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HERMITIAN POSITIVE DEFINITE SOLUTIONS OF THE MATRIX EQUATION $X^s + A^*X^{-t}A = Q$

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Abstract. In this paper, the Hermitian positive definite solutions of the matrix equation $X^s + A^*X^{-t}A = Q$, where Q is an $n \times n$ Hermitian positive definite matrix, A is an $n \times n$ nonsingular complex matrix and $s, t \in [1, \infty)$ are discussed. We find a matrix interval which contains all the Hermitian positive definite solutions of this equation. Also, a necessary and sufficient condition for the existence of these solutions is presented. Iterative methods for obtaining the maximal and minimal Hermitian positive definite solutions are proposed. The theoretical results are illustrated by numerical examples.

1. Introduction and preliminaries

We consider Hermitian positive definite solutions of the nonlinear matrix equation

$$X^s + A^*X^{-t}A = Q, \tag{1.1}$$

where, A is an $n \times n$ nonsingular complex matrix, Q is an $n \times n$ Hermitian positive definite matrix and $s, t \in [1, \infty)$.

This form of the nonlinear matrix equation and same configuration to them, can be appeared in control theory [11, 13], ladder networks [2, 3], dynamic programming [19], quantum mechanics [17], stochastic filtering and statistics [5]. The existence of Hermitian positive definite solutions of the matrix equation (1.1), has been investigated in some special cases. The case $s = t = 1$ has been systematically investigated by several authors [2, 3, 10, 11]. The cases $s = 1, t \in \mathbb{N}$ in [16], $s = 1, t \in (0, \infty)$ in [18, 20], $s = 1, t \geq 1$ in [9], $s, t \in \mathbb{N}$ in [6, 7, 8, 21, 22] and $s > 0, t > 0$ in [24] have been studied.

In this paper, we consider the Hermitian positive definite solutions of the matrix equation (1.1), where $s \geq 1$ and $t \geq 1$. Also, we find a matrix interval

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which contains all the Hermitian positive definite solutions of the matrix equation (1.1). Indeed by using the Brouwer's fixed point theorem [1, Theorem 4.3] and the Banach's fixed point theorem [1, Theorem 1.1], we obtain sufficient conditions regarding to the existence and uniqueness of the Hermitian positive definite solutions of equation (1.1). Also, we obtain a necessary and sufficient condition for the existence of these solutions. Iterative methods for obtaining the extremal Hermitian positive definite solutions of the matrix equation (1.1) are presented. Moreover, we show that [8, Theorem 2.2], [9, Theorem 2.2], [14, Theorem 4], and [22, Theorem 2.2] are not formulated correctly, because some of the assumptions are vacuous, see Section 2. Finally, the theoretical results are illustrated by numerical examples.

The following notations are used throughout this paper. The notations M_n denotes the algebra of $n \times n$ complex matrices. For $A \in M_n$, the matrices A^T and A^* denote the transpose and conjugate transpose of A, respectively. The symbol I denotes the $n \times n$ identity matrix. Let A be an $m \times n$ matrix and B be an $p \times q$ matrix. Then the Kronecker product A and B denoted by $A \otimes B$ that is the $mp \times nq$ block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \cdots & \cdots & \cdots \\ a_{n1}B & \cdots & a_{mn}B \end{bmatrix}$$

If $m = p$ and $n = q$, then $A \circ B$ denotes Schur product A and B with elements given by $(A \circ B)_{ij} = (A)_{ij}(B)_{ij}$. For Hermitian matrix A, we write $A \geq 0$ ($A > 0$), if A is a positive semi-definite (definite) matrix. For two Hermitian matrices A and B, the notation $A \geq B$ ($A > B$) means that $A - B \geq 0$ ($A - B > 0$). We define a matrix interval by $[A, B] = \{X \mid A \leq X \leq B\}$ and $(A, B) = \{X \mid A < X < B\}$. Symbols $\|A\|$ and $\|A\|_F$ are used, respectively, for the spectral norm and Frobenius norm. Let A be a nonsingular matrix. We indicate the condition number of A with $\operatorname{cond}(A)$. Let $\{\lambda_i(A)\}_{i=1}^n$ be the spectrum of a Hermitian matrix A. Then we assume that $\lambda_n(A) \leq \cdots \leq \lambda_2(A) \leq \lambda_1(A)$. We use the notations X_S and X_L for the minimal and maximal Hermitian positive definite solutions of equation (1.1), respectively. By the HPD solution of Eq. (1.1), we mean the Hermitian positive definite solution of equation (1.1).

Let $A > 0$ and $A = U^*DU$ be the spectral decomposition of the matrix A. We define $A^r := U^*D^rU$, where $r \in \mathbb{R}$. In the following, we state inequalities between A^r and B^r, where $0 < A \leq B$ and $r \in \mathbb{R}$.

Lemma 1.1 ([23, Theorem 1.1](Löwner-Heinz))**.** *If* $0 \leq A \leq B$ *and* $0 \leq r \leq 1$, *then* $0 \leq A^r \leq B^r$.

Lemma 1.2 ([12, Theorem 2.1])**.** *Let* A *and* B *be positive operators on a Hilbert space* $\mathcal{H}$ *such that* $M_1I \geq A \geq m_1I > 0$, $M_2I \geq B \geq m_2I > 0$ *and*

$0 < A \le B$. *Then, for all* $r \ge 1$

$$A^r \le \left(\frac{M_1}{m_1}\right)^{r-1} B^r, \; A^r \le \left(\frac{M_2}{m_2}\right)^{r-1} B^r. \tag{1.2}$$

Using [4, Proposition V.1.6], we find the similar inequalities as Lemma 1.1 and Lemma 1.2 with opposite direction, for $r \in (-\infty, 0)$.

Now, we are going to find some bounds for $\|A^r - B^r\|_F$, where $r \in \mathbb{R}$. Let J be an open interval in $\mathbb{R}$. We say that $f \in \mathcal{C}^1(J)$, if the real function f is continuously differentiable on J.

Theorem 1.3. *Let* $f \in \mathcal{C}^1(J)$ *and* $[\alpha, \beta] \subset J$. *If* $A, B \in [\alpha I, \beta I]$, *then*

$$\|f(A) - f(B)\|_F \le \max_{\alpha \le c \le \beta} |f'(c)| \, \|A - B\|_F. \tag{1.3}$$

Proof. Suppose that $A, B \in [\alpha I, \beta I]$ and $L_t = tA + (1-t)B$ for all $0 \le t \le 1$. Then $L_t \in [\alpha I, \beta I]$. Using [4, Theorem X.4.5], we have

$$\|f(A) - f(B)\|_F \le \sup_{0 \le t \le 1} \|\mathcal{D}f(L_t)\|_F \|A - B\|_F,$$

where $\mathcal{D}f(A)$ is denoted the Frechet derivative of the function f at A. Let $L_t = U_t D_t U_t^*$ for all $0 \le t \le 1$, where D_t and U_t are diagonal and unitary matrices, respectively. Suppose that $f^{[1]}(A)$ is denoted the first divided difference of f at A [4, p. 123]. Then, by using [4, Theorem V.3.3] and the mean value theorem,

$$\begin{aligned}
\|\mathcal{D}f(L_t)\|_F &= \sup_{\|H\|_F = 1} \|\mathcal{D}f(L_t)(H)\|_F = \sup_{\|H\|_F = 1} \|f^{[1]}(D_t) \circ U_t^* H U_t\|_F \\
&\le \sup_{\|H\|_F = 1} \left(\max_{i,j} \left| \left(f^{[1]}(D_t)\right)_{ij} \right| \|H\|_F \right) = \max_{i,j} \left| \left(f^{[1]}(D_t)\right)_{ij} \right| \\
&\le \max_{\lambda_n(L_t) \le c \le \lambda_1(L_t)} |f'(c)| \le \max_{\alpha \le c \le \beta} |f'(c)|,
\end{aligned}$$

where $\circ$ is denoted the Schur product. Therefore,

$$\|f(A) - f(B)\|_F \le \max_{\alpha \le c \le \beta} |f'(c)| \, \|A - B\|_F. \qquad \square$$

Corollary 1.4. *Let* $A, B \in [\alpha I, \beta I]$ *and* $\alpha > 0$. *Then*

$$r\alpha^{r-1}\|A - B\|_F \le \|A^r - B^r\|_F \le r\beta^{r-1}\|A - B\|_F; \; r \ge 1, \tag{1.4}$$

$$r\beta^{r-1}\|A - B\|_F \le \|A^r - B^r\|_F \le r\alpha^{r-1}\|A - B\|_F; \; 0 < r \le 1. \tag{1.5}$$

Proof. Let $r > 0$ and $f(x) = x^r$ be defined on the interval $J := (0, \infty)$. So $f \in \mathcal{C}^1(J)$ and is increasing on J. We know that f is convex (concave) for $r \ge 1$ $(0 < r \le 1)$. Therefore $\max_{\alpha \le c \le \beta} |f'(c)| = f'(\beta)$ for $r \ge 1$ and $\max_{\alpha \le x \le \beta} |f'(x)| = f'(\alpha)$ for $0 < r \le 1$. Using (1.3), the right hand side of (1.4) and (1.5) are derived. By replacing $A^r \to A$, $B^r \to B$, and $\frac{1}{r} \to r$, the left hand sides of the inequalities (1.4) and (1.5) are obtained by the right hand sides of the inequalities (1.5) and (1.4), respectively. $\square$

Corollary 1.5. *Let* $A, B \in [\alpha I, \beta I]$ *and* $\alpha > 0$*. Then for all* $r \in (-\infty, 0)$*,*

$$-r\beta^{r-1}\|A-B\|_F \leq \|A^r - B^r\|_F \leq -r\alpha^{r-1}\|A-B\|_F. \tag{1.6}$$

Proof. Let $r \in (-\infty, 0)$ and $f(x) = x^r$ be defined on the interval $J := (\alpha, \infty)$ with $\alpha > 0$. We have, $f \in \mathcal{C}^1(J)$ and f is convex and decreasing on J. Therefore, $\max\limits_{\alpha \leq c \leq \beta} |f'(c)| = -f'(\alpha) = -r\alpha^{r-1}$. Hence, by using (1.3), we have

$$\|A^r - B^r\|_F = \|f(A) - f(B)\|_F \leq -r\alpha^{r-1}\|A-B\|_F.$$

By replacing $A^r \to A$, $B^r \to B$, and $\frac{1}{r} \to r$, the left hand side of the inequality (1.6) is obtained by the right hand side of (1.6) and [4, Proposition V.1.6]. □

2. Necessary conditions and sufficient conditions

Let X be an HPD solution of Eq. (1.1) and $s, t \in [1, \infty)$. It is readily seen that

$$X \in \left[\left(AQ^{-1}A^*\right)^{\frac{1}{t}}, Q^{\frac{1}{s}}\right]. \tag{2.1}$$

This interval was obtained in [22, Theorem 2.1] for $s, t \in N$. Now, we are going to obtain a better interval for HPD solutions of Eq. (1.1).

Theorem 2.1. *Let* $F(P) = \left(A\left(Q - \operatorname{cond}(P)^{1-s}P^s\right)^{-1}A^*\right)^{\frac{1}{t}}$ *and* X *be an HPD solution of Eq.* (1.1)*. Then*

$$\left(AQ^{-1}A^*\right)^{\frac{1}{t}} < F\left(\left(AQ^{-1}A^*\right)^{\frac{1}{t}}\right) \leq X.$$

Proof. Let X be an HPD solution of Eq. (1.1). First, we will show that for all matrix P such that the conditions (i) $0 < P \leq X$ and (ii) $Q < \operatorname{cond}(P)^{1-s} P^s + A^*P^{-t}A$ hold, then $P < F(P) \leq X$.

Let $0 < P \leq X$. Since $\lambda_n(P)I \leq P \leq \lambda_1(P)I$, by using (1.2), we obtain that

$$\begin{aligned} P^s &\leq \left(\frac{\lambda_1(P)}{\lambda_n(P)}\right)^{s-1} X^s = \left(\lambda_1(P)\lambda_1\left(P^{-1}\right)\right)^{s-1} X^s \\ &= \left(\|P\|\,\|P^{-1}\|\right)^{s-1} X^s = \operatorname{cond}(P)^{s-1}\, X^s. \end{aligned}$$

Since X is an HPD solution of Eq. (1.1),

$$F(P) = \left(A\left(Q - \operatorname{cond}(P)^{1-s}P^s\right)^{-1}A^*\right)^{\frac{1}{t}} \leq \left(A(Q - X^s)^{-1}A^*\right)^{\frac{1}{t}} = X.$$

Thus, $F(P) \leq X$.

Now, let $\operatorname{cond}(P)^{1-s}P^s + A^*P^{-t}A > Q$. Therefore,

$$F(P) = \left(A\left(Q - \operatorname{cond}(P)^{1-s}P^s\right)^{-1}A^*\right)^{\frac{1}{t}} > \left(P^t\right)^{\frac{1}{t}} = P$$

and so $P < F(P) \leq X$.

Choose $P = \left(AQ^{-1}A^*\right)^{\frac{1}{t}}$. By using (2.1), we obtain that $0 < P \leq X$. Also, we have

$$\operatorname{cond}(P)^{1-s} P^s + A^* P^{-t} A = \operatorname{cond}(P)^{1-s} P^s + Q > Q.$$

Therefore, $P = \left(AQ^{-1}A^*\right)^{\frac{1}{t}}$ holds in conditions (i) and (ii). Hence

$$\left(AQ^{-1}A^*\right)^{\frac{1}{t}} < F\left(\left(AQ^{-1}A^*\right)^{\frac{1}{t}}\right) \leq X. \qquad \square$$

The matrix X is an HPD solution of Eq. (1.1) if and only if $Y := X^{-1}$ is an HPD solution of Eq.

$$Y^t + A^{-*}Y^{-s}A^{-1} = A^{-*}QA^{-1}. \tag{2.2}$$

Remark 2.2. We see that Eq. (2.2) is the same as Eq. (1.1) by replacing $A^{-*}QA^{-1} \to Q$, $A^{-1} \to A$, $t \to s$, and $s \to t$.

We are using auxiliary Eq. (2.2) to find an upper bound for HPD solutions of Eq. (1.1) which is sharper than (2.1).

Theorem 2.3. *Let X be an HPD solution of Eq.* (1.1)*. Then*

$$X \leq G\left(Q^{\frac{1}{s}}\right) < Q^{\frac{1}{s}},$$

where $G(P) = \left(Q - \operatorname{cond}(P)^{1-t} A^* P^{-t} A\right)^{\frac{1}{s}}$.

Proof. Let X be an HPD solution of Eq. (1.1). Therefore $Y := X^{-1}$ is an HPD solution of Eq. (2.2). Using Remark 2.2 and Theorem 2.1, we have $Q^{\frac{-1}{s}} < F(Q^{\frac{-1}{s}}) \leq Y$, where

$$\begin{aligned} F(P) &= \left(A^{-1}\left(A^{-*}QA^{-1} - \operatorname{cond}(P)^{1-t}P^t\right)^{-1}A^{-*}\right)^{\frac{1}{s}} \\ &= \left(Q - \operatorname{cond}(P)^{1-t}A^*P^tA\right)^{\frac{-1}{s}}. \end{aligned}$$

By choosing $G(P) = F^{-1}\left(P^{-1}\right)$, the proof is completed. $\square$

Corollary 2.4. *Let F and G be the same as in Theorem 2.1 and Theorem 2.3, respectively. If X is an HPD solution of Eq.* (1.1)*, then*

$$X \in \left[F\left(\left(AQ^{-1}A^*\right)^{\frac{1}{t}}\right), G\left(Q^{\frac{1}{s}}\right)\right] \subset \left[\left(AQ^{-1}A^*\right)^{\frac{1}{t}}, Q^{\frac{1}{s}}\right].$$

By using Corollary 2.4, we will present an iterative method for obtaining the minimal (maximal) HPD solution of Eq. (1.1), when $t \geq s \geq 1$ ($s \geq t \geq 1$), in Section 3.

In the following, we study sufficient conditions for the existence of HPD solutions of Eq. (1.1). Some sufficient conditions, for various values of $s, t \in [1, \infty)$, was presented in [14, Theorem 4], [8, Theorem 2.2], [9, Theorem 2.2], and [22, Theorem 2.2]. But some of the assumptions of these theorems are vacuous, because, by choosing $X = \left(AQ^{-1}A^*\right)^{\frac{1}{t}}$, we obtain that $A^*X^{-t}A = Q >$

$Q-\left(AQ^{-1}A^*\right)^{\frac{s}{t}}$. Therefore, we can not assume $A^*X^{-t}A\leq Q-\left(AQ^{-1}A^*\right)^{\frac{s}{t}}$ for all $X\in\left[\left(AQ^{-1}A^*\right)^{\frac{1}{t}},P\right]$. Now, in the following, we are going to improve these results.

Theorem 2.5. *Let $s,t\in[1,\infty)$ and there exist $k>1$ such that*

$$\lambda_1\left(Q^{-\frac{1}{2}}\left(AQ^{-1}A^*\right)^{\frac{s}{t}}Q^{-\frac{1}{2}}\right)\leq\left(1-k^{-1}\right)k^{\frac{-s}{t}}. \tag{2.3}$$

Then, $\left(kAQ^{-1}A^*\right)^{\frac{1}{t}}\leq Q^{\frac{1}{s}}$. Moreover, if $X^t\geq k\left(AQ^{-1}A^*\right)$ for all $X\in\Omega:=\left[\left(kAQ^{-1}A^*\right)^{\frac{1}{t}},Q^{\frac{1}{s}}\right]$, then Eq. (1.1) *has an HPD solution in Ω.*

Proof. Using (2.3), we obtain that

$$Q^{-\frac{1}{2}}\left(AQ^{-1}A^*\right)^{\frac{s}{t}}Q^{-\frac{1}{2}}\leq\lambda_1\left(Q^{-\frac{1}{2}}\left(AQ^{-1}A^*\right)^{\frac{s}{t}}Q^{-\frac{1}{2}}\right)I\leq\left(1-k^{-1}\right)k^{\frac{-s}{t}}I.$$

Since $k>1$, we have

$$\left(kAQ^{-1}A^*\right)^{\frac{s}{t}}\leq\left(1-k^{-1}\right)Q\leq Q. \tag{2.4}$$

So $\left(kAQ^{-1}A^*\right)^{\frac{1}{t}}\leq Q^{\frac{1}{s}}$.

Now, let $\Omega=\left[\left(kAQ^{-1}A^*\right)^{\frac{1}{t}},Q^{\frac{1}{s}}\right]$ and $X^t\geq k\left(AQ^{-1}A^*\right)$ for all $X\in\Omega$. It is readily seen that Ω is a closed, convex and bounded set. We define $G\left(X\right)=\left(Q-A^*X^{-t}A\right)^{\frac{1}{s}}$ on Ω. Suppose that $X\in\Omega$. Using (2.4), we have

$$\begin{aligned}G(X)^s&=Q-A^*X^{-t}A\geq Q-A^*\left(k^{-1}A^{-*}QA^{-1}\right)A\\&=\left(1-k^{-1}\right)Q\geq\left(kAQ^{-1}A^*\right)^{\frac{s}{t}}.\end{aligned} \tag{2.5}$$

Therefore, $G\left(X\right)\geq\left(kAQ^{-1}A^*\right)^{\frac{1}{t}}$.

On the other hand $G\left(X\right)=\left(Q-A^*X^{-t}A\right)^{\frac{1}{s}}\leq Q^{\frac{1}{s}}$. So $G\left(\Omega\right)\subseteq\Omega$ and since G is continuous on $(0,\infty)$, by using the Brouwer's fixed point theorem and [1, Remark 4.1], the map G on Ω has a fixed point. So, the matrix Eq. (1.1) has an HPD solution in Ω. ☐

Lemma 2.6. *If $A\in M_m$, $B\in M_{m\times n}$, and $C\in M_n$, then*

$$\left\|ABC\right\|_F\leq\left\|A\right\|\left\|C\right\|\left\|B\right\|_F. \tag{2.6}$$

Proof. Let $\operatorname{vec}\left(A\right):=\left[a_1^T,a_2^T,\ldots,a_n^T\right]^T$, where $a_i\;(1\leq i\leq n)$ are the columns of the matrix A. By [15, Lemma 4.3.1], $\operatorname{vec}\left(ABC\right)=\left(C^T\otimes A\right)\operatorname{vec}\left(B\right)$. So

$$\begin{aligned}\left\|ABC\right\|_F&=\left\|\operatorname{vec}\left(ABC\right)\right\|=\left\|\left(C^T\otimes A\right)\operatorname{vec}\left(B\right)\right\|\\&\leq\left\|C^T\otimes A\right\|\left\|\operatorname{vec}(B)\right\|\leq\left\|A\right\|\left\|C\right\|\left\|B\right\|_F.\end{aligned}$$

☐

In the following, we study uniqueness of the solutions of Eq. (1.1) in Ω.

Corollary 2.7. *Let the assumptions of Theorem 2.5 hold and*

$$a = \frac{t}{s}\frac{\lambda_1(A^*A)}{((1-k^{-1})\lambda_n(Q))^{1-\frac{1}{s}}(k\lambda_n(AQ^{-1}A^*))^{1+\frac{1}{t}}} < 1.$$

Then, the matrix X_L *is the unique HPD solution of Eq.* (1.1) *in* Ω *and the sequence*

$$X_{k+1} = \left(Q - A^* X_k^{-t} A\right)^{\frac{1}{s}}, \ k \geq 1,$$

for any $X_1 \in \Omega$*, is convergent to the* X_L*. Also, for all* $k \geq 1$*,*

$$\|X_{k+1} - X_L\| \leq \frac{a^k}{1-a}\|X_2 - X_1\|,$$
$$\|X_{k+1} - X_L\| \leq a^k\|X_1 - X_L\|.$$

Proof. Let $G(X) = (Q - A^*X^{-t}A)^{\frac{1}{s}}$ on Ω. Suppose that $G(X)^s, G(Y)^s \geq \beta I$ and $X, Y \geq \alpha I$. Using (1.5), (2.6) and (1.6), respectively, we have

$$\begin{aligned} \|G(X) - G(Y)\|_F &= \left\|(G(X)^s)^{\frac{1}{s}} - (G(Y)^s)^{\frac{1}{s}}\right\|_F \\ &\leq \frac{1}{s}\beta^{\frac{1}{s}-1}\|G(X)^s - G(Y)^s\|_F \\ &= \frac{1}{s}\beta^{\frac{1}{s}-1}\|A^*\left(X^{-t} - Y^{-t}\right)A\|_F \\ &\leq \frac{1}{s}\beta^{\frac{1}{s}-1}\|A\|^2\|X^{-t} - Y^{-t}\|_F \\ &\leq \frac{1}{s}\beta^{\frac{1}{s}-1}\lambda_1(A^*A)\left(t\alpha^{-t-1}\|X - Y\|_F\right). \end{aligned} \tag{2.7}$$

Since $X, Y \in \Omega$, we have

$$X, Y \geq \left(kAQ^{-1}A^*\right)^{\frac{1}{t}} \geq \left(k\lambda_n\left(AQ^{-1}A^*\right)\right)^{\frac{1}{t}} I,$$

and by using (2.5), we obtain that

$$G(X)^s, G(Y)^s \geq \left(1 - k^{-1}\right)Q \geq \left(1 - k^{-1}\right)\lambda_n(Q)\, I.$$

Let $\alpha := \left(k\lambda_n(AQ^{-1}A^*)\right)^{\frac{1}{t}}$ and $\beta := (1-k^{-1})\lambda_n(Q)$. By replacing α, β in (2.7), we have $\|G(X) - G(Y)\|_F \leq a\|X - Y\|_F$, where $a < 1$. Hence, G is a contraction map on Ω. Since $G(\Omega) \subseteq \Omega$, by Banach's fixed point theorem, G has the unique fixed point $\bar{X}$ in Ω and so Eq. (1.1) has the unique HPD solution $\bar{X} \in \Omega$. Also, for any $X_1 \in \Omega$, sequence

$$X_{k+1} = G(X_k) = \left(Q - A^*X_k^{-t}A\right)^{\frac{1}{s}}; \ k \geq 1,$$

is convergent to the $\bar{X}$ and for all $k \geq 1$,

$$\|X_{k+1} - \bar{X}\| \leq \frac{a^k}{1-a}\|X_2 - X_1\|,$$
$$\|X_{k+1} - \bar{X}\| \leq a^k\|X_1 - \bar{X}\|.$$

Now, let X be an HPD solution of Eq. (1.1) and $\bar{X} \leq X$. By using (2.1), we obtain that $\bar{X} \leq X \leq Q^{\frac{1}{s}}$ and so $X \in \Omega$. Since $\bar{X}$ is the unique HPD solution of Eq. (1.1) in Ω, we have $\bar{X} = X$ and hence $\bar{X}$ is the maximal HPD solution of Eq. (1.1). □

Using Remark 2.2, Theorem 2.5, and choosing $l = k^{-1}$, we obtain the following:

Corollary 2.8. *Let $s, t \in [1, \infty)$ and there exist $0 < l < 1$ such that*

$$\lambda_1 \left(Q^{-\frac{t}{2s}} A Q^{-1} A^* Q^{-\frac{t}{2s}} \right) \leq (1-l)\, l^{\frac{t}{s}}. \tag{2.8}$$

Then $\left(AQ^{-1}A^*\right)^{\frac{1}{t}} \leq (lQ)^{\frac{1}{s}}$. Moreover, if for all $X \in \Lambda = \left[\left(AQ^{-1}A^*\right)^{\frac{1}{t}}, (lQ)^{\frac{1}{s}} \right]$ we have $X^s \leq lQ$, then Eq. (1.1) has an HPD solution in Λ.

Corollary 2.9. *Let the assumptions of Corollary 2.8 hold and*

$$b = \frac{s}{t} \frac{\left((1-l)^{-1} \lambda_1 \left(AQ^{-1}A^*\right) \right)^{1-\frac{1}{t}} \left(l \lambda_1 (Q) \right)^{1+\frac{1}{s}}}{\lambda_n(A^*A)} < 1.$$

Then, the matrix X_S is the unique HPD solution of Eq. (1.1) in Λ. Also, the sequence

$$X_{k+1} = \left(A \left(Q - X_k^s\right)^{-1} A^* \right)^{\frac{1}{t}} \ ; \ k \geq 1,$$

for any $X_1 \in \Lambda$, is convergent to the X_S and for all $k \geq 1$, we have

$$\|X_{k+1} - X_S\| \leq \frac{b^k}{1-b} \|X_2 - X_1\|,$$
$$\|X_{k+1} - X_S\| \leq b^k \|X_1 - X_S\|.$$

Proof. Let $F(X) = \left(A \left(Q - X^s\right)^{-1} A^* \right)^{\frac{1}{t}}$ on Λ. Suppose that $F(X), F(Y) \leq \beta I$ and $X, Y \leq \alpha I$. Therefore, by using (2.6) and (1.4), we have

$$\begin{aligned}
\|F(X) - F(Y)\|_F &= \left\| F(X) \left(F(X)^{-1} - F(Y)^{-1} \right) F(Y) \right\|_F \\
&\leq \|F(X)\| \, \|F(Y)\| \left\| F(X)^{-1} - F(Y)^{-1} \right\|_F \\
&\leq \beta^2 \left\| F(X)^{-1} - F(Y)^{-1} \right\|_F \\
&\leq \frac{1}{t} \beta^{1+t} \left\| F(X)^{-t} - F(Y)^{-t} \right\|_F \\
&= \frac{1}{t} \beta^{1+t} \left\| A^{-*} \left(X^s - Y^s\right) A^{-1} \right\|_F \\
&\leq \frac{1}{t} \beta^{1+t} \|A^{-1}\|^2 \|X^s - Y^s\|_F \\
&\leq \frac{s}{t} \beta^{1+t} \alpha^{s-1} \lambda_1(A^{-*}A^{-1}) \|X - Y\|_F
\end{aligned}$$

$$= \frac{s}{t}\frac{\beta^{1+t}\alpha^{s-1}}{\lambda_n(A^*A)}\left\|X-Y\right\|_F . \tag{2.9}$$

Since $X, Y \in \Lambda$, we have

$$X, Y \leq (lQ)^{\frac{1}{s}} \leq \left(l\lambda_1\left(Q\right)\right)^{\frac{1}{s}} I,$$

and the same as (2.6), we obtain that

$$F\left(X\right), F\left(Y\right) \leq \left(\left(1-l\right)^{-1}\left(AQ^{-1}A^*\right)\right)^{\frac{1}{t}} \leq \left(\left(1-l\right)^{-1}\lambda_1\left(AQ^{-1}A^*\right)\right)^{\frac{1}{t}} I.$$

Let $\alpha := \left(l\lambda_1\left(Q\right)\right)^{\frac{1}{s}}$ and $\beta := \left(\left(1-l\right)^{-1}\lambda_1\left(AQ^{-1}A^*\right)\right)^{\frac{1}{t}}$. By replacing α, β in (2.9), we have $\left\|F\left(X\right)-F\left(Y\right)\right\|_F \leq b\|X-Y\|_F$, where $b<1$. The same as the proof of Corollary 2.7, proof is completed. ☐

Let $\gamma \in \mathbb{R}, \theta > 0$. Consider $f_{\theta,\gamma}\left(x\right) = x^{s+t} - \theta x^t + \gamma$ on $(0,\infty)$. Then $f_{\theta,\gamma}$ on $\left[\left(\frac{t}{s+t}\theta\right)^{\frac{1}{s}}, \infty\right)$ is increasing and $\min f_{\theta,\gamma}\left(x\right) = \gamma - \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\theta^{1+\frac{t}{s}}$. If $\min f_{\theta,\gamma}\left(x\right) = \gamma - \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\theta^{1+\frac{t}{s}} \leq 0$, then equation $f_{\theta,\gamma}\left(x\right)=0$ has a unique solution α in $\left[\left(\frac{t}{s+t}\theta\right)^{\frac{1}{s}}, \infty\right)$. Consider the following functions on $(0,\infty)$.

$$\begin{aligned} f\left(x\right) &= x^{s+t} - \lambda_n\left(Q\right)x^t + \lambda_1\left(A^*A\right), \\ g\left(x\right) &= x^{s+t} - \lambda_1\left(Q\right)x^t + \lambda_n\left(A^*A\right). \end{aligned}$$

Let $\lambda_1\left(A^*A\right) \leq \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\lambda_n^{1+\frac{t}{s}}\left(Q\right)$. Therefore

$$\lambda_n\left(A^*A\right) \leq \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\lambda_1^{1+\frac{t}{s}}\left(Q\right).$$

So we have $\min f\left(x\right) \leq 0$ and $\min g\left(x\right) \leq 0$. Therefore equations $f\left(x\right)=0$ and $g\left(x\right)=0$ have a unique solution α and β in $\left[\left(\frac{t}{s+t}\theta\right)^{\frac{1}{s}}, \infty\right)$, respectively. Since $f\left(x\right) \geq g\left(x\right)$, we have $\left(\frac{t}{s+t}\lambda_n\left(Q\right)\right)^{\frac{1}{s}} \leq \alpha \leq \beta$. Consider the matrix interval $\Omega := [\alpha I, \beta I]$.

Theorem 2.10. *If one of the following inequalities hold, then Eq.* (1.1) *has an HPD solution.*

(1) $\lambda_1(A^*A) \leq \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\lambda_n^{1+\frac{t}{s}}(Q)$.

(2) $\lambda_1(AQ^{-1}A^*) \leq \left(\frac{t}{s+t}\right)^{\frac{t}{s+t}}\left(\frac{s}{s+t}\right)^{\frac{s}{s+t}}\lambda_n^{\frac{t}{s+t}}(AA^*)$.

Proof. Let inequality (1) be holds and $G(X) = (Q - A^*X^{-t}A)^{\frac{1}{s}}$ on $\Omega = [\alpha I, \beta I]$. If $X \in \Omega$, then

$$\begin{aligned} \lambda_n\left(G^s\left(X\right)\right) &= \lambda_n\left(Q - A^*X^{-t}A\right) \geq \lambda_n\left(Q - A^*\alpha^{-t}A\right) \\ &\geq \lambda_n\left(Q\right) - \lambda_1\left(A^*A\right)\alpha^{-t} = \alpha^s. \end{aligned}$$

$$\begin{aligned} \lambda_1\left(G^s\left(X\right)\right) &= \lambda_1\left(Q - A^*X^{-t}A\right) \leq \lambda_1\left(Q - A^*\beta^{-t}A\right) \\ &\leq \lambda_1\left(Q\right) - \lambda_n\left(A^*A\right)\beta^{-t} = \beta^s. \end{aligned}$$

Hence $G(\Omega) \subseteq \Omega$. By using Brouwer's fixed point theorem, the map G on Ω has a fixed point and so Eq. (1.1) has an HPD solution.

For the second one, we know that Eq. (1.1) has an HPD solution if and only if Eq. (2.2) has an HPD solution. Let inequality (2) be holds. Therefore

$$\begin{aligned}\lambda_1(A^{-*}A^{-1}) = \frac{1}{\lambda_n(AA^*)} &\leq \left((\frac{t}{s+t})^{\frac{t}{s+t}} (\frac{s}{s+t})^{\frac{s}{s+t}} \frac{1}{\lambda_1(AQ^{-1}A^*)} \right)^{\frac{s+t}{t}} \\ &= \frac{t}{s+t}(\frac{s}{s+t})^{\frac{s}{t}} \lambda_n^{1+\frac{s}{t}}(A^{-*}QA^{-1}).\end{aligned}$$

Now, by using Remark 2.2 and inequality (1), Eq. (2.2) has an HPD solution Y. Therefore, the matrix $X := Y^{-1}$ is an HPD solution of Eq. (1.1). □

Corollary 2.11. *If* $\lambda_1(A^*A) < \frac{s}{s+t}(\frac{t}{s+t})^{\frac{t}{s}} \lambda_n^{1+\frac{t}{s}}(Q)$, *then Eq.* (1.1) *has the unique HPD solution in* Ω.

Proof. Let G be the same as in the proof of Theorem 2.10. By the proof of this Theorem, we see that $G(\Omega) \subseteq \Omega$. Let $X, Y \in \Omega$. Therefore $G^s(X), G^s(Y) \geq \alpha^s I$ and $X, Y \geq \alpha I$. Using (2.7), we have

$$\|G(X) - G(Y)\|_F \leq \frac{t}{s} \frac{\lambda_1(A^*A)}{\alpha^{s+t}} \|X - Y\|_F.$$

Since $\alpha \geq \left(\frac{t}{s+t}\lambda_n(Q)\right)^{\frac{1}{s}}$, we have

$$\|G(X) - G(Y)\|_F \leq \frac{\lambda_1(A^*A)}{\frac{s}{s+t}(\frac{t}{s+t})^{\frac{t}{s}}\lambda_n^{1+\frac{t}{s}}(Q)} \|X - Y\|_F.$$

Hence, G is contraction on the set Ω and by Banach's fixed point theorem, G has the unique fixed point on Ω. So Eq. (1.1) has the unique HPD solution in Ω. □

Theorem 2.12. *If Eq.* (1.1) *has an HPD solution, then for* $s, t \in [1, \infty)$ *we have*

(1) $\lambda_n(A^*A) \leq \frac{s}{s+t}(\frac{t}{s+t})^{\frac{t}{s}} \lambda_1^{1+\frac{t}{s}}(Q)$,

(2) $\lambda_n(AQ^{-1}A^*) \leq (\frac{t}{s+t})^{\frac{t}{s+t}}(\frac{s}{s+t})^{\frac{s}{s+t}} \lambda_1^{\frac{t}{s+t}}(A^*A)$.

Proof. The first inequality is obtained by the same method as in [8, Theorem 3.3]. For the second one, let X be an HPD solution of Eq. (1.1). Then $Y := X^{-1}$ is an HPD solution of Eq. (2.2). Now, by using Remark 2.2 and inequality (1), we have

$$\lambda_n(A^{-*}A^{-1}) \leq \frac{t}{s+t}(\frac{s}{s+t})^{\frac{s}{t}} \lambda_1^{1+\frac{s}{t}}(A^{-*}QA^{-1}).$$

Therefore,

$$\lambda_n(AQ^{-1}A^*) = \frac{1}{\lambda_1(A^{-*}QA^{-1})} \leq \left(\frac{t}{s+t}(\frac{s}{s+t})^{\frac{s}{t}} \frac{1}{\lambda_n(A^{-*}A^{-1})} \right)^{\frac{t}{s+t}}$$

$$= (\frac{t}{s+t})^{\frac{t}{s+t}} (\frac{s}{s+t})^{\frac{s}{s+t}} \lambda_1^{\frac{t}{s+t}} (A^*A). \qquad \square$$

3. Iterative methods

In this section, we will present iterative methods for obtaining the extremal HPD solution of Eq. (1.1). Let F be an operator on Λ and $A, B \in \Lambda$. We say that F is an operator monotone on Λ, if $F(A) \geq F(B)$, whenever $A \geq B$.

Theorem 3.1. *Let $t = 1, s \geq 1$ and Eq.* (1.1) *has an HPD solution. Then the sequence*

$$P_1 = Q^{\frac{1}{s}}, \quad P_{k+1} = \big(Q - A^* P_k^{-1} A\big)^{\frac{1}{s}}; \ k \geq 1, \tag{3.1}$$

is monotonically decreasing and converges to the matrix X_L.

Proof. Let X be an HPD solution of Eq. (1.1). By considering $t = 1$ in Theorem 2.3, we have $G(P) = \big(Q - A^* P^{-1} A\big)^{\frac{1}{s}}$. In this case, we see that G is an operator monotone on $(0, \infty)$. By induction, we will show that $X \leq P_{k+1} < P_k$, for $k \in \mathbb{N}$. Using Theorem 2.3, we have $X \leq P_2 = G(P_1) < P_1$. Now, let $X \leq P_k < P_{k-1}$. Since G is an operator monotone map, we obtain that

$$X = G(X) \leq P_{k+1} = G(P_k) < P_k = G(P_{k-1}).$$

Thus, $X \leq P_{k+1} < P_k$ for $k \in \mathbb{N}$. Therefore, the sequence $\{P_k\}$ is decreasing and bounded sequence and hence it is convergent. Let $\lim P_k = P$. Since G is continuous on $(0, \infty)$, we have $G(P) = P$ and $P \geq X$. Therefore, P is a solution of Eq. (1.1) and $P \geq X$. Hence $P = X_L$. $\square$

Theorem 3.2. *Let $s \geq t \geq 1$ and Eq.* (1.1) *has an HPD solution. Then the sequence*

$$P_1 = Q^{\frac{1}{s}}, \quad P_{k+1} = \big(Q - A^* P_k^{-t} A\big)^{\frac{1}{s}}; \ k \geq 1, \tag{3.2}$$

is monotonically decreasing and converges to the matrix X_L. Moreover, for all $k \in \mathbb{N}$,

$$\|P_{k+1} - X_L\|_F \leq \frac{t}{s} \frac{\lambda_1(A^*A)}{\lambda_n^{s+t}(X_L)} \|P_k - X_L\|_F . \tag{3.3}$$

Proof. Let X be an HPD solution of Eq. (1.1). Therefore $Y = X^t$ is an HPD solution of equation

$$Y^{\frac{s}{t}} + A^* Y^{-1} A = Q. \tag{3.4}$$

Since $\frac{s}{t} \geq 1$, by Theorem 3.1, the sequence $\{P_k^t\}$ is monotonically decreasing and converges to the matrix Y_L, where Y_L is the maximal HPD solution of Eq. (3.4). Let X be an arbitrary HPD solution of Eq. (1.1). So $Y = X^t$ is an HPD solution of Eq. (3.4). Since Y_L is the maximal HPD solution of Eq. (3.4), we have $Y_L \geq Y = X^t$. So $Y_L^{\frac{1}{t}} \geq X$. Therefore $X_L = Y_L^{\frac{1}{t}}$ and the sequence (3.2) is monotonically decreasing and converges to the matrix X_L.

Let $G(P) = (Q - A^*P^{-t}A)^{\frac{1}{s}}$ on $(0, \infty)$. Therefore $P_{k+1} = G(P_k)$ and $X_L = G(X_L)$. By using (2.7), for all $k \in \mathbb{N}$, we obtain that

$$\begin{aligned} \|P_{k+1} - X_L\|_F &= \|G(P_k) - G(X_L)\|_F \\ &\leq \frac{t}{s}\beta^{\frac{1}{s}-1}\alpha^{-t-1}\lambda_1(A^*A)\|P_k - X_L\|_F, \end{aligned} \tag{3.5}$$

where $P_k,\ X_L \geq \alpha I$ and $G(P_k)^s,\ G(X_L)^s \geq \beta I$. But $G(P_k)^s,\ G(X_L)^s \geq \lambda_n^s(X_L)I$ and $P_k \geq X_L \geq \lambda_n(X_L)I$ for all $k \in \mathbb{N}$. Therefore, by replacing $\alpha = \lambda_n(X_L)$ and $\beta = \lambda_n^s(X_L)$ in (3.5),

$$\|P_{k+1} - X_L\|_F \leq \frac{t}{s}\frac{\lambda_1(A^*A)}{\lambda_n^{s+t}(X_L)}\|P_k - X_L\|_F . \qquad \square$$

Theorem 3.3. *Let $t \geq s \geq 1$ and Eq.* (1.1) *has an HPD solution. Then the sequence*

$$P_1 = (AQ^{-1}A^*)^{\frac{1}{t}}, \quad P_{k+1} = (A(Q - P_k^s)^{-1}A^*)^{\frac{1}{t}};\ k \geq 1, \tag{3.6}$$

is monotonically increasing and converges to the matrix X_S. Moreover, for all $k \in \mathbb{N}$, we have

$$\|P_{k+1} - X_S\|_F \leq \frac{s}{t}\frac{\lambda_1^{s+t}(X_S)}{\lambda_n(A^*A)}\|P_k - X_S\|_F .$$

Proof. Let X be an HPD solution of Eq. (1.1). So, the matrix $Y := X^{-1}$ is an HPD solution of Eq. (2.2). Using Remark 2.2 and Theorem 3.2, the sequence $\{P_k^{-1}\}$ is monotonically decreasing and converges to the matrix Y_L, where Y_L is the maximal HPD solution of Eq. (2.2). Therefore, the sequence $\{P_k\}$ is monotonically increasing and converges to the matrix $X_S = Y_L^{-1}$.

Now, let $F(P) = \left(A\,(Q - P^s)^{-1}A^*\right)^{\frac{1}{t}}$ on $(0, Q^{\frac{1}{s}})$. Hence $P_{k+1} = F(P_k)$ and $X_S = F(X_S)$. By using (2.9), for all $k \in \mathbb{N}$, we obtain that

$$\|P_{k+1} - X_S\|_F = \|F(P_k) - F(X_S)\|_F \leq \frac{s}{t}\frac{\beta^{1+t}\alpha^{s-1}}{\lambda_n(A^*A)}\|P_k - X_S\|_F , \tag{3.7}$$

where $P_k, X_S \leq \alpha I$ and $F(P_k), F(X_S) \leq \beta I$. For all $k \in \mathbb{N}$, we have $P_k \leq X_S \leq \lambda_1(X_S)I$ and $F(P_k) \leq F(X_S) \leq \lambda_1(X_S)I$. So, by replacing $\alpha = \lambda_1(X_S)$ and $\beta = \lambda_1(X_S)$ in (3.7),

$$\|P_{k+1} - X_S\|_F \leq \frac{s}{t}\frac{\lambda_1(X_S)^{s+t}}{\lambda_n(A^*A)}\|P_k - X_S\|_F . \qquad \square$$

In the following, we present a necessary and sufficient condition for the existence of HPD solutions of Eq. (1.1), when $s, t \in [1, \infty)$.

Proposition 3.4. *If $s \geq t \geq 1$, then Eq.* (1.1) *has an HPD solution if and only if the sequence* (3.2) *is convergent to the Hermitian positive definite matrix P.* (*In this case, by Theorem 3.2, the sequence* (3.2) *is monotonically decreasing and converges to the maximal HPD solution X_L.*)

Also, if $t \geq s \geq 1$*, then Eq.* (1.1) *has an HPD solution if and only if the sequence* (3.6) *is convergent to the Hermitian positive definite matrix* P*.* (*In this case, by Theorem 3.3, the sequence* (3.6) *is monotonically increasing and converges to the minimal HPD solution* X_S*.*)

Remark 3.5. By Proposition 3.4, if $s \geq t \geq 1$ ($t \geq s \geq 1$) and the sequence (3.2) (sequence (3.6)) is not decreasing (not increasing) sequence or there exists $k \in \mathbb{N}$ such that P_k is not an Hermitian positive definite, then Eq. (1.1) has not HPD solution.

Proposition 3.6. *Let* $s = t \geq 1$*. Then Eq.* (1.1) *has an HPD solution if and only if the sequence* (3.2) *is monotonically decreasing and converges to the matrix* X_L *if and only if the sequence* (3.6) *is monotonically increasing and converges to the matrix* X_S*.*

4. Numerical examples

In this section, by some numerical examples, the convergence of the above iterative sequences are studied. All the tests are performed by MATLAB with machine precision around 10^{-10}. We continue the iterative sequences up to step k, where $\|P_{k+1} - P_k\|_F \leq 1.0e-10$.

Example 4.1. Consider Eq. (1.1) with $s = \frac{5}{3}, t = 1$,

$$A=\begin{bmatrix} 0.75 & -0.75 & 0 & 0 \\ 0.80 & 0.80 & 0 & 0 \\ 0 & 0 & 0.85 & 0.85 \\ 0 & 0 & -0.90 & 0.90 \end{bmatrix}, \; Q=\begin{bmatrix} 2.405 & 0.155 & 0 & 0 \\ 0.155 & 2.405 & 0 & 0 \\ 0 & 0 & 3.065 & -0.175 \\ 0 & 0 & -0.175 & 3.065 \end{bmatrix}.$$

The matrix A is a nonsingular and Q is a Hermitian positive definite matrix. By choosing $k = 2$, we obtain that

$$\lambda_1(Q^{-\frac{1}{2}}(AQ^{-1}A^*)^{\frac{s}{t}}Q^{-\frac{1}{2}}) = 0.1400 \leq 0.1575 = \frac{k-1}{k^{\frac{s+t}{t}}}.$$

So, by Theorem 2.5, $2AQ^{-1}A^* \leq Q^{\frac{3}{5}}$. Moreover if $\Omega := [2AQ^{-1}A^*, Q^{\frac{3}{5}}]$, Eq. (1.1) has an HPD solution in Ω. Since

$$a = \frac{t}{s}\frac{\lambda_1(A^*A)}{\left((1-k^{-1})\,\lambda_n\,(Q)\right)^{1-\frac{1}{s}}\left(k\lambda_n\,(AQ^{-1}A^*)\right)^{1+\frac{1}{t}}} = 0.9273 < 1,$$

by Corollary 2.7, Eq. (1.1) has the unique HPD solution X_L in Ω and sequence

$$P_{k+1} = \left(Q - A^*P^{-t}A\right)^{\frac{1}{s}}, \; P_1 = \frac{1}{2}\left(\left(kAQ^{-1}A^*\right)^{\frac{1}{t}} + Q^{\frac{1}{s}}\right) \in \Omega,$$

is convergent to the X_L. After $k = 25$ step, we see that $\|P_{25} - P_{24}\|_F = 7.7435e-011$. Therefore

$$X_L \simeq P_{25} = \begin{bmatrix} 1.2575 & 0.0412 & 0 & 0 \\ 0.0412 & 1.2241 & 0 & 0 \\ 0 & 0 & 1.5625 & -0.0491 \\ 0 & 0 & -0.0491 & 1.5344 \end{bmatrix}.$$

Example 4.2. Consider the matrix Eq. (1.1) with $s = \sqrt{3}$, $t = \sqrt{2}$,

$$A = \begin{bmatrix} 4 & 1 & 9 & 4 \\ 5 & 6 & 9 & 8 \\ 0 & 4 & 1 & 5 \\ 2 & 7 & 5 & 1 \end{bmatrix}, \ Q = \begin{bmatrix} 686 & 441 & 392 & 441 \\ 441 & 931 & 588 & 686 \\ 392 & 588 & 686 & 392 \\ 441 & 686 & 392 & 735 \end{bmatrix}.$$

Since

$$\lambda_1(A^*A) - \frac{s}{s+t}(\frac{t}{s+t})^{\frac{t}{s}}\lambda_n^{1+\frac{t}{s}}(Q) = -627.6697 \leq 0,$$

by Theorem 2.10 (1), Eq. (1.1) has an HPD solution and so by using Theorem 3.2, the iterative sequence (3.2) is convergent to the maximal HPD solution of Eq. (1.1). For $k = 6$, we have $\|P_6 - P_5\|_F = 6.7641e - 12$. Hence

$$X_L \simeq P_6 = \begin{bmatrix} 39.1792 & 11.8409 & 12.4479 & 14.1989 \\ 11.8409 & 43.2579 & 19.6539 & 23.7551 \\ 12.4479 & 19.6539 & 37.7836 & 9.9658 \\ 14.1989 & 23.7551 & 9.9658 & 37.6148 \end{bmatrix}.$$

Example 4.3. Consider the matrix Eq. (1.1) with $s = \frac{3}{2}$, $t = 3$,

$$A = \begin{bmatrix} 4 & 2 & -2 & 0 & 1 \\ 1 & 5 & -1 & 0 & 3 \\ 2 & 0 & 5 & 1 & 0 \\ -3 & 1 & 5 & -7 & 5 \\ 0 & 0 & -4 & 1 & 8 \end{bmatrix}, \ Q = \begin{bmatrix} 32 & 10 & -14 & 23 & -8 \\ 10 & 32 & -4 & -7 & 22 \\ -14 & -4 & 73 & -34 & -12 \\ 23 & -7 & -34 & 53 & -27 \\ -8 & 22 & -12 & -27 & 101 \end{bmatrix}.$$

Since

$$\lambda_1(AQ^{-1}A^*) - (\frac{t}{s+t})^{\frac{t}{s+t}}(\frac{s}{s+t})^{\frac{s}{s+t}}\lambda_n(A^*A)^{\frac{t}{s+t}} = -0.9199 \leq 0,$$

by Theorem 2.10(2), Eq. (1.1) has an HPD solution and so by Theorem 3.3, the iterative sequence (3.6) is convergent to the minimal HPD solution of Eq. (1.1). For $k = 9$, we have $\|P_9 - P_8\|_F = 5.3450e - 011$. Therefore

$$X_S \simeq P_9 = \begin{bmatrix} 0.9692 & 0.0170 & 0.0033 & -0.0065 & 0.0020 \\ 0.0170 & 0.9776 & -0.0003 & 0.0042 & 0.0036 \\ 0.0033 & -0.0003 & 0.9853 & 0.0026 & -0.0044 \\ -0.0065 & 0.0042 & 0.0026 & 0.9951 & 0.0012 \\ 0.0020 & 0.0036 & -0.0044 & 0.0012 & 0.9930 \end{bmatrix}.$$

Example 4.4. Let $s = 3, t = 2$ and A, Q be the same as in Example 4.3. Replacing $2A \to A$ and consider the sequence (3.2). Since P_3 is not an Hermitian positive definite, by using Remark 3.5, the Eq. (1.1) has not HPD solution.

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