, \end{cases} \qquad (y^s)^c = \begin{cases} (y^s) \textit{ for } 0 \le s \le n \\ (y^n) \textit{ for } s > n, \end{cases}$$

$$\mathfrak{P}^c_{r\,s} = \begin{cases} \mathfrak{P}_{r\,s} \textit{ for all } 0 \le r \le m-1 \textit{ and all } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,(n-1)} \textit{ for } 0 \le r \le i \textit{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \textit{ for } r > i \textit{ and } s \ge n \\ \mathfrak{P}_{(m-1)\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le n \\ \mathfrak{P}_{(m-1)\,n} \textit{ for } r \ge m-1 \textit{ and } s \ge n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \textit{ for } (r,s,b) \notin T \\ \mathfrak{P}_{r\,s} \textit{ for } (r,s,b) \in T, 0 \le r \le m-1 \textit{ and } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,(n-1)} \textit{ for } 0 \le r \le i \textit{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \textit{ for } r > i \textit{ and } s \ge n \\ \mathfrak{P}_{(m-1)\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le n \\ \mathfrak{P}_{(m-1)\,n} \textit{ for } r \ge m-1 \textit{ and } s \ge n. \end{cases}$$

(8) *For $i < m$: $c = c(m(i), n(n)', T)$ where $(0)^c = (x^m)$,*

$$(x^r)^c = \begin{cases} (x^r) \textit{ for } 0 \le r \le m \\ (x^m) \textit{ for } r > m, \end{cases} \qquad (y^s)^c = \begin{cases} \mathfrak{P}_{(m-1)\,s} \textit{ for } 0 \le s \le n \\ \mathfrak{P}_{(m-1)\,n} \textit{ for } s > n, \end{cases}$$

$$\mathfrak{P}^c_{r\,s} = \begin{cases} \mathfrak{P}_{r\,s} \textit{ for all } 0 \le r \le m-1 \textit{ and all } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,(n-1)} \textit{ for } 0 \le r \le i \textit{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \textit{ for } r > i \textit{ and } s \ge n \\ \mathfrak{P}_{(m-1)\,s} \textit{ for } r \ge m-1 \textit{ and } 0 \le s \le n \\ \mathfrak{P}_{(m-1)\,n} \textit{ for } r \ge m-1 \textit{ and } s \ge n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \text{ for } (r,s,b) \notin T \\ \mathfrak{P}_{r\,s} \text{ for } (r,s,b) \in T, 0 \le r \le m-1 \text{ and } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,(n-1)} \text{ for } 0 \le r \le i \text{ and } s \ge n-1 \\ \mathfrak{P}_{r\,n} \text{ for } r > i \text{ and } s \ge n \\ \mathfrak{P}_{(m-1)\,s} \text{ for } r \ge m-1 \text{ and } 0 \le s \le n \\ \mathfrak{P}_{(m-1)\,n} \text{ for } r \ge m-1 \text{ and } s \ge n. \end{cases}$$

(9) $c = c(m(m), n(n), T)$ *where* $(0)^c = (0)$,

$$(x^r)^c = \begin{cases} (x^r) \text{ for } 0 \le r \le m \\ (x^m) \text{ for } r > m, \end{cases} \qquad (y^s)^c = \begin{cases} (y^s) \text{ for } 0 \le s \le n \\ (y^n) \text{ for } s > n, \end{cases}$$

$$\mathfrak{P}_{r\,s}^c = \begin{cases} \mathfrak{P}_{r\,s} \text{ for all } 0 \le r \le m-1 \text{ and all } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,(n-1)} \text{ for } 0 \le r \le m-1 \text{ and } s \ge n-1 \\ \mathfrak{P}_{(m-1)\,s} \text{ for } r \ge m-1 \text{ and } 0 \le s \le n-1 \\ \mathfrak{P}_{(m-1)\,(n-1)} \text{ for } r \ge m-1 \text{ and } s \ge n-1, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \text{ for } (r,s,b) \notin T \\ \mathfrak{P}_{r\,s} \text{ for } (r,s,b) \in T, 0 \le r \le m-1 \text{ and } 0 \le s \le n-1 \\ \mathfrak{P}_{r\,(n-1)} \text{ for } 0 \le r \le m-1 \text{ and } s \ge n-1 \\ \mathfrak{P}_{(m-1)\,s} \text{ for } r \ge m-1 \text{ and } 0 \le s \le n-1 \\ \mathfrak{P}_{(m-1)\,(n-1)} \text{ for } r \ge m-1 \text{ and } s \ge n-1. \end{cases}$$

(10) $c = c_p(m, n, a)$ *where* $a \in k^\times$ *where* $(0)^c = (0)$,

$$(x^r)^c = \begin{cases} (x^r) \text{ for } 1 \le r \le m \\ (x^m) \text{ for } r > m, \end{cases} \qquad (y^s)^c = \begin{cases} (y^s) \text{ for } 1 \le s \le n \\ (y^n) \text{ for } s > n, \end{cases}$$

$$\mathfrak{P}_{r\,s}^c = \begin{cases} \mathfrak{P}_{r\,s} \text{ for } 1 \le r \le m-1 \text{ and } 1 \le s \le n-1 \\ \mathfrak{P}_{r\,(n-1)} \text{ for } 1 \le r \le m-1 \text{ and } s \ge n-1 \\ \mathfrak{P}_{(m-1)\,s} \text{ for } r \ge m-1 \text{ and } 1 \le s \le n-1 \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \text{ for } r \ge m \text{ and } s \ge n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \text{ for } 1 \le r \le m-2 \text{ and } 1 \le s \le n-2 \\ \quad \text{or } r = m-1, s = n-1 \text{ and } b = a \\ \mathfrak{P}_{r\,(n-1)} \text{ for } 1 \le r \le m-1 \text{ and } s \ge n \\ \mathfrak{P}_{(m-1)\,s} \text{ for } r \ge m \text{ and } 1 \le s \le n-1 \\ \mathfrak{P}_{(m-1)\,(n-1)} \text{ for } r = m-1, s = n-1 \text{ and } b \ne a \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \text{ for } r \ge m \text{ and } s \ge n. \end{cases}$$

(11) $c = c_p(m', n, a)$ *where* $a \in k^\times$ *where* $(0)^c = (y^n)$,

$$(x^r)^c = \begin{cases} \mathfrak{P}_{r\,(n-1)} \text{ for } 1 \le r \le m-1 \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \text{ for } r \ge m, \end{cases} \qquad (y^s)^c = \begin{cases} (y^s) \text{ for } 1 \le s \le n \\ (y^n) \text{ for } s > n, \end{cases}$$

$$\mathfrak{P}_{r\,s}^c = \begin{cases} \mathfrak{P}_{r\,s} \ \textit{for} \ 1 \leq r \leq m-1 \ \textit{and} \ 1 \leq s \leq n-1 \\ \mathfrak{P}_{r\,(n-1)} \ \textit{for} \ 1 \leq r \leq m-1 \ \textit{and} \ s \geq n-1 \\ \mathfrak{P}_{(m-1)\,s} \ \textit{for} \ r \geq m-1 \ \textit{and} \ 1 \leq s \leq n-1 \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \ \textit{for} \ r \geq m \ \textit{and} \ s \geq n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \ \textit{for} \ 1 \leq r \leq m-2 \ \textit{and} \ 1 \leq s \leq n-2 \\ \quad \textit{or} \ r = m-1, s = n-1 \ \textit{and} \ b = a \\ \mathfrak{P}_{r\,(n-1)} \ \textit{for} \ 1 \leq r \leq m-1 \ \textit{and} \ s \geq n-1 \\ \mathfrak{P}_{(m-1)\,s} \ \textit{for} \ r \geq m-1 \ \textit{and} \ 1 \leq s \leq n-1 \\ \mathfrak{P}_{(m-1)\,(n-1)} \ \textit{for} \ r = m-1, s = n-1 \ \textit{and} \ b \neq a \\ \quad \textit{or} \ r = m-1 \ \textit{and} \ s \geq n \ \textit{or} \ r \geq m \ \textit{and} \ s = n-1 \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \ \textit{for} \ r \geq m \ \textit{and} \ s \leq n. \end{cases}$$

(12) $c = c_p(m, n', a)$ *where* $a \in k^\times$ *where* $(0)^c = (x^m)$,

$$(x^r)^c = \begin{cases} (x^r) \ \textit{for} \ 1 \leq r \leq m \\ (x^m) \ \textit{for} \ r > m, \end{cases} \qquad (y^s)^c = \begin{cases} \mathfrak{P}_{(m-1)\,s} \ \textit{for} \ 1 \leq s \leq n-1 \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \ \textit{for} \ s \geq n, \end{cases}$$

$$\mathfrak{P}_{r\,s}^c = \begin{cases} \mathfrak{P}_{r\,s} \ \textit{for} \ 1 \leq r \leq m-1 \ \textit{and} \ 1 \leq s \leq n-1 \\ \mathfrak{P}_{r\,(n-1)} \ \textit{for} \ 1 \leq r \leq m-1 \ \textit{and} \ s \geq n-1 \\ \mathfrak{P}_{(m-1)\,s} \ \textit{for} \ r \geq m-1 \ \textit{and} \ 1 \leq s \leq n-1 \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \ \textit{for} \ r \geq m \ \textit{and} \ s \geq n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \ \textit{for} \ 1 \leq r \leq m-2 \ \textit{and} \ 1 \leq s \leq n-2 \\ \quad \textit{or} \ r = m-1, s = n-1 \ \textit{and} \ b = a \\ \mathfrak{P}_{r\,(n-1)} \ \textit{for} \ 1 \leq r \leq m-1 \ \textit{and} \ s \geq n-1 \\ \mathfrak{P}_{(m-1)\,s} \ \textit{for} \ r \geq m-1 \ \textit{and} \ 1 \leq s \leq n-1 \\ \mathfrak{P}_{(m-1)\,(n-1)} \ \textit{for} \ r = m-1, s = n-1 \ \textit{and} \ b \neq a \\ \quad \textit{or} \ r = m-1 \ \textit{and} \ s \geq n \ \textit{or} \ r \geq m \ \textit{and} \ s = n-1 \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \ \textit{for} \ r \geq m \ \textit{and} \ s \leq n. \end{cases}$$

(13) $c = c_p(m', n', a)$ *where* $a \in k^\times$ *where* $(0)^c = \mathfrak{A}_{m\,n}(a)$,

$$(x^r)^c = \begin{cases} \mathfrak{P}_{r\,(n-1)} \ \textit{for} \ 1 \leq r \leq m-1 \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \ \textit{for} \ r \geq m, \end{cases} \qquad (y^s)^c = \begin{cases} \mathfrak{P}_{(m-1)\,s} \ \textit{for} \ 1 \leq s \leq n-1 \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \ \textit{for} \ s \geq n, \end{cases}$$

$$\mathfrak{P}_{r\,s}^c = \begin{cases} \mathfrak{P}_{r\,s} \ \textit{for} \ 1 \leq r \leq m-1 \ \textit{and} \ 1 \leq s \leq n-1 \\ \mathfrak{P}_{r\,(n-1)} \ \textit{for} \ 1 \leq r \leq m-1 \ \textit{and} \ s \geq n-1 \\ \mathfrak{P}_{(m-1)\,s} \ \textit{for} \ r \geq m-1 \ \textit{and} \ 1 \leq s \leq n-1 \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \ \textit{for} \ r \geq m \ \textit{and} \ s \geq n, \end{cases}$$

$$\mathfrak{A}_{r\,s}(b)^c = \begin{cases} \mathfrak{A}_{r\,s}(b) \text{ for } 1 \leq r \leq m-2 \text{ and } 1 \leq s \leq n-2 \\ \quad \text{or } r = m-1, s = n-1 \text{ and } b = a \\ \mathfrak{P}_{r\,(n-1)} \text{ for } 1 \leq r \leq m-1 \text{ and } s \geq n-1 \\ \mathfrak{P}_{(m-1)\,s} \text{ for } r \geq m-1 \text{ and } 1 \leq s \leq n-1 \\ \mathfrak{P}_{(m-1)\,(n-1)} \text{ for } r = m-1, s = n-1 \text{ and } b \neq a \\ \quad \text{or } r = m-1 \text{ and } s \geq n \text{ or } r \geq m \text{ and } s = n-1 \\ \mathfrak{A}_{(m-1)\,(n-1)}(a) \text{ for } r \geq m \text{ and } s \leq n. \end{cases}$$

Proof. Suppose that c is a semiprime operation which is weakly (x, y)-bounded. Then by Proposition 3.6, c is weakly (x)-bounded and weakly (y)-bounded. There are 6 possibilities for the closure of (0). Either (I) $(0)^c = R$; (II) $(0)^c = \mathfrak{A}_{r\,s}(a)$ some r, s and $a \in k^\times$, (III) $(0)^c = \mathfrak{P}_{r\,s}$ some r and s, (IV) $(0)^c = (x^r)$ some r, (V) $(0)^c = (y^s)$ some s, or (VI) $(0)^c = (0)$.

(I) If $(0)^c = R$, the $I^c = R$ for all ideals $I \subseteq R$ by extension. Thus c corresponds to the closure defined in (9) with $m = 1 = n$ and $T = \mathbb{N}^2 \times k^\times$.

(II) Let $r = m-1$ and $s = n-1$. If $(0)^c = \mathfrak{A}_{m-1\,n-1}(a)$ for some natural numbers m and n and some $a \in k^\times$, then $\mathfrak{P}_{r\,n-1} = \mathfrak{A}_{m-1\,n-1}(a) = (0)^c \subseteq \mathfrak{P}^c_{r\,s} \subseteq \mathfrak{A}_{m-1\,n-1}(a)$ for all $r \geq m$ and $s \geq n$. Lemma 4.2 classifies the closures of $\mathfrak{P}_{r\,s}$ for all natural numbers r and s. Since $\mathfrak{A}_{m-1\,n-1}(a) + (x^r) \subseteq (x^r)^c$ for all r and $\mathfrak{P}_{m-1\,s} = \mathfrak{A}_{m-1\,n-1}(a) + (y^s) \subseteq (y^s)^c$ for all s, we obtain that c must be the closure described in (13).

(III) Let $r = m$ and $s = n$. If $(0)^c = \mathfrak{P}_{m\,n}$ for some natural numbers m and n, then for all $r > m$ and $s > n$ $\mathfrak{P}_{m\,n} = (0)^c \subseteq \mathfrak{P}^c_{r\,s} \subseteq \mathfrak{P}_{m\,n}$. Lemma 4.1 gives us a classification of the closures of $\mathfrak{P}_{r\,s}$ for all r and s. Since $\mathfrak{P}_{m\,n} = (0)^c \subseteq (x^r)^c \subseteq \mathfrak{P}^c_{r\,n}$ for all r and $\mathfrak{P}_{m\,n} = (0)^c \subseteq (y^s)^c \subseteq \mathfrak{P}^c_{m\,s}$ for all s, the we obtain the closure described in (4).

(IV) If $(0)^c = (x^r)$ for some natural number r, then by Proposition 4.3(2) gives the closures of the ideals (x^i) for all i and Lemma 4.4(1) give the closures of the ideal (y^j) for all j. Also Lemma 4.1 gives us a classification of the closures of $\mathfrak{P}_{r\,s}$ for all r and s and Lemma 3.19 determines the closures of the (x, y)-primary principal ideals and we obtain the closures described in (3), (8) and (12).

(V) If $(0)^c = (y^s)$ for some natural number s, then by Proposition 4.3(1) (y^j) for all j and Lemma 4.4(2) give the closures of (x^i) for all i. Also Lemma 4.1 gives us a classification of the closures of $\mathfrak{P}_{r\,s}$ for all r and s and Lemma 3.19 determines the closures of the (x, y)-primary principal ideals and we obtain the closures described in (2), (6) and (11).

(VI) If $(0)^c = (0)$, then since c is weakly (x, y)-bounded, then c is both weakly (x)-bounded and (y)-bound by Proposition 3.6. Lemma 4.5 classifies the closures of the ideals (x^r), (y^s) and $\mathfrak{P}_{r\,s}$ for all natural numbers r and s and the closures of the of (x, y)-primary principal ideals are easily determined using Lemma 3.19. We conclude that the closures obtained are those described in (1), (5), (7), (9) and (10). $\square$

5. Some non-Noetherian examples

Recall that the closures on the nodal curve which were both (x)-bounded and (y)-bounded, were also (x, y)-bounded. However, there are examples of closures c on commutative rings where c is P-bounded for the dimension one primes but not $\mathfrak{m}$-bounded for a maximal ideal $\mathfrak{m}$.

Example 5.1. Let $S = k[y, x, \frac{x}{y}, \frac{x}{y^2}, \ldots]$ and $\mathfrak{m} = (y, x, \frac{x}{y}, \frac{x}{y^2}, \ldots)$. Set $R = S_{\mathfrak{m}}$. R is a local 2-dimensional valuation ring and in fact $\mathfrak{m}R = (y)$ is principal. Set $\bigcap_{n\geq 1} (y^n) = (x, \frac{x}{y}, \frac{x}{y^2}, \ldots) = P$. Note that $P^{n+1} \subseteq (x^n) \subseteq (\frac{x^n}{y}) \subseteq \cdots \subseteq P^n$ and $\bigcap_{n\geq 1} P^n = (0)$. The only prime ideals in R are (0), P and $\mathfrak{m}R$. We can define a closure operation c on R which is semiprime and P-bounded but not $\mathfrak{m}R$-bounded. Suppose $(P^n)^c = P$ for all $n \geq 1$. Since the ideals $(\frac{x^n}{y})$ are sandwiched between P^{n+1} and P^n then $\left(\frac{x^n}{y}\right)^c = P$. Note that $(y^m)P^n = P^n$. Thus if $(y^m)^c = (y^m)$ for all $m \geq 1$, then

$$P = (P^n)^c = ((y^m)P^n)^c \supseteq (y^m)^c(P^n)^c = (y^m)P = P.$$

Also

$$P = \left(\frac{x^n}{y^k}\right)^c = \left((y^m)\left(\frac{x^n}{y^{m+k}}\right)\right)^c \supseteq (y^m)^c\left(\frac{x^n}{y^{m+k}}\right)^c = (y^m)P = P.$$

Thus c is P-bounded but not $\mathfrak{m}R$-bounded.

Example 5.2. Let R be as above. Define c instead to be $(y^m)^c = (y)$ for all $m \geq 1$ and $I^c = I$ for all P-primary ideals. c is an example of a closure operation which is $\mathfrak{m}R$-bounded but not P-bounded. Note that c is not a semiprime operation since $\left(\frac{x^n}{y^k}\right) = (y^m)\left(\frac{x^n}{y^{m+k}}\right)$ and

$$(y^m)^c\left(\frac{x^n}{y^{m+k}}\right)^c = \left(\frac{x^n}{y^{m+k-1}}\right) \not\subseteq \left((y^m)\left(\frac{x^n}{y^{m+k}}\right)\right)^c = \left(\frac{x^n}{y^k}\right)$$

for $m > 1$.

Example 5.3. Let R be again as above. Define c instead to be $(y^m)^c = (y)$ for all $m \geq 1$, $(P^n)^c = P^n$ for all $n \geq 1$ and $\left(\frac{x^n}{y^k}\right)^c = P^n$ for all $k \geq 1$. c is an example of a closure operation which is $\mathfrak{m}R$-bounded but not P-bounded. In comparison to the example above, c is actually semiprime since

$$(y^m)^c\left(\frac{x^n}{y^{m+k}}\right)^c = (y)P^n = P^n = \left(\frac{x^n}{y^k}\right)^c = \left((y^m)\left(\frac{x^n}{y^{m+k}}\right)\right)^c$$

for $k \geq 1$, $m \geq 1$ and $n \geq 1$.

This type of behavior of the closure operations c most likely will only be exhibited in non-Noetherian rings. In Noetherian local rings, if $\mathfrak{m}I = I$, then we know that $I = (0)$ by Nakayama's Lemma. Also for any Noetherian domain R of dimension 2 or more, R has infinitely many height 1 primes. So most likely if $(R, \mathfrak{m})$ is a local ring of dimension two or more, if P and Q are height one prime ideals such that $P + Q = \mathfrak{m}$ and c is a closure operations which is both P-bounded and Q-bounded, then c will be a closure operation which is at least weakly $\mathfrak{m}$-bounded. For example,

Example 5.4. Let $R = k[x, y]_{(x,y)}$. Suppose c is a closure operation such that $(x^m)^c = (x)$ for all $m \geq 1$ and $(y^n)^c = (y)$ for all $n \geq 1$. Then

$$(\mathfrak{m}^n)^c \supseteq (x^n, y^n)^c \supseteq (x^n)^c + (y^n)^c \supseteq (x, y) = \mathfrak{m}$$

which implies that $(\mathfrak{m}^n)^c = \mathfrak{m}^c$ for all $m, n \geq 1$. Since all $\mathfrak{m}$-primary ideals are sandwiched between powers of $\mathfrak{m}$, this implies that all $\mathfrak{m}$-primary ideals have the same closure as $\mathfrak{m}$. Thus c is weakly $\mathfrak{m}$-bounded.

However, it may be the case that $\mathfrak{m}$-boundedness of a closure operation c may not imply that c is P-bounded for primes P of dimension one even if c is semiprime.

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