J. Korean Math. Soc. **54** (2017), No. 6, pp. 1683–1698
https://doi.org/10.4134/JKMS.j160579
pISSN: 0304-9914 / eISSN: 2234-3008

THE EXTREMAL PROBLEM ON HUA DOMAIN

Sujuan Long

Abstract. In this paper, we study the Carathéodory extremal problems on the Hua domain of the first three types. We give the explicit formula for the Carathéodory extremal problems between the first three types of Hua domain and the unit ball, which improves the works done on Hua domain and Cartan-egg domain and super-Cartan domain.

1. Introduction

Let $\mathcal{M}$ and $\mathcal{N}$ be two domains in $\mathbb{C}^N$ with $p \in \mathcal{M}$ and $q \in \mathcal{N}$. Let $\mathrm{Hol}(\mathcal{M}_p, \mathcal{N}_q)$ denote the set of holomorphic maps f from $\mathcal{M}$ to $\mathcal{N}$ such that $f(p) = q$, and let $|J_f(p)|$ be the Jacobian $|\det df(p)|$ of f at the point p. We define the Carathéodory extremal value, simply C-extremal value as follows:

$$c_{(\mathcal{M},p;\mathcal{N},q)} = \sup\{|J_g(p)| : g \in Hol(\mathcal{M}_p; \mathcal{N}_q)\}. \tag{1.1}$$

A map $f \in \mathrm{Hol}(\mathcal{M}_p, \mathcal{N}_q)$ is said to be Carathéodory extremal map (C-extremal map), if

$$|J_f(p)| = c_{(\mathcal{M},p;\mathcal{N},q)}. \tag{1.2}$$

The classical extremal problem is to find Carathéodory extremal maps from a domain to the unit ball in $\mathbb{C}^N$, and the problem is an analogue of the classical Schwarz Lemma [14]. It was first studied by Carathéodory in [1], where he obtained the explicit formula for the C-extremal mapping from the unit polydisc into the unit ball. Kubota and Travaglini [2–5] got the explicit formula for the C-extremal mapping from symmetric domains to the ball. Ma [8, 9] obtained the C-extremal mapping from the complex ellipsoids to the ball. Recently, with the construction of the super-Cartan domain, Cartan-egg domain and Hua domain given by Yin in [16–18], many researches have been done on the classical extremal problem from these classes of domains to the ball (see [6,10,11,15,19]).

Received September 1, 2016; Revised April 2, 2017; Accepted August 4, 2017.
2010 *Mathematics Subject Classification.* 32H02, 32F45.
Key words and phrases. extremal problem, Hua domain, minimal circumscribed Hermitian ellipsoid.
This research was supported by the Natural Science Foundation of Guangxi (Grant No. 2015jjBA10049), Research Foundation of Fujian Province for Young and Middle-aged Teachers (Grant No. JAT170467), Research Foundation of Minjiang University for the Introduction of Talents.

Let D be a domain in $\mathbb{C}^N$. We say that D is a balanced domain if $\lambda z \in D$ for $z \in D$ and $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$. It is clear that $0 \in D$. The classical extremal problem between two balanced domains D_1 and D_2 in $\mathbb{C}^N$ which maps 0 to 0 was studied by Carathéodory [1], who proved the following theorem.

Theorem 1.1. *Assume D_1, D_2 are two balanced domains in $\mathbb{C}^N$ with D_2 a domain of holomorphy. If $f \in Hol((D_1;0);(D_2;0))$, then $df(0)(D_1) \subset D_2$. Moreover,*

$$c_{(D_1,0;D_2,0)} = \sup\{|\det(l)| : l \text{ complex linear map};\ l(D_1) \subset D_2\}. \tag{1.3}$$

If $D_1 = D$ is a holomorphically convex balanced domain in $\mathbb{C}^N$, $D_2 = B_N$, for simplicity, we let

$$\begin{aligned} c_D &:= c_{(D,0;B_N,0)} \\ &= \sup\{|\det A| : A \in M^{(N,N)}(\mathbb{C}),\ \|Az\| < 1 \text{ for all } z \in D\}, \end{aligned} \tag{1.4}$$

where $M^{(N,N)}(\mathbb{C})$ is the set of all $N \times N$ matrices with entries in $\mathbb{C}$.

One of the balanced holomorphically convex domain is Hua domain. Let $n \ge 2$ be a positive integer, and for $j = 1, \dots, n-1$, let N_j be a positive integer and $p_j > 0$ be a real number. We use the following notation

$$\mathbf{N} = (N_1, \dots, N_{n-1}),\ \mathbb{C}^{\mathbf{N}} = \mathbb{C}^{N_1} \times \cdots \times \mathbb{C}^{N_{n-1}} \ \text{ and } \ \mathbf{p} = (p_1, \dots, p_{n-1}). \tag{1.5}$$

It is well known that the first three types of Hua domain are:

$$\begin{aligned} HE_{\mathcal{A}}(n, \mathbf{N}, \mathbf{p}; q, \ell) := \{&(w_1, \dots, w_{n-1}, Z) \in \mathbb{C}^{\mathbf{N}} \times R_{\mathcal{A}}(q,\ell) : \\ &\|w_1\|^{2p_1} + \cdots + \|w_{n-1}\|^{2p_{n-1}} < \det(I - ZZ^*)\}, \end{aligned} \tag{1.6}$$

where $R_{\mathcal{A}}(q,\ell)$ is the classical bounded symmetric domain of type $\mathcal{A}$ ($\mathcal{A} = I, II, III$) (see [7]), consisting of elements in $M^{(q,\ell)}(\mathbb{C})$, the set of all $q \times \ell$ matrices with $q \le \ell$.

Su and Yin [13] solved the C-extremal problem for $HE_I(n, \mathbf{N}, \mathbf{p}; q, \ell)$ and obtained the C-extremal map from $HE_I(n, \mathbf{N}, \mathbf{p}; q, \ell)$ into the unit ball in $\mathbb{C}^{\mathbf{N}} \times \mathbb{C}^{q\ell}$ when $p_j > q$. For the special case $n = 3$, Su and Li [12] considered the first type of Cartan-egg domain.

$$\begin{aligned} CE_I(k, q, \ell) \ = \{&(w_1, w_2, Z) \in \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times R_I(q,\ell) : \\ &\|w_1\|^2 + \|w_2\|^{2k} < \det(I - ZZ^*)\}. \end{aligned}$$

They obtained the explicit formula for the extremal map from $CE_I(k,q,\ell)$ to the unit ball in $\mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \mathbb{C}^{q\ell}$ with $k > 0$.

Main purpose of the paper is to solve the extremal problem for general $\mathbf{p} = (p_1, \dots, p_{n-1})$ with $p_j > 0$ and $n \ge 2$, which improves the works done by Park [10], Su and Yin [13], Su and Li [12], Yin and Su [19] and Wang et al. [15]. Our main results will be stated in Section 3.

2. Preliminaries

Let $A = [a_{ij}]$ be an $N \times N$ self-adjoint positive definite matrix over $\mathbb{C}$. The Hermitian ellipsoid is defined as follows:

$$E_A := \{z \in \mathbb{C}^N : \sum_{j,k=1}^{n} a_{jk} z_j \overline{z}_k < 1\}. \tag{2.1}$$

The following basic properties of Hermitian ellipsoid were given in [9, 15].

Proposition 2.1. *Let A and B be two self-adjoint positive definite matrices over $\mathbb{C}$. Then*

$$Vol(E_A) = \frac{1}{\det A} Vol(B_N). \tag{2.2}$$

Moreover, if $E_A = E_B$, then $A = B$.

Proposition 2.2 ([9])**.** *Let D be a domain of dimension N, containing* 0. *If l is a complex linear map such that $l(D) \subset B_N$, then $l^{-1}(B_N)$ is an Hermitian ellipsoid containing D. If l is a solution of the extremal problem*

$$|\det l| = \sup\{|\det l_1| : l_1 \text{ complex linear map; } l_1(D) \subset B_N\}, \tag{2.3}$$

then $l^{-1}(B_N)$ is a circumscribed Hermitian ellipsoid of D of least volume, or a minimal circumscribed Hermitian ellipsoid.

Proposition 2.3 ([9])**.** *Let D be a bounded balanced domain. Then D has minimal circumscribed Hermitian ellipsoids, and the minimal circumscribed Hermitian ellipoid of D is unique.*

Proposition 2.4 ([9])**.** *For two bounded domains D_1 and D_2, let $P(D_j)$ $(j = 1, 2)$ be the minimal circumscribed Hermitian ellipsoid of D_j. If $l \in GL(N; \mathbb{C})$ and $D_2 = l(D_1)$, then $l(P(D_1)) = P(D_2)$, where $GL(N; \mathbb{C})$ denotes the set of $N \times N$ invertible matrices in $\mathbb{C}$ and is called N-th-order general linear group.*

Let I_i^m denote $m \times m$ diagonal matrix whose ith diagonal element is -1, others being 1. Let I_{ij}^m be the $m \times m$ matrix obtained by exchanging ith row and jth row from the identity matrix I^m.

Proposition 2.5. *Let $D \subset \mathbb{C}^{N_1 \times N_2 \times \cdots \times N_n}$ be a balanced holomorphically convex domain such that*

$$I_i^N D \subset D, \quad I_{k,j}^N D \subset D \tag{2.4}$$

for $1 \leq i \leq N$, $\sum_{i=0}^{\ell-1} N_i < j, k \leq \sum_{i=0}^{\ell} N_i$ for $1 \leq \ell \leq n$, where $N = \sum_{j=1}^{n} N_j$. Then the minimal circumscribed Hermitian ellipsoid of D is the ellipsoid of the following form:

$$\{(Z_1, \ldots, Z_n) \in \mathbb{C}^{\mathbf{N}} \times \mathbb{C}^{N_n} : a_1\|Z_1\|^2 + \cdots + a_n\|Z_n\|^2 < 1\}, \tag{2.5}$$

where $a_1, \ldots, a_n > 0$.

Proof. Let E_A be the minimal circumscribed Hermitian ellipoid of D with a self-adjoint positive definite matrix A. By Proposition 2.4 and (2.4), one has that

$$E_{I_i^N A I_i^N} = E_A.$$

By Proposition 2.1, one has $A = I_i^N A I_i^N$ for all $1 \le i \le N$, which implies that A is a diagonal matrix. By (2.4), one has that

$$E_{(I_{kj}^N)^* A I_{kj}^N} = E_A \text{ for all } \sum_{i=0}^{\ell-1} N_i < k, j \le \sum_{i=0}^{\ell} N_i,\ \ell = 1, \dots, n.$$

By Proposition 2.1 again, one has

$$(I_{kj}^N)^* A I_{kj}^N = A \text{ for all } \sum_{i=0}^{\ell-1} N_i < k, j \le \sum_{i=0}^{\ell} N_i,\ \ \ell = 1, \dots, n.$$

Since A is diagonal, it is easy to verify that

$$a_\ell := a_{jj} = a_{kk} \text{ if } \sum_{i=0}^{\ell-1} N_i \le k, j \le \sum_{i=0}^{\ell} N_i,\ \ell = 1, \dots, n.$$

This proves that

$$E_A = \{(Z_1, \dots, Z_n) : \sum_{j=1}^{n} a_j \|Z_j\|^2 < 1\},$$

where $a_1, \dots, a_n > 0$. The proof of the lemma is complete. □

Corollary 2.6. *If D is a holomorphically convex balanced domain in $\mathbb{C}^N$ with $N = \sum_{j=1}^n N_j$ and satisfies (2.4), then the minimal circumscribed Hermitian ellipsoid of D is given*

$$E_{(a_1,\dots,a_n)} = \{(Z_1, \dots, Z_n) \in \mathbb{C}^{\mathbf{N}} \times \mathbb{C}^{N_n} : a_1\|Z_1\|^2 + \cdots + a_n\|Z_n\|^2 < 1\}, \tag{2.6}$$

where $a_1, \dots, a_n > 0$ are uniquely determined by

$$D \subset E_{(a_1,\dots,a_n)} \tag{2.7}$$

and $\prod_{j=1}^n a_j^{N_j}$ attains its maximum (this is equivalent to $c_D^2 = \prod_{j=1}^n a_j^{N_j}$).

Proof. The proof of this corollary follows directly from Proposition 2.5 and the fact that

$$\begin{aligned} c_D^2 = \max\{&\det A : A \text{ is positive definite and } E_A \text{ is Hermitian ellipsoid},\\ &D \subset E_A\}. \end{aligned} \tag{2.8}$$

When $E_{(a_1,\dots,a_n)}$ is the minimal circumscribed Hermitian ellipsoid of D given by (2.6), one has

$$c_D^2 = a_1^{N_1} \cdots a_n^{N_n}. \tag{2.9}$$

□

For $A, B \in M^{N,N}(\mathbb{C})$, let $A \sim B :=A$ is similar with B. We define

$$D_h := \left\{ (Z_1, \ldots, Z_n) \in \mathbb{C}^{\mathbf{N}} \times \mathbb{C}^{N_n};\ \begin{array}{l} f(Z_1, \ldots, Z_{n-1}, t_1, \ldots, t_m) < 0, \\ Z_n Z_n^* \sim cD[t_1, t_2, \ldots, t_m] \end{array} \right\}, \tag{2.10}$$

where c is positive constant, $D[t_1, t_2, \ldots, t_m]$ denote $m \times m$ diagonal matrix whose ith diagonal element is t_i, $t_i \in [0, 1)$ $(j = 1, \ldots, m)$,

$$f(Z_1, \ldots, Z_{n-1}, t_1, \ldots, t_m) = \sum_{j=1}^{n-1} \|Z_j\|^{2p_j} - \prod_{k=1}^{m} (1 - t_k). \tag{2.11}$$

Remark 1. We may check that D_h defined by (2.10) and (2.11) is a holomorphically convex balanced domain, and the Hua domain defined by (1.6) can be identified with D_h if we view Z_n as a matrix in $R_{\mathcal{A}}$ $(\mathcal{A} = I, II, III)$, and the eigenvalues of $Z_n Z_n^*$ as $t_1, \ldots, t_m$.

Next we will find the minimal circumscribed Hermitian ellipsoid of D_h. By applying Proposition 2.5 and Corollary 2.6 to D_h, the minimal circumscribed Hermitian ellipsoid of D_h has the form given by (2.6). In order to find the minimal $E_{(a_1, \ldots, a_n)}$, we only need to find the maximum value of $c_D^2 = a_1^{N_1} \cdots a_n^{N_n}$. Let $\|Z_j\|^2 = r_j$, then

$$f(r_1, \ldots, r_{n-1}, t_1, \ldots, t_m) = \sum_{j=1}^{n-1} r_j^{p_j} - \prod_{k=1}^{m} (1 - t_k)$$

is the defining function of D_h.

Now we are considering the extremal problem with $f(r, t) \geq 0$:

$$\begin{cases} \text{Minimize: } f(r_1, \ldots, r_{n-1}, t_1, \ldots, t_m) = \displaystyle\sum_{j=1}^{n-1} r_j^{p_j} - \prod_{j=1}^{m} (1 - t_j), \\ \text{Subject to: } \displaystyle\sum_{j=1}^{n-1} a_j r_j + c a_n (t_1 + \cdots + t_m) = 1,\ 0 \leq r_j \leq 1,\ 0 \leq t_j \leq 1. \end{cases} \tag{2.12}$$

Notice that when $t_1 + \cdots + t_m$ is fixed, we know $\prod_{j=1}^{m} (1 - t_j)$ attains its maximum at where $t_1 = t_2 = \cdots = t_m =: t$. Therefore, to solve Problem (2.12), it is equivalent to solve

$$\begin{cases} \text{Minimize: } f(r, t) := f(r_1, \ldots, r_{n-1}, t) = \displaystyle\sum_{j=1}^{n-1} r_j^{p_j} - (1 - t)^m, \\ \text{Subject to: } g(r, t) := \displaystyle\sum_{j=1}^{n-1} a_j r_j + c a_n m t = 1,\ 0 \leq r_j \leq 1,\ 0 \leq t \leq 1. \end{cases} \tag{2.13}$$

Assume that Problem (2.13) takes its minimum value 0 at point $(r^0, t^0) := (r_1^0, \ldots, r_{n-1}^0, t^0)$.

Let ∂D denote the boundary of domain D. Then (r^0, t^0) must be located at where ∂D_h and $\partial E_{(a_1, \ldots, a_n)}$ meets tangentially. Notice that $f(r, t)$ is the defining function of D_h, so that

$$f(r_1^0, \ldots, r_{n-1}^0, t^0) = \Big(\sum_{j=1}^{n-1} r_j^{p_j} - (1-t)^m\Big)\Big|_{(r^0, t^0)} = 0. \tag{2.14}$$

And a_j $(1 \le j \le n)$ as well as t^0 can be expressed as functions of $r_1^0, \ldots, r_{n-1}^0$, which are uniquely determined so that $\prod_{j=1}^n a_j^{N_j}$ is maximum. Finally, we obtain the minimal $E_{(a_1,\ldots,a_n)}$ of D_h.

Proposition 2.7. *If $E_{(a_1,\ldots,a_n)}$ is a circumscribed Hermitian ellipsoid for D_h defined in* (2.10) *and* (2.11)*, then*

(i) $a_0 := cma_n \le 1$;
(ii) $a_j \le 1$ *for* $1 \le j \le n-1$;
(iii) *If* $a_j = 1$*, then* $a_0 \le \frac{m}{p_j}$;
(iv) *If* $p_j \le 1$ *for* $1 \le j \le n-1$*, then* $E_{(1,\ldots,1,\frac{1}{mc})}$ *is the minimal circumscribed Hermitian ellipsoid for* D_h.

Proof. If $a_0 > 1$, we choose $t_0 = \frac{1}{a_0} < 1$ such that $cma_nt_0 = 1$. Then $(0, t_0) = (0, \ldots, 0, t_0) \in \partial E_{(a_1,\ldots,a_n)}$, but

$$f(0, t_0) = -(1-t_0)^m < 0,$$

which contradicts the assumption $E_{(a_1,\ldots,a_n)}$ is the circumscribed Hermitian ellipsoid for D_h. So Part (i) is proved.

Similarly, one can prove Part (ii).

For Part (iii), we consider $(0, \ldots, 0, r_j, 0, \ldots, 0, t)$ satisfying

$$r_j + a_0t = 1.$$

Then

$$\begin{aligned} h(r_j, t) :&= f(0, \ldots, 0, r_j, 0, \ldots, 0, t) \\ &= r_j^{p_j} - (1-t)^m = (1-a_0t)^{p_j} - (1-t)^m \\ &= 1 - p_ja_0t + O(t^2) - 1 + mt \\ &= (m - p_ja_0)t + O(t^2) \\ &\ge 0. \end{aligned}$$

This implies that

$$m - p_ja_0 \ge 0 \quad \text{or} \quad a_0 \le m/p_j.$$

This proves (iii).

To prove (iv), we take $a_j = 1$ for $1 \le j \le n-1$ and $a_n = \frac{1}{cm}$. Then

$$f(r_1, \ldots, r_{n-1}, t) = \sum_{j=1}^{n-1} r_j^{p_j} - (1-t)^m \ge \sum_{j=1}^{m} r_j - (1-t)^m = (1-t) - (1-t)^m \ge 0,$$

where $\sum_{j=1}^{n-1} r_j + t = 1$. Therefore, this ellipsoid $E_{(1,\ldots,1,\frac{1}{cm})}$ is the minimal circumscribed Hermitian ellipsoid for D_h. □

Proposition 2.8. *If $n = 2$, then the following statements hold.*

(i) *If $0 < p_1 \le m$, then the minimal circumscribed Hermitian ellipsoid of D_h is $E_{(1,\frac{1}{mc})}$.*

(ii) *If $p_1 > m$, then $E_{(a_1,\frac{1}{cm})}$ with $a_1 \le 1$ is not the minimal circumscribed Hermitian ellipsoid of D_h.*

Proof. By Proposition 2.7, we have $a_1 \le 1$ and $a_2 \le \frac{1}{cm}$. Then $a_1^{N_1} a_2^{N_2} \le (cm)^{-N_2}$. Let $a_1 = 1$ and $a_2 = \frac{1}{cm}$. Then on the boundary of $E_{(1,\frac{1}{cm})}$, we have

$$r_1 = 1 - t.$$

Since $p_1 \le m$, we have

$$f(r_1, t) = (1-t)^{p_1} - (1-t)^m \ge 0, \quad r_1, t \in [0,1].$$

Therefore, $E_{(1,1/mc)}$ is the minimal circumscribed Hermitian ellipsoid of D_h.

If $a_1 = 1$, Part (ii) was proved by Yin and Su [19]. When $a_1 < 1$, for the points (r_1, t) satisfying $a_1 r_1 + t = 1$, it is easy to check that there exist $r_1 \in (0, a_1^{\frac{m}{p_1 - m}})$ and $t \in (0,1)$ such that $a_1 r_1 + t = 1$, but $r_1^{p_1} - (1-t)^m < 0$. Thus $E_{(a_1,\frac{1}{cm})}$ can not be the minimal circumscribed Hermitian ellipsoid of D_h. □

Proposition 2.9. *If $n \ge 2$, $E_{(1,a_2,\dots,a_n)}$ with $a_n < \frac{1}{cm}$ is the circumscribed Hermitian ellipsoid of D_h, then there exists another D_h's circumscribed Hermitian ellipsoid $E_{(b,a_2,\dots,a_{n-1},a)}$ with $b < 1$, $a_n < a < \frac{1}{cm}$ and $b^{N_1} a^{N_n} > a_n^{N_n}$.*

Proof. Considering $(r_1, \dots, r_{n-1}, t) \in \partial E_{(b,a_2,\dots,a_{n-1},a)}$ and

$$br_1 + a_2 r_2 + \dots + a_{n-1} r_{n-1} + cmat = 1, \quad 1/cm > a > a_n, b < 1. \tag{2.15}$$

We need to find a and b such that $b^{N_1} a^{N_n} > a_2^{N_n}$ and

$$\sum_{j=1}^{n-1} r_j^{p_j} - (1-t)^m \ge 0 \text{ for } (r_1, \dots, r_{n-1}, t) \in (0, \dots, 0, 1) \text{ satisfying (2.15).}$$

Let $a_0 = cma_n$, $a' = cma$. We want to find $x \in (0,1)$ such that

$$\begin{aligned} 1 &= br_1 + \sum_{j=2}^{n-1} a_j r_j + a' t \\ &= xr_1 + \sum_{j=2}^{n-1} a_j r_j + a_0 \Big[1 - \Big((1 - |x|^{p_1}) \sum_{j=2}^{n-1} r_j^{p_j} + x^{p_1} (1-t)^m \Big)^{1/m} \Big]. \end{aligned} \tag{2.16}$$

Notice that (2.16) holds if and only if

$$a_0 \Big[1 - \Big((1 - |x|^{p_1}) \sum_{j=2}^{n-1} r_j^{p_j} + x^{p_1} (1-t)^m \Big)^{1/m} \Big] + \sum_{j=2}^{n-1} a_j r_j + xr_1 - 1 = 0. \tag{2.17}$$

Next we show that the equality of (2.17) has a positive solution x. Let

$$h(x) := a_0\Big[1 - \Big((1-|x|^{p_1})\sum_{j=2}^{n-1} r_j^{p_j} + x^{p_1}(1-t)^m\Big)^{1/m}\Big] + \sum_{j=2}^{n-1} a_j r_j + x r_1 - 1.$$

Since $h(0) = -1 + a_0 < 0$ and $br_1 + \sum_{j=2}^{n-1} a_j r_j + a't = 1$ implies that

$$r_1 \geq \frac{1 - a_2 - \cdots - a_{n-1} - a'}{b}.$$

If $b < \frac{1-a'-\sum_{j=2}^{n-1} a_j}{1-a_0-\sum_{j=2}^{n-1} a_j}$, we have

$$\begin{aligned} h(1) &= r_1 + \sum_{j=2}^{n-1} a_j r_j + a_0 t - 1 \\ &= r_1 + br_1 + \sum_{j=2}^{n-1} a_j r_j + a't - br_1 - a't + a_0 t - 1 \\ &= (1-b)r_1 - (a' - a_0)t \\ &\geq (1-b)\frac{1 - a_2 - \cdots - a_{n-1} - a'}{b} - (a' - a_0) > 0. \end{aligned}$$

Let

$$r_1' = x r_1 \in (0,1),\ \ r_j' = r_j \text{ for } j = 2, \ldots, n-1, \tag{2.18}$$

$$t' = 1 - \Big((1-|x|^{p_1})\sum_{j=2}^{n-1} r_j^{p_j} + x^{p_1}(1-t)^m\Big)^{1/m}.$$

Then there exists $x \in (0, 1/r)$ such that $r_1' + \sum_{j=2}^{n-1} a_j r_j' + a_0 t' = 1$. Thus

$$0 \leq \sum_{j=1}^{n-1} r_j'^{p_j} - (1-t')^m = x^{p_1}(\sum_{j=1}^{n-1} r_j^{p_j} - (1-t)^m), \tag{2.19}$$

which implies that $\sum_{j=1}^{n-1} r_j^{p_j} - (1-t)^m \geq 0$ for $(r,t) \in \partial E_{(b,a_2,\ldots,a_{n-1},a)}$.

Moreover, for any $(r_1', \ldots, r_{n-1}', t') \in (0,1]^n$ such that $r_1' + \sum_{j=2}^{n-1} a_j r_j' + a_0 t' = 1$ and $\sum_{j=1}^{n-1} r_j'^{p_j} - (1-t')^m = 0$. We can reverse the above process by choosing $x > 0$ such that

$$\tilde{r}_1 = r_1' x^{-1},\ \ \tilde{r}_j = r_j' \text{ for } j = 2, \ldots, n-1,$$

$$\tilde{t} = 1 - \Big(x^{-p_1}(1-t')^m - (x^{-p_1} - 1)\sum_{j=2}^{n-1} r_j'^{p_j}\Big)^{1/m} \in [0,1].$$

Moreover, there exist a', b such that

$$1 = b\tilde{r}_1 + \sum_{j=2}^{n-1} a_j \tilde{r}_j + a'\tilde{t}$$

$$= x\tilde{r}_1 + a_0\Big[1 - \Big((1-|x|^{p_1})\sum_{j=2}^{n-1}\tilde{r}_j^{p_j} + x^{p_1}(1-\tilde{t})^m\Big)^{1/m}\Big].$$

By (2.18) and (2.19), we have

$$\Big(\sum_{j=1}^{n-1} {r_j}^{p_j} - (1-t)^m\Big)\Big|_{(\tilde{r}_1,\ldots,\tilde{r}_{n-1},\tilde{t})} = 0.$$

For any $a_0 < a' < 1$, we choose

$$1 > \frac{1 - a' - \sum_{j=2}^{n-1} a_j}{1 - a_0 - \sum_{j=2}^{n-1} a_j} > b > \Big(\frac{a_0}{a'}\Big)^{N_2/N_1}.$$

Then

$$a^{N_2} b^{N_1} > a_2^{N_2}.$$

The proof of Part (iii) is complete. □

To solve Problem (2.13), we will divide our argument into the following cases:

Case 1: $(r^0, t^0) \in [0,1)^n$.

By the Lagrange multiplier method, the minimizers of extremal problem (2.13) must be located in the solutions of

$$\begin{cases} p_j r_j^{p_j-1} = a_j\lambda, j = 1,\ldots,n-1; (1-t)^{m-1} = ca_n\lambda, \\ \sum_{j=1}^{n-1} a_j r_j + ca_n mt = 1. \end{cases} \tag{2.20}$$

From (2.20), we have

$$p_j r_j^{p_j} = a_j r_j \lambda, \quad 1 \le j \le n-1; \quad mt(1-t)^{m-1} = cma_n t\lambda, \tag{2.21}$$

and

$$\sum_{j=1}^{n-1} p_j r_j^{p_j} + mt(1-t)^{m-1} = \lambda. \tag{2.22}$$

Therefore,

$$a_j = p_j r_j^{p_j-1}/\lambda,\ j = 1,\ldots,n-1; \quad a_n = c^{-1}(1-t)^{m-1}/\lambda. \tag{2.23}$$

Let

$$\begin{aligned} F(r_1,\ldots,r_{n-1},t) &:= \det A = a_1^{N_1}\cdots a_n^{N_n} \\ &= \prod_{j=1}^{n-1}\Big(\frac{p_j r_j^{p_j-1}}{\lambda}\Big)^{N_j}\big(\frac{(1-t)^{m-1}}{c\lambda}\big)^{N_n} \\ &= C_D\prod_{j=1}^{n-1} r_j^{N_j(p_j-1)}(1-t)^{N_n(m-1)}\lambda^{-N}, \end{aligned} \tag{2.24}$$

where

$$C_D = \prod_{j=1}^{n-1} p_j^{N_j} c^{-N_n}, \quad N = \sum_{j=1}^{n} N_j. \tag{2.25}$$

By (2.14) and (2.22), we have

$$\begin{aligned}\lambda &= \sum_{j=1}^{n-1} p_j r_j^{p_j} + m(1 - (\sum_{j=1}^{n-1} r_j^{p_j})^{1/m})(\sum_{j=1}^{n-1} r_j^{p_j})^{\frac{m-1}{m}} \\ &= \sum_{j=1}^{n-1}(p_j - m) r_j^{p_j} + m(\sum_{j=1}^{n-1} r_j^{p_j})^{\frac{m-1}{m}}.\end{aligned} \tag{2.26}$$

Let

$$y_j = r_j^{p_j}, \quad j = 1, 2, \ldots, n-1. \tag{2.27}$$

Then

$$\lambda = \sum_{j=1}^{n-1} p_j y_j - m \sum_{j=1}^{n-1} y_j + m(\sum_{j=1}^{n-1} y_j)^{(m-1)/m}. \tag{2.28}$$

Thus

$$\frac{\partial \lambda}{\partial y_j} = p_j - m + (m-1)(\sum_{j=1}^{n-1} y_j)^{-1/m}. \tag{2.29}$$

Then

$$\sum_{j=1}^{n-1} y_j \frac{\partial \lambda}{\partial y_j} = \lambda - (\sum_{j=1}^{n-1} y_j)^{(m-1)/m}. \tag{2.30}$$

Thus

$$G := \log F = -N \log \lambda + \log C_D + \sum_{j=1}^{n-1} N_j(1-1/p_j) \log y_j + \frac{m-1}{m} N_n \log(\sum_{j=1}^{n-1} y_j) \tag{2.31}$$

and

$$\frac{\partial G}{\partial y_j} = -\frac{N}{\lambda}\frac{\partial \lambda}{\partial y_j} + \frac{1}{y_j} N_j(1-1/p_j) + \frac{m-1}{m} N_n \frac{1}{\sum_{j=1}^{n-1} y_j} = 0. \tag{2.32}$$

Thus

$$-N y_j \frac{\partial \lambda}{\partial y_j} + N_j(1-1/p_j)\lambda + \frac{m-1}{m}\frac{y_j}{\sum_{j=1}^{n-1} y_j} N_n \lambda = 0. \tag{2.33}$$

Summing up (2.33) for $j = 1, \ldots, n-1$, one has

$$-N\lambda + N(\sum_{j=1}^{n-1} y_j)^{(m-1)/m} + \sum_{j=1}^{n-1} N_j(1-1/p_j)\lambda + \frac{m-1}{m} N_n \lambda = 0. \tag{2.34}$$

Therefore,

$$-\left[\frac{N_n}{m}+\sum_{j=1}^{n-1}\frac{N_j}{p_j}\right]\lambda+N(\sum_{j=1}^{n-1}y_j)^{(m-1)/m}=0 \tag{2.35}$$

and

$$\lambda=\frac{N(\sum_{j=1}^{n-1}y_j)^{(m-1)/m}}{\frac{N_n}{m}+\sum_{j=1}^{n-1}\frac{N_j}{p_j}}=\frac{Nb^{(m-1)/m}}{N(p)}, \tag{2.36}$$

where

$$b=\sum_{j=1}^{n-1}y_j \quad \text{and } N(p):=\frac{N_n}{m}+\sum_{j=1}^{n-1}\frac{N_j}{p_j}. \tag{2.37}$$

By (2.29), (2.33) and (2.37), one has

$$-Ny_j[(p_j-m)+(m-1)b^{-1/m}]+N_j(1-1/p_j)\lambda+\frac{m-1}{m}\frac{N_n\lambda}{b}y_j=0. \tag{2.38}$$

By (2.36) and (2.38), one has

$$y_j[N(p_j-m)+N(m-1)b^{-1/m}-\frac{m-1}{m}\frac{N_nNb^{-1/m}}{N(p)}]=N_j(1-1/p_j)\lambda. \tag{2.39}$$

Therefore,

$$\begin{aligned} y_j&=\frac{N_j(1-1/p_j)b}{N(p)(p_j-m)b^{1/m}+(m-1)-\frac{m-1}{m}\frac{N_n}{N(p)}} \\ &=\frac{N_j(1-1/p_j)b}{N(p)(p_j-m)b^{1/m}+(m-1)\sum_{k=1}^{n-1}N_k/p_k}. \end{aligned} \tag{2.40}$$

Summing up (2.40) for $j=1,\dots,n-1$, we have

$$b=\sum_{j=1}^{n-1}\frac{N_j(1-1/p_j)b}{N(p)(p_j-m)b^{1/m}+(m-1)\sum_{k=1}^{n-1}N_k/p_k}.$$

Then

$$\sum_{j=1}^{n-1}\frac{N_j/p_j(p_j-1)}{N(p)(p_j-m)b^{1/m}+(m-1)\sum_{k=1}^{n-1}N_k/p_k}=1. \tag{2.41}$$

It is clear that

$$b^{1/m}=\frac{\sum_{k=1}^{n-1}N_k/p_k}{N(p)} \tag{2.42}$$

is a solution of (2.41).

Let

$$I=\{1,\dots,n-1\},\quad J=\{j:p_j=1,j\in I\}. \tag{2.43}$$

By (2.40) and (2.42), one has

$$y_j = \frac{N_j}{p_j N(p)}\Big(\frac{\sum_{k=1}^{n-1} N_k/p_k}{N(p)}\Big)^{m-1}, \quad j \in I \setminus J. \tag{2.44}$$

By (2.23), (2.27), (2.36) and (2.44), we have

$$\begin{aligned} a_j &= p_j\Big[\frac{N_j}{p_j N(p)}\Big(\frac{\sum_{k=1}^{n-1} N_k/p_k}{N(p)}\Big)^{m-1}\Big]^{1-1/p_j} \frac{N(p)}{N}\Big(\frac{N(p)}{\sum_{k=1}^{n-1} N_k/p_k}\Big)^{m-1} \\ &= \frac{p_j^{1/p_j} N_j^{1-1/p_j} N(p)^{m/p_j}}{N}\Big(\sum_{k=1}^{n-1} N_k/p_k\Big)^{-(m-1)/p_j}, \quad j \in I \setminus J, \end{aligned} \tag{2.45}$$

and

$$a_n = \frac{(1-t)^{m-1}}{c\lambda} = \frac{b^{\frac{(m-1)}{m}}}{c\frac{N}{N(p)} b^{(m-1)/m}} = \frac{N(p)}{cN}. \tag{2.46}$$

For $j \in J$, by (2.23), we have $a_j = 1/\lambda$. Thus (2.36) and (2.42) yield

$$a_j = \frac{N(p)^m}{N}\Big(\sum_{k=1}^{n-1} N_k/p_k\Big)^{(1-m)} \quad j \in J. \tag{2.47}$$

Case 2. Either $t^0 = 1$ or there exist some $r_j^0 = 1$

1. If $t^0 = 1$, by (2.13), one has that $r_j^0 = 0$ for all $1 \le j \le n-1$, then

$$cma_n = 1. \tag{2.48}$$

When $mN(p) < N$, one can easily see that there exists at least one $p_j > m$, we may assume that $p_1 > m$. For any point $(r_1, 0, \dots, 0, t)$ satisfying $a_1 r_1 + t = 1$, by (2.13), we have $r_1^{p_1} - (1-t)^m \ge 0$, which is impossible by the proof of Part (ii) of Proposition 2.8. Therefore, this case can not happen when $mN(p) < N$.

2. If there exist some $r_j^0 = 1$, then by (2.13), there is only one $r_j^0 = 1$, we may also assume that $r_1^0 = 1$, then $r_k^0 = 0$ for $k \neq 1$ and $t^0 = 0$ such that

$$a_1 = 1. \tag{2.49}$$

When $mN(p) < N$, by the discussion above, we have $cma_n < 1$. For $n \ge 2$, Then $E_{(1,a_2,\dots,a_n)}(a_n < \frac{1}{cm})$ is the minimal circumscribed Hermitian ellipsoid of D_h, which contradicts statement of Proposition 2.9. Therefore, this case also can not happen.

As a summary, we have proved the following theorem.

Theorem 2.10. *Let m, n, $N_1, \dots, N_n$ be positive integers and $p_1, \dots, p_{n-1}$ are positive real numbers such that $mN(p) < N$ and (2.41) has unique solution b given by (2.42). Let D_h be defined by (2.10) and (2.11). Then the minimal circumscribed Hermitian ellipsoid of D_h is given by*

(2.50)

$$E_{(a_1,\dots,a_n)} := \{(Z_1, \dots, Z_n) \in \mathbb{C}^{N_1} \times \dots \times \mathbb{C}^{N_n} : a_1\|Z_1\|^2 + \dots + a_n\|Z_n\|^2 < 1\},$$

where a_j *are given by* (2.45)-(2.47).

Remark 2. (i) If we substitute $p_j = 1$ into (2.45), then we obtain (2.47) eventually.

(ii) Let

$$\begin{aligned} J_1 &:= \{j \,|\, p_j < 1,\ j \in I\}, \\ J_2 &:= \{j \,|\, 1 < p_j < m,\ j \in I\}, \\ J_3 &:= \{j \,|\, p_j > m,\ j \in I\}. \end{aligned} \tag{2.51}$$

By (2.41), we let

$$H(t) = \sum_{j=1}^{n-1} \frac{N_j/p_j(p_j-1)}{N(p)(p_j-m)t + (m-1)\sum_{k=1}^{n-1} N_k/p_k}. \tag{2.52}$$

Notice that

$$H'(t) = N(p) \sum_{j=1}^{n-1} \frac{N_j/p_j(p_j-1)(p_j-m)}{[N(p)(p_j-m)t + (m-1)\sum_{k=1}^{n-1} N_k/p_k]^2}. \tag{2.53}$$

If $I \setminus J = J_i$ for either $i = 1, 2, 3$, then $H(t)$ is monotonic function and (2.41) has unique solution b given by (2.42).

3. Explicit formula for C-extremal maps

Let

$$N_n = \begin{cases} q\ell, & \text{if } \mathcal{A} = I; \\ \frac{q(q+1)}{2}, & \text{if } \mathcal{A} = II; \\ \frac{q(q-1)}{2}, & \text{if } \mathcal{A} = III. \end{cases} \tag{3.1}$$

$$m = \begin{cases} q, & \text{if } \mathcal{A} = I \text{ or } II; \\ 2[\frac{q}{2}], & \text{if } \mathcal{A} = III. \end{cases} \qquad c = \begin{cases} 1, & \text{if } \mathcal{A} = I \text{ or } II; \\ \frac{1}{2}, & \text{if } \mathcal{A} = III. \end{cases} \tag{3.2}$$

By Remark 1 and Theorem 2.10, we can obtain the minimal circumscribed Hermitian ellipsoid of Hua domain, which promotes us to give the C-extremal map for the Hua domain in this section. When $m = 1$, Hua domain is ellipsoid, we may consult [8] for discussion of the C-extremal problem. Therefore, in this section, we may assume that $m > 1$.

We are going to view $HE_{\mathcal{A}}(n, \mathbf{N}, \mathbf{p}, q, \ell)$ as a domain in $\mathbb{C}^{\mathbf{N}} \times \mathbb{C}^{N_n}$ and $R_{\mathcal{A}}(q, \ell)$ ($\mathcal{A} = I, II, III$) as a domain in $\mathbb{C}^{N_n}$ as follows:

(i) For $Z \in R_I(q, \ell)$, we let

$$z = (z_{11}, \ldots, z_{1q}, z_{21}, \ldots, z_{2q}, \ldots, z_{q\ell}) \in \mathbb{C}^{q\ell} \tag{3.3}$$

and $\|z\|^2 = \|Z\|^2 = \mathrm{tr}(ZZ^*)$.

(ii) For $X = [x_{jk}] \in R_{II}(q)$, we consider

$$\hat{R}_{II}(q) := \{Z : x_{jk} = \frac{z_{jk}}{\sqrt{2}p_{jk}}, X \in R_{II}(q)\}, \tag{3.4}$$

where

$$p_{jk} = \begin{cases} 1, & \text{if } j \neq k; \\ \frac{1}{\sqrt{2}}, & \text{if } j = k. \end{cases} \tag{3.5}$$

For $Z \in \hat{R}_{II}(q)$, let

$$z = (z_{11}, \ldots, z_{1q}, z_{22}, \ldots, z_{2q}, \ldots, z_{qq}) \in \mathbb{C}^{\frac{q(q+1)}{2}}. \tag{3.6}$$

Then

$$\|z\|^2 = \|Z\|^2 = \mathrm{tr}(XX^*).$$

Instead of $HE_{II}(n, \mathbf{N}, \mathbf{p}, q)$, we define

$$\hat{HE}_{II}(n, \mathbf{N}, \mathbf{p}; q) := \Big\{(w_1, \ldots, w_{n-1}, Z) \in \mathbb{C}^{\mathbf{N}} \times \hat{R}_{II}(q) : \|w_1\|^{2p_1} + \cdots + \|w_{n-1}\|^{2p_{n-1}} < \det(I - ZZ^*)\Big\}, \tag{3.7}$$

where $Z = [z_{jk}] \in \hat{R}_{II}(q)$.

(iii) For $Z \in R_{III}(q)$, we let

$$z = (z_{12}, \ldots, z_{1q}, z_{23}, \ldots, z_{2q}, \ldots, z_{(q-1)q}) \in \mathbb{C}^{\frac{q(q-1)}{2}} \tag{3.8}$$

and $2\|z\|^2 = 2\|Z\|^2 = \mathrm{tr}(ZZ^*)$.

Let I, J be defined by (2.43). Now we can state the main results of the paper. By Proposition 2.2, Theorem 2.10 and Remark 2(i), It is easy to prove the following theorem.

Theorem 3.1. *Let $HE_{\mathcal{A}}(n, \mathbf{N}, \mathbf{p}, q, \ell)$ $(\mathcal{A} = I, III)$ (or $\hat{HE}_{II}(n, \mathbf{N}, \mathbf{p}; q)$) be Hua domain defined by* (1.6) *(or* (3.7)*) with $N = \sum_{j=1}^{n} N_j$ and $p_1, \ldots, p_{n-1}$ are positive real numbers. If $mN(p) < N$ and* (2.41) *has unique solution b given by* (2.42)*. Then the C-extremal map $F : HE_{\mathcal{A}}(n, \mathbf{N}, \mathbf{p}, q, \ell)$ (or $\hat{HE}_{II}(n, \mathbf{N}, \mathbf{p}; q)$) $\to B_N$ is given by*

$$F(w_1, \ldots, w_{n-1}, Z) = (\sqrt{a_1} w_1, \ldots, \sqrt{a_{n-1}} w_{n-1}, \sqrt{a_n} Z), \tag{3.9}$$

where

$$a_j = \frac{p_j^{1/p_j} N_j^{1-1/p_j} N(p)^{1/p_j}}{N} \Big(\sum_{k=1}^{n-1} N_k/p_k\Big)^{-(m-1)/p_j} \quad j \in I; \qquad a_n = \frac{N(p)}{cN}. \tag{3.10}$$

Theorem 3.2. *Let $HE_{II}(n, \mathbf{N}, \mathbf{p}, q)$ be Hua domain defined by* (1.6) *and $p_1, \ldots, p_{n-1}$ are positive real numbers, $mN(p) < N$ and* (2.41) *has a unique solution b given by* (2.42)*. Let $F(w_1, \ldots, w_{n-1}, Z)$ be the map defined by* (3.9)-(3.10) *and*

$$h : HE_{II}(n, \mathbf{N}, \mathbf{p}, q) \to \hat{HE}_{II}(n, \mathbf{N}, \mathbf{p}, q), \tag{3.11}$$

$$(w_1, \ldots, w_{n-1}, [x_{jk}]) \mapsto (w_1, \ldots, w_{n-1}, [\sqrt{2} p_{jk} x_{jk}]). \tag{3.12}$$

Then the C-extremal map $T(w_1,\ldots,w_{n-1},X) : HE_{II}(n,\mathbf{N},\mathbf{p},q) \to B_N$ *is given by*

$$T = F \circ h. \tag{3.13}$$

Proof. By Theorem 3.1, the C-extremal map $F(w_1,\ldots,w_{n-1},Z)$ from $\hat{HE}_{II}(n,\mathbf{N},\mathbf{p};q)$ to B_N is given by (3.9)-(3.10). Let $X = [x_{jk}] \in R_{II}(q)$, $[Z] = [\sqrt{2}p_{jk}x_{jk}] \in \hat{R}_{II}(q)$, we consider

$$h : HE_{II}(n,\mathbf{N},\mathbf{p},q) \to \hat{HE}_{II}(n,\mathbf{N},\mathbf{p},q)$$

$$(w_1,\ldots,w_{n-1},[x_{jk}]) \mapsto (w_1,\ldots,w_{n-1},[\sqrt{2}p_{jk}x_{jk}]).$$

Obviously, h is a biholomorphic mapping from $HE_{II}(n,\mathbf{N},\mathbf{p},q)$ to $\hat{HE}_{II}(n,\mathbf{N},\mathbf{p},q)$. Let $T = F\circ h$, then $T(w_1,\ldots,w_{n-1},X)$ is the C-extremal holomorphic map from $HE_{II}(n,\mathbf{N},\mathbf{p};q)$ to B_N. □

By Theorem 3.1 and Remark 2, it is easy to prove the following corollary.

Corollary 3.3. *Let* $HE_{\mathcal{A}}(n,\mathbf{N},\mathbf{p},q,\ell)$ $(\mathcal{A} = I, III)$ *(or* $\hat{HE}_{II}(n,\mathbf{N},\mathbf{p};q)$*) be Hua domain defined by* (1.6) *(or* (3.7)*) with* $N = \sum_{j=1}^n N_j$. *Then the C-extremal holomorphic map* $F : HE_{\mathcal{A}}(n,\mathbf{N},\mathbf{p},q,\ell)$ *(or* $\hat{HE}_{II}(n,\mathbf{N},\mathbf{p};q)$*)* $\to B_N$ *is given by*

$$F(w_1,\ldots,w_{n-1},Z) = (\sqrt{a_1}w_1,\ldots,\sqrt{a_{n-1}}w_{n-1},\sqrt{a_n}Z).$$

Let $K = \{k \,|\, p_k = m, k \in I\}$. *If* $I \setminus K = J_3$ *is non-empty or* $I \setminus J = J_3$ *such that* $mN(p) < N$, *then* (3.9)-(3.10) *hold.*

Remark 3. For Hua domain of the first type $HE_I(n,\mathbf{N},\mathbf{p},q,\ell)$ with $N = \sum_{j=1}^n N_j$.

(i) When all $p_j > m$, the results of [10] and [13] are included in statement of Corollary 3.3.

(ii) When $n = 2$, $p_1 = k$, the results of [19] and [15] are included in statements of Proposition 2.8 and Corollary 3.3.

(iii) When $HE_I(n,\mathbf{N},\mathbf{p},q,\ell) = CE_I(k,q,\ell)$ with $n = 3$, $N_2 = q\ell$, $p_1 = 1, p_2 = k$, the result of [12] is included in Proposition 2.7 and Corollary 3.3.

Acknowledgements. The author gratefully acknowledge her appreciation to professor Song-Ying Li for his supervision.

References

[1] C. Carathéodory, *Über die Abbildungen, die durch Systeme von analytischen Funktionen von mehreren Veranderlichen erzeugt werden*, Math. Z. **34** (1932), no. 1, 758–792.

[2] Y. Kubota, *An extremal problem on the classical Cartan domains*, Kodai Math. J. **4** (1981), no. 2, 278–287.

[3] ———, *An extremal problem on the classical Cartan domains. II*, Kodai Math. J. **5** (1982), no. 2, 218–224.

[4] ———, *An extremal problem on the classical Cartan domains. III*, Kodai Math. J. **5** (1982), no. 3, 402–407.

[5] ———, *An extremal problem on bounded symmetric domains*, Bull. London Math. Soc. **15** (1983), no. 2, 126–130.
[6] H. T. Li, Y. Y. Wang, and J. B. Su, *Extremal problem on the second type of Cartan-egg domain*, Adv. Math. **42** (2013), no. 2, 243–255.
[7] Q. K. Lu, *The Classical Manifolds and Classical Domains*, Shanghai Scienti Technical Publishers, 1963.
[8] D.-W. Ma, *Boundary behavior of invariant metrics and volume forms on strongly pseudoconvex domains*, Duke Math. J. **63** (1991), no. 3, 673–697.
[9] ———, *Carathéodory extremal maps of ellipsoids*, J. Math. Soc. Japan **49** (1997), no. 4, 723–739.
[10] J. D. Park, *A new algorithmic method for finding the Caratheodory extremal maps*, J. Math. Anal. Appl. **349** (2009), no. 2, 475–483.
[11] J. B. Su, *A note on extremal problems on Cartan domains*, J. Henan Univ. (Natural Sci.) **36** (2006), DOI: 10.3969/j.issn.1003-4978.2006.04.001.
[12] J. B. Su and H. T. Li, *The Extremal problem on Cartan-egg domain of the first type*, Chin. Quart. J. Math. **26** (2011), 343–349.
[13] J. B. Su and W. P. Yin, *Extremal problems on the Hua domain of the first type*, Sci. China Ser. A Math. **48** (2005), suppl., 343–354.
[14] G. Travaglini, *An analogue of the Schwarz lemma for bounded symmetric domains*, Proc. Amer. Math. Soc. J. **83** (1983), no. 1, 85–88.
[15] A. Wang, X. Zhao, and Z. Y. Liu, *Extremal problems on the Cartan-Hartogs domains*, J. Korean Math. Soc. **44** (2007), no. 6, 1291–1312.
[16] W. P. Yin, *The Bergman kernels on Cartan-Hartogs domains*, Chinese Sci. Bull. **44** (1999), no. 21, 1947–1951.
[17] ———, *The Bergman kernels on CartanEgg domain of the first type*, Northeast. Math. J. **17** (2001), no. 2, 210–220.
[18] ———, *The Bergman kernel functions on four types of Cartan-egg domains*, Adv. Math. (China) **30** (2001), no. 6, 533–542.
[19] W. P. Yin and J. B. Su, *Extremal problems on super-Cartan domain of the first type*, Complex Var. Theory Appl. **48** (2003), no. 5, 441–452.

Sujuan Long
Department of Mathematics
Minjiang University
Fuzhou 350108, P. R. China
and
School of Mathematics and Computing Science
Guilin University of Electronic Technology
Guilin 541004, P. R. China
E-mail address: lsj_math@163.com