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Functional central limit theorems for ARCH(∞) models

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Abstract

In this paper, we study ARCH(∞) models with either geometrically decaying coefficients or hyperbolically decaying coefficients. Most popular autoregressive conditional heteroscedasticity (ARCH)-type models such as various modified generalized ARCH (GARCH) (p, q), fractionally integrated GARCH (FIGARCH), and hyperbolic GARCH (HYGARCH). can be expressed as one of these cases. Sufficient conditions for L_2-near-epoch dependent (NED) property to hold are established and the functional central limit theorems for ARCH(∞) models are proved.

Keywords: functional central limit theorem, ARCH(∞) model, L_2-NED property

1. Introduction

Introduced by the seminal work of Engle (1982) and Bollerslev (1986), autoregressive conditional heteroscedasticity (ARCH)-type processes are widely used for modelling dynamics in different fields such as in econometric studies. A number of modifications of the classical generalized ARCH (GARCH) model such as various modified-GARCH(p, q) models, integrated GARCH (IGARCH), fractionally IGARCH (FIGARCH), hyperbolic GARCH (HYGARCH), and fractionally integrated asymmetric power ARCH (FIAPARCH) models were proposed to account for long memory property, asymmetry, leverage effect, and other stylized facts. ARCH(∞) models was first introduced by Robinson (1991) in the context of testing for strong serial correlation. It can often be helpful to view a GARCH(p, q) process as an ARCH(∞) processes. In particular, from the ARCH(∞) representation we can easily read off the conditional variance given its infinite past. All of these aforementioned models can be represented as ARCH(∞) models. When we consider a time series model as a data generating process, one of the important properties to show is the (functional) central limit theorem (CLT). Functional central limit theorem (FCLT) is applied for statistical inference in time series to establish the asymptotics of various statistics concerned. For example, the FCLT has been employed in the theory of detecting structural breaks in GARCH-type models. Probabilistic and statistical properties of ARCH(∞) models have been studied by many authors (e.g., Davidson, 2004; Giraitis *et al.*, 2000; Kazakevičius and Leipus, 2002; Zaffaroni, 2004) and the references therein.

A process y_t is said to obey the FCLT if

$$Y_n(\xi) = \sigma_n^{-1} \sum_{t=1}^{[n\xi]} (y_t - E(y_t)), \quad 0 \le \xi \le 1,$$

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converges weakly to standard Brownian motion, where $\sigma_n^2 = \mathrm{Var}(\sum_{t=1}^n y_t)$. If $\xi = 1$,then this convergence implies the CLT. There are numerous literatures considering the CLT and FCLT for various GARCH family models (Berkes *et al.*, 2008; Billingsley, 1968; Davidson, 2002; De Jong and Davidson, 2000; Herrndorf, 1984; Lee, 2014a, 2014b).

This paper is to find some sufficient conditions under which the FCLT holds for the partial sums processes of the given ARCH(∞) models. The typical approach to obtaining the FCLT for time series models is to show that a specific dependence property such as various mixing conditions, L_p-NED (near-epoch dependent), association or $\theta, \mathcal{L}$ or ψ-weak dependence holds. In order to prove such dependence properties, rather restrictive conditions such as distributional assumptions on errors and higher order moment are required (Dedecker *et al.*, 2007; Doukhan and Wintenberger, 2007).

Our proof is based on the L_2-NED condition among various dependence conditions. The proofs, in part, rely on results in Davidson (2004) and Giraitis *et al.* (2000).

Definition 1. *y_t is said to be L_2-NED on $\{e_t\}$ of size $-\lambda_0$ if*

$$\left\| y_t - E_{t-m}^{t+m}(y_t) \right\|_2 \leq d_t \nu(m),$$

where d_t is a sequence of positive constants and $\nu(m) = O(m^{-\lambda})$ for $\lambda > \lambda_0$. If $\nu(m) = O(e^{-\delta m})$ for some $\delta > 0$, we say that the process is geometrically L_2-NED. Here we define $E_{t-m}^{t+m}(y_t) = E(y_t|\sigma(e_{t-m}, \ldots, e_{t-1}, e_t, e_{t+1}, \ldots, e_{t+m}))$.

Theorem 1. (Davidson, 2002) *Suppose that the following Assumptions (1)–(3) hold: (1) y_t is L_2-NED of size $-1/2$ on the underlying i.i.d. process $\{e_t\}$. (2) $\sup_t E|y_t - E(y_t)|^r < \infty$ for some $r \geq 2$. (3) $\sigma_n^2/n \to \sigma^2 > 0$ as $n \to \infty$. Then the FCLT holds for y_t.*

2. Functional central limit theorem for ARCH(∞) model

In this section, we consider a nonnegative coefficients ARCH(∞) model defined as follows.

$$u_t = \sigma_t e_t, \tag{2.1}$$

and

$$\sigma_t^2 = \omega + \sum_{i=1}^{\infty} \theta_i u_{t-i}^2, \tag{2.2}$$

where $\omega > 0, \theta_i \geq 0$ for all $i \geq 1$ are constants and $\{e_t\}$ is a sequence of independent and identically distributed random variables with mean 0 and variance 1.

Repeated substitution of the equations (2.1) and (2.2) leads to, for given m,

$$\begin{aligned}\sigma_t^2 = \omega &\left(1 + \sum_{p=1}^{m} \sum_{j_1, j_2, \ldots, j_p = 1}^{\infty} \theta_{j_1}\theta_{j_2} \cdots \theta_{j_p} e_{t-j_1}^2 e_{t-j_1-j_2}^2 \cdots e_{t-j_1-j_2-\cdots-j_p}^2 \right) \\ &+ \sum_{j_1, j_2, \ldots, j_{m+1} = 1}^{\infty} \theta_{j_1} \cdots \theta_{j_{m+1}} e_{t-j_1}^2 \cdots e_{t-j_1-j_2-\cdots-j_{m+1}}^2 \sigma_{t-j_1-j_2-\cdots-j_{m+1}}^2 . \end{aligned} \tag{2.3}$$

For notational simplicity, we let $S = \sum_{i=1}^{\infty} \theta_i$, $\mu_n = E(|e_t|^n)$, and $M_n = E(\sigma_t^n)$ for $n = 1, 2, \ldots$. For a process y_t, define $E_{t-m}^{t+m}(y_t) = E(y_t|\mathcal{F}_{t-m}^{t+m})$, where $\mathcal{F}_s^t = \sigma(e_s, \ldots, e_t)$ is a sigma field generated by $\{e_j, s \leq j \leq t\}$.

We first write the lemma due to Giraitis *et al.* (2000) and Zaffaroni (2004) in which the stationarity and finite moment condition of the process is obtained.

Lemma 1. *Assume $S < 1$. Then σ_t^2 given by*

$$\sigma_t^2 = \omega\left(1 + \sum_{p=1}^{\infty} \sum_{j_1, j_2, \ldots, j_p = 1}^{\infty} \theta_{j_1}\theta_{j_2}\cdots\theta_{j_p} e_{t-j_1}^2 e_{t-j_1-j_2}^2 \cdots e_{t-j_1-j_2-\cdots-j_p}^2\right) \tag{2.4}$$

is the unique nonanticipative strictly stationary solution to (2.1) and (2.2) with finite first moment $E(\sigma_t^2)$. If, in addition, $\mu_4^{1/2} S < 1$, then (2.4) is also unique weakly stationary solution to (2.1) and (2.2).

The following lemma will be used to derive some sufficient conditions for L_2-NED property and then to prove the FCLT for the process u_t.

Lemma 2. *Consider the processes u_t and σ_t^2 given by (2.1) and (2.2). Then we obtain that*

$$\left\| u_t - E_{t-m}^{t+m}(u_t) \right\|_2^2 \le E\left|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right|.$$

Proof: We have that

$$\begin{aligned} \left\| u_t - E_{t-m}^{t+m}(u_t) \right\|_2^2 &= \left\| \sigma_t - E_{t-m}^{t+m}(\sigma_t) \right\|_2^2 \\ &\le \left\| \left(\sigma_t^2\right)^{\frac{1}{2}} - \left(E_{t-m}^{t+m}\left(\sigma_t^2\right)\right)^{\frac{1}{2}} \right\|_2^2 \\ &\le E\left|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right|. \end{aligned} \tag{2.5}$$

The first inequality in (2.5) follows from the relation: $\|Y - E(Y|X)\|_2 \le \|Y - g(X)\|_2$ for any measurable function g. For the second inequality in (2.5), we apply the relation $(\sqrt{a} - \sqrt{b})^2 \le |a - b|$ $(a > 0, b > 0)$.
□

Next, we define for some $m \ge p \ge 1$,

$$T_p = \sum_{j_1=1}^{\infty} \cdots \sum_{j_p=1}^{\infty} I_{\{j_1+\cdots+j_p > m\}}(j_1, j_2, \ldots, j_p)\theta_{j_1}\theta_{j_2}\cdots\theta_{j_p}, \tag{2.6}$$

where $I_A(\cdot)$ is the indicator function of a set A. Then combining (2.3) and (2.6), we can easily show that (Davidson, 2004, p.26)

$$E\left|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right| \le 2\omega \sum_{p=1}^{m} T_p + 2M_2 S^{m+1}, \tag{2.7}$$

and

$$\left\|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right\|_2 \le 2\omega \sum_{p=1}^{m} T_p \mu_4^{\frac{p}{2}} + 2M_4^{\frac{1}{2}} \mu_4^{\frac{m+1}{2}} S^{m+1}. \tag{2.8}$$

Now, we first consider the processes which have hyperbolically decaying lag coefficients.

(A1) $0 \leq \theta_j \leq Cj^{-1-\delta}$ for $j \geq 1, C > 0, \delta > 0$, and $S < 1$.

Theorem 2.

(1-1) If the Assumption (A1) holds, then u_t given by (2.1) and (2.2) is L_2-NED on $\{e_t\}$, of size $-\lambda/2$, with $\delta > \lambda > 0$.

(1-2) If the assumption (A1) and $\mu_4^{1/2} S < 1$ hold, then u_t, u_t^2, and σ_t^2 are L_2-NED on $\{e_t\}$ of size $-\lambda$, with $\delta > \lambda > 0$.

Proof: After simple calculation, we derive that

$$T_2 \leq 2\left(\sum_{j=[m/p]+1}^{\infty} \theta_j\right) S$$

and then use mathematical induction to obtain that

$$\begin{aligned} T_p &\leq \left(\sum_{j=[m/p]+1}^{\infty} \theta_j\right) S^{p-1} + S \sum_{j_1=1}^{\infty} \cdots \sum_{j_{p-1}=1}^{\infty} I_{\{j_1+\cdots+j_{p-1}>(p-1)m/p\}}(j_1, \ldots, j_{p-1}) \theta_{j_1} \cdots \theta_{j_{p-1}} \\ &= \left(\sum_{j=[m/p]+1}^{\infty} \theta_j\right) S^{p-1} + (p-1)\left(\sum_{j=[m/p]+1}^{\infty} \theta_j\right) S^{p-1} \\ &= p\left(\sum_{j=[m/p]+1}^{\infty} \theta_j\right) S^{p-1}, \end{aligned} \tag{2.9}$$

where $[x]$ denotes the largest integer which is less than or equal to x.

Moreover, if the Assumption (A1) holds, then

$$\sum_{j=[m/p]+1}^{\infty} \theta_j \leq C \int_{\frac{m}{p}}^{\infty} x^{-1-\delta} dx = \left(\frac{C}{\delta}\right) m^{-\delta} p^{\delta}. \tag{2.10}$$

Combining the Assumption (A1), the equations (2.9) and (2.10) yields that

$$T_p \leq O\left(m^{-\delta} p^{\delta+1} S^{p-1}\right), \quad p \geq 1. \tag{2.11}$$

Proof of Theorem 2(1-1): Note that

$$\sum_{p=1}^{\infty} p^{\delta+1} S^{p-1} \leq \frac{1}{S} \int_0^{\infty} x^{\delta+1} S^x dx = \frac{\Gamma(\delta+2)}{S(-\log S)^{\delta+2}} < \infty. \tag{2.12}$$

Then the equations (2.7), (2.11), and (2.12) give that

$$\begin{aligned} E\left|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right| &\leq 2\omega \sum_{p=1}^{m} T_p + 2M_2 S^{m+1} \\ &= O\left(m^{-\delta} \sum_{p=1}^{m} p^{\delta+1} S^{p-1}\right) \\ &= O\left(m^{-\delta}\right). \end{aligned} \tag{2.13}$$

Use Lemma 2 and the equation (2.13) to obtain that

$$\left\|u_t - E_{t-m}^{t+m}(u_t)\right\|_2 \leq O\left(m^{-\frac{\delta}{2}}\right),$$

and the conclusion follows.

Proof of Theorem 2(1-2): Combine, similarly the equations (2.8), (2.11), and (2.12) to have that

$$\left\|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right\|_2 = O\left(m^{-\delta}\sum_{p=1}^{m} p^{\delta+1}\left(\mu_4^{\frac{1}{2}}S\right)^{p-1}\right) = O\left(m^{-\delta}\right), \tag{2.14}$$

since $\mu_4^{1/2}S < 1$. Also, we have

$$\left\|u_t^2 - E_{t-m}^{t+m}\left(u_t^2\right)\right\|_2 = \mu_4^{\frac{1}{2}}\left\|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right\|_2. \tag{2.15}$$

Hence u_t^2 and σ_t^2 hold the L_2-NED property of size $-\lambda$ with $\delta > \lambda > 0$.

On the other hand, using the inequalities $\sigma_t^2 \geq \omega$, $E_{t-m}^{t+m}\sigma_t^2 \geq \omega$, and $|\sqrt{a} - \sqrt{b}| \leq |a-b|$ if $a, b \geq 1$ yields that

$$\left|\sigma_t - \left(E_{t-m}^{t+m}\left(\sigma_t^2\right)\right)^{\frac{1}{2}}\right| \leq \omega^{-\frac{1}{2}}\left|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right|. \tag{2.16}$$

Then from (2.16) and the first inequality in (2.5), we obtain that

$$\left\|u_t - E_{t-m}^{t+m}(u_t)\right\|_2 \leq \left\|\sigma_t - \left(E_{t-m}^{t+m}\left(\sigma_t^2\right)\right)^{\frac{1}{2}}\right\|_2 \leq \omega^{-\frac{1}{2}}\left\|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right\|_2. \tag{2.17}$$

Therefore, L_2-NED property of u_t follows from (2.14) and (2.17). □

Next, consider the case where the process has geometrically decaying coefficients. We make the Assumption (A2):

(A2) $0 \leq \theta_j \leq Cr^j$, for $j \geq 1, C > 0, 0 < r < 1$, and $S < 1$.

Theorem 3.

(2-1) If the Assumption (A2) holds, then u_t is L_2-NED on $\{e_t\}$ of size $-1/2$.

(2-2) If the Assumption (A2) and $\mu_4^{1/2}S < 1$ hold, then u_t^2 and σ_t^2 are L_2-NED on $\{e_t\}$ of size $-1/2$.

(2-3) If the Assumption (A2) and $0 < rC < 1$ hold, then u_t is geometrically L_2-NED on $\{e_t\}$.

(2-4) If the Assumption (A2), $\mu_4^{1/2}S < 1$, $0 < rC < 1$, and $\mu_4^{1/2}rC < 1$ hold, then u_t^2 and σ_t^2 are geometrically L_2-NED on $\{e_t\}$.

Proof: Proof of Theorem 3(2-1): From the Assumption (A2) and the equation (2.9), we have

$$\begin{aligned}T_p &\leq p\left(\sum_{j=[m/p]+1}^{\infty}\theta_j\right)S^{p-1} \\ &\leq Cp\left(\sum_{j=[m/p]+1}^{\infty}r^j\right)S^{p-1} \\ &\leq Kr^{\frac{m}{p}}pS^p.\end{aligned} \tag{2.18}$$

Throughout this paper, K denotes a generic constant. There is no loss of generality in setting $r > S$. Choose $0 < \epsilon < 1$. Then we have that

$$\sum_{p=1}^{m} T_p \leq \sum_{p=1}^{m} K r^{\left(\frac{m}{p}\right)+\epsilon p} p S^{(1-\epsilon)p}.$$

Define $f(p) = (m/p) + \epsilon p$ $(1 \leq p \leq m)$. Then the minimum value of $f(p) = 2\sqrt{\epsilon}\sqrt{m}$ if $m > 1/\epsilon$. Thus we have that, for sufficiently large m,

$$\begin{aligned} E\left|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right| &\leq 2\omega \sum_{p=1}^{m} T_p + 2M_2 S^{m+1} \\ &\leq K r^{2\sqrt{\epsilon}\sqrt{m}} \sum_{p=1}^{m} p S^{(1-\epsilon)p} + 2M_2 S^{m+1} \\ &\leq K\left((r')^{\sqrt{m}} \vee S^m\right) \\ &\leq K\left(r' \vee S\right)^{\sqrt{m}}. \end{aligned} \tag{2.19}$$

Therefore,

$$\begin{aligned} \left(E\left|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right|\right)^{\frac{1}{2}} &\leq K\left(r' \vee S\right)^{\frac{\sqrt{m}}{2}} \\ &= O\left(m^{-\frac{1}{2}-\eta}\right), \end{aligned} \tag{2.20}$$

for some $\eta > 0$ where $r' = r^{2\sqrt{\epsilon}} < 1$ and $a \vee b = \max\{a, b\}$. The equality in the equation (2.20) is obtained from $(1/2)\sqrt{m}\log(r' \vee S) \leq (-1/2 - \eta)\log m$ for large enough m. Combining Lemma 2 and the equation (2.20) yields the conclusion.

Proof of Theorem 3(2-2): Let $S_0 = \mu_4^{1/2} S < 1$. Without loss of generality we assume that $r > S_0$ and choose $0 < \epsilon < 1$. Then from (2.8) and (2.18)

$$\begin{aligned} \left\|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right\|_2 &\leq K \sum_{p=1}^{m} r^{\frac{m}{p}} p S_0^p + 2M_4^{\frac{1}{2}} S_0^{m+1} \\ &\leq K \sum_{p=1}^{m} r^{\frac{m}{p}+\epsilon p} p S_0^{(1-\epsilon)p} + 2M_4^{\frac{1}{2}} S_0^{m+1}. \end{aligned}$$

Then by the same method used to prove the equation (2.19) and (2.20), we obtain that

$$\left\|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right\|_2 \leq K\left(r' \vee S_0\right)^{\sqrt{m}} = O\left(m^{-\frac{1}{2}-\eta}\right), \tag{2.21}$$

for some $\eta > 0$ and sufficiently large m. From (2.15) and (2.21), u_t^2 and σ_t^2 are L_2-NED of size $-1/2$.

Proof of Theorem 3(2-3): The baseline of the proof of Theorem 3(2-3) is the same as that of Theorem 2(a) in Davidson (2004). Choose $\epsilon > 0$ such that $\tilde{S} = \sum_{j=1}^{\infty} \theta_j^{1-\epsilon} < 1$. Then

$$\begin{aligned} T_p &= \sum_{j_1=1}^{\infty} \cdots \sum_{j_p=1}^{\infty} I_{\{j_1+\cdots+j_p>m\}}(j_1, \ldots, j_p) |\theta_{j_1} \cdots \theta_{j_p}|^{\epsilon} |\theta_{j_1} \cdots \theta_{j_p}|^{1-\epsilon} \\ &\leq C^{\epsilon p} r^{\epsilon m} \tilde{S}^p. \end{aligned} \tag{2.22}$$

By using (2.7) and (2.22), we have that

$$\begin{aligned} E\left|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right| &\leq 2\omega r^{\epsilon m}\sum_{p=1}^{m} C^{\epsilon p}\tilde{S}^p + 2M_2 S^{m+1} \\ &\leq K r^{\epsilon m} C^{\epsilon m}\left(C^{-\epsilon m} - \tilde{S}^m\right) + 2M_2 S^{m+1} \\ &= O\left(\alpha^m\right) \\ &= O\left(e^{-\rho m}\right), \end{aligned} \tag{2.23}$$

where $\alpha = r^{\epsilon} \vee (rC)^{\epsilon} \vee S < 1$ and $\rho = -\log\alpha > 0$. To prove the first equality in (2.23), note that

$$r^{\epsilon m} C^{\epsilon m}\left(C^{-\epsilon m} - \tilde{S}^m\right) \leq \begin{cases} 2r^{\epsilon m}, & \text{if } C^{-\epsilon} \geq \tilde{S}, \\ 2(rC)^{\epsilon m}, & \text{if } C^{-\epsilon} < \tilde{S}. \end{cases}$$

Then apply Lemma 2 to get $\|u_t - E_{t-m}^{t+m}(u_t)\|_2 \leq O(e^{-(1/2)\rho m})$, which implies that u_t holds the geometric L_2-NED property.

Proof of Theorem 3(2-4): Choose $\epsilon > 0$ such that $\tilde{S} = \sum_{j=1}^{\infty}\theta_j^{1-\epsilon} < 1$, and $\mu_4^{(1-\epsilon)/2}\tilde{S} < 1$. Now use the equations (2.8) and (2.22) to have that

$$\begin{aligned} \left\|\sigma_t^2 - E_{t-m}^{t+m}\left(\sigma_t^2\right)\right\|_2 &\leq K r^{\epsilon m}\left(1 - \left(C^{\epsilon}\mu_4^{\frac{1}{2}}\tilde{S}\right)^m\right) + 2M_4^{\frac{1}{2}}\left(\mu_4^{\frac{1}{2}}S\right)^{m+1} \\ &= O\left(\beta^m\right) \\ &= O\left(e^{-\rho^* m}\right), \end{aligned} \tag{2.24}$$

where $\beta = r^{\epsilon} \vee (\mu_4^{1/2} rC)^{\epsilon} \vee \mu_4^{1/2} S < 1$ and $\rho^* = -\log\beta > 0$. Combine the equation (2.15) and (2.24) to obtain the geometric L_2-NED property of u_t^2 and σ_t^2. Note that the first equality in (2.24) is obtained from the following inequality:

$$r^{\epsilon m}\left(1 - \left(C^{\epsilon}\mu_4^{\frac{1}{2}}\tilde{S}\right)^m\right) \leq \begin{cases} 2r^{\epsilon m}, & \text{if } \left(C\mu_4^{\frac{1}{2}}\right)^{-\epsilon} \geq \mu_4^{\frac{1-\epsilon}{2}}\tilde{S}, \\ 2\left(rC\mu_4^{\frac{1}{2}}\right)^{\epsilon m}, & \text{if } \left(C\mu_4^{\frac{1}{2}}\right)^{-\epsilon} < \mu_4^{\frac{1-\epsilon}{2}}\tilde{S}. \end{cases}$$

□

Remark 1. Compared to the results in Davidson (2004), Theorems 2 and 3 weaken sufficient conditions for L_2-NED property of u_t, u_t^2, and σ_t^2.

Theorem 4. *If one of the following conditions (a)–(c) is satisfied then the FCLT holds for the process u_t given by (2.1) and (2.2):*

(a) the Assumption (A1) with $\delta > 1$,

(b) the Assumption (A1) with $\delta > 1/2$ and $\mu_4^{1/2} S < 1$,

(c) the Assumption (A2).

Proof: Lemma 1 ensures that $S < 1$ implies the strict stationarity of σ_t^2 with $E(\sigma_t^2) < \infty$. In Theorems 2 and 3, it is shown that u_t is either L_2-NED of size $-1/2$ or geometrically L_2-NED under one of the above assumptions (a)–(c). Also, $\sigma_n^2 = \mathrm{Var}(\sum_{t=1}^n u_t) = nE(\sigma_t^2)$. Apply Theorem 1 to obtain the FCLT for u_t. $\square$

Theorem 5.

(3-1) If the Assumption (A1) with $\delta > 1/2$ and $\mu_4^{1/2}S < 1$, then the FCLT holds for u_t^2 and σ_t^2.

(3-2) If the Assumption (A2) and $\mu_4^{1/2}S < 1$, then the FCLT holds for u_t^2 and σ_t^2.

Proof: Lemma 1 shows that the condition $\mu_4^{1/2}S < 1$ is sufficient for the existence of $E(u_t^4)$ and the existence of weakly stationary solution of the process u_t^2. Moreover, by Proposition 3.1 in Giraitis *et al.* (2000), $\mu_4^{1/2}S < 1$ implies that

$$\sum_{t=1}^{\infty} \mathrm{Cov}\left(u_t^2, u_0^2\right) < \infty. \tag{2.25}$$

Also, from weak stationarity of u_t^2,

$$\mathrm{Var}\left(\sum_{t=1}^{n} u_t^2\right) = \sum_{t=1}^{n} \mathrm{Var}\left(u_t^2\right) + 2\sum_{t=1}^{n}(n-t)\mathrm{Cov}\left(u_t^2, u_0^2\right). \tag{2.26}$$

From (2.25) and (2.26), as $n \to \infty$,

$$\frac{1}{n}\mathrm{Var}\left(\sum_{t=1}^{n} u_t^2\right) \longrightarrow \mathrm{Var}\left(u_0^2\right) + 2\sum_{t=1}^{\infty} \mathrm{Cov}\left(u_t^2, u_0^2\right) < \infty. \tag{2.27}$$

Proof of Theorem 5(3-1): Theorem 2(1-2) shows that under the assumptions, u_t^2 and σ_t^2 are L_2-NED of size $-1/2$. Therefore, the FCLT for u_t^2 and σ_t^2 follows from (2.14), (2.15), (2.27) and Theorem 1.

Proof of Theorem 5(3-2): In Theorem 3(2-2), it is shown that under the given assumptions, u_t^2 and σ_t^2 are L_2-NED of size $-1/2$. Then the FCLT for u_t^2 and σ_t^2 are obtained from (2.15), (2.21), (2.27), and Theorem 1. $\square$

Remark 2. Assume $\mu_4^{1/2}S < 1$. It is known that if the exponential decay of the coefficient θ_j in (2.2) implies the exponential decay of the covariance function of the sequence $\{u_t^2\}$. On the other hand, if $\theta_j \leq Cj^{-1-\delta}$, $\delta > 0$, then the hyperbolic decay of the covariance function of u_t^2 is proved, that is, there exists $K > 0$ such that for $t \geq 1$, $\mathrm{Cov}(u_t^2, u_0^2) \leq Kt^{-1-\delta}$ (Giraitis *et al.*, 2000; Zaffaroni, 2004).

Example 1. Under proper constraints, conditional variance σ_t^2 of various GARCH-type process can be rewritten as an ARCH(∞) model. The FCLT for various GARCH-type model including augmented GARCH, asymmetric power GARCH (APGARCH), vector GARCH (VGARCH), exponential GARCH (EGARCH) as well as the classical GARCH model is studied in Lee (2014a). For the classical GARCH model

$$u_t = \sigma_t e_t, \quad \sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i u_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \quad (\omega > 0,\ \alpha_i \geq 0,\ \beta_j \geq 0),$$

recall that if $\sum \alpha_i + \sum \beta_j < 1$, then the process satisfies the Assumption (A2) and $S < 1$. Thus, Theorem 3(2-3) ensures that $\sum \alpha_i + \sum \beta_j < 1$ is sufficient for u_t to be L_2-NED of size $-1/2$ and the FCLT holds for u_t. Lee (2014a) shows that $\sum \|\alpha_i e_t^2 + \beta_i\|_2 < 1$ is sufficient for the FCLT for u_t^2 and σ_t^2. Note that $\mu_4 \geq 1$ and $\sum \|\alpha_i e_t^2 + \beta_i\|_2 \leq \sum(\mu_4^{1/2}\alpha_i + \beta_i) < 1$ if $\mu_4^{1/2}S < 1$.

Example 2. Results obtained in this section can be easily extended to a general ARCH(∞) model. Consider the following process

$$u_t = \sigma_t e_t, \quad \sigma_t^d = \omega + \sum_{j=1}^{\infty} \theta_j |u_{t-j}|^d \quad (d > 0,\ \omega > 0,\ \theta_j \geq 0). \tag{2.28}$$

If $E|e_0|^{2d} < \infty$ and $(E|e_0|^{2d})^{1/2} \sum \theta_j < 1$, then a unique strictly stationary and weak stationary solution to (2.28) with $E|u_t|^{2d} < \infty$ exists. If θ_j in (2.28) satisfies the condition (A1) (or (A2)) and $(E|e_0|^{2d})^{1/2} \sum \theta_j < 1$, then the FCLT holds for $|u_t|^d$ and σ_t^d. If θ_j satisfies the condition (A2), then the FCLT holds for $|u_t|^{d/2}$.

Example 3. Consider the HYGARCH model which is given by

$$u_t = \sigma_t e_t, \quad \sigma_t^2 = \omega + \theta(L) u_t^2, \quad (\omega > 0) \tag{2.29}$$

where $\theta(L) = 1 - (\delta(L)/\beta(L))(1 + \alpha((1-L)^\delta - 1))$ $(\alpha \geq 0, \delta \geq 0)$. Here L is the lag operator defined by $Ly_t = y_{t-1}$. HYGARCH model given by (2.29) includes IGARCH, FIGARCH, and classical GARCH models depending on the values of α and δ. If $\delta > 0$, then $S = 1 - (\delta(1)/\beta(1))(1-\alpha)$. When δ in (2.29) is not too large, then this model will correspond closely to the following case

$$\theta(L) = 1 - \frac{\delta(L)}{\beta(L)}(1 - \alpha\phi(L)), \quad \phi(L) = \zeta(1+\delta)^{-1} \sum_{j=1}^{\infty} j^{-1-\delta} L^j, \quad (\delta > 0) \tag{2.30}$$

and $\zeta(\cdot)$ is the Riemann zeta function (Davidson, 2004). Note that $\delta > 1$ in (2.29) gives rise to negative coefficients where as δ in (2.30) can take any positive values. Let $\delta > 1$ in (2.30) and $S = 1 - (\delta(1)/\beta(1))(1-\alpha) < 1$, then Theorem 4(a) yields the FCLT for u_t in (2.29) with $\theta(L)$ given by (2.30).

Example 4. For an ARCH(∞) model in order to $\sigma_t^2 \geq 0$ with probability 1, all its coefficients are expected to be nonnegative. In general, nonnegative coefficients condition for HYGARCH model are more complicated than those of FIGARCH (Conrad and Haag, 2006; Conrad, 2010). Li *et al.* (2015) suggests the following so called HGARCH process

$$u_t = \sigma_t e_t, \quad \sigma_t^2 = \frac{\gamma}{\beta(1)} + \omega\left\{1 - \frac{\delta(L)}{\beta(L)}(1-L)^\delta\right\} u_t^2, \quad (0 < \delta \leq 1, \omega > 0, \gamma > 0). \tag{2.31}$$

The process given by (2.31) allows the existence of finite variance as in HYGARCH models, while it has a form nearly as simple as FIGARCH models. σ_t^2 in (2.31) can be rewritten as $\sigma_t^2 = \gamma/\beta(1) + \sum_{j=1}^{\infty} \theta_j u_{t-j}^2$. When $\omega < 1$, $S = \sum \theta_j = \omega < 1$ and there exists a unique strictly stationary solution u_t^2 to (2.31) with $E(u_t^2) < \infty$. If in addition $\mu_4^{1/2}\omega < 1$, then applying Theorem 5 yields the FCLT for u_t^2 and σ_t^2.

3. Simulations

3.1. Structural breaks of the ARCH(∞) model

As an application of the FCLT, we consider the cumulative sum (CUSUM) tests for mean break and variance break.

H_0 : no structural breaks versus H_1 : not H_0.

The following CUSUM statistics are the most often used statistics to test for the stability of $\{f(u_t) : 1 \le t \le n\}$:

$$Q_n^M = \frac{1}{\hat{\sigma}_n \sqrt{n}} \max_{1\le k\le n} \left| \sum_{1\le i\le k} f(u_i) - \frac{k}{n} \sum_{1\le i\le n} f(u_i) \right|, \quad f(u_i) = u_i$$

and

$$Q_n^V = \frac{1}{\hat{\sigma}_n \sqrt{n}} \max_{1\le k\le n} \left| \sum_{1\le i\le k} f(u_i) - \frac{k}{n} \sum_{1\le i\le n} f(u_i) \right|, \quad f(u_i) = u_i^2,$$

where

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{j=1}^{n} \left(f(u_j) - \overline{f(u_n)}\right)^2 + \frac{2}{n} \sum_{j=1}^{q} \left(1 - \frac{j}{q+1}\right) \sum_{i=1}^{n-j} \left(f(u_i) - \overline{f(u_n)}\right)\left(f(u_{i+j}) - \overline{f(u_n)}\right), \quad q < n$$

and $\overline{f(u_n)} = (1/n)\sum_{i=1}^{n} f(u_i)$, $0 \le i \le n$. According to Theorem 4 and 5, asymptotic null distributions of Q_n^M and Q_n^V are all standard Brownian bridges (Csörgő and Horváth, 1997; Hwang and Shin, 2013).

3.2. A Monte-Carlo study

We conduct a simulation to examine the finite sample sizes and powers of the CUSUM test for breaks. In this simulation study, we perform a test at a nominal level $\alpha = 0.05$. The empirical sizes and powers are calculated as the rejection number of the null hypothesis out of 1,000 repetitions. In order to see the performance of Q_n, we generate data by approximating ARCH(∞) by ARCH(10) model

$$u_t = \sigma_t e_t, \quad \sigma_t^2 = \omega + \sum_{i=1}^{10} \theta_i u_{t-i}^2,$$

where $\{e_t\}$ is a sequence of independent standard normal errors. We evaluate Q_n with sample sizes $n = 1{,}000, 2{,}000$, and $4{,}000$. For power study of mean break tests, we add 0.002 to u_t for all $t > n/2$. For power study of variance breaks test, we multiply 1.1 to e_t for all $t > n/2$. The parameters for the ARCH model are chosen as in Table 1: D_1, D_2, and D_3 for ARCH(10) models with $\sum_{j=1}^{10}\theta_j = 0.86$, $\sum_{j=1}^{10}\theta_j = 0.84$, and $\sum_{j=1}^{10}\theta_j = 0.90$, respectively which are estimation results for three data sets that will be analyzed in Subsection 3.3 below.

The finite sample performance depends on the sample size n as well as the bandwidth parameter q used to estimate the long-run variance and covariance. Since the optimal bandwidth is $O(n^{1/3})$ for the Bartlett kernel and the tests are very sensitive to q, we consider wide range of q values that are 1/3-order bandwidth: $q_1 = [2n^{1/3}]$ and $q_2 = [4n^{1/3}]$. Table 2 summarizes the empirical sizes and powers of mean break tests.

Table 1: Parameters for DGP

DGP	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
D_1	0.054	0.130	0.071	0.098	0.153	0.038	0.074	0.048	0.105	0.084
D_2	0.046	0.135	0.092	0.105	0.074	0.062	0.074	0.093	0.091	0.063
D_3	0.156	0.135	0.074	0.102	0.106	0.019	0.049	0.055	0.071	0.131

Table 2: Size (%) and power (%) of CUSUM test Q_n^M

n	q	D_1		D_2		D_3	
		Size	Power	Size	Power	Size	Power
1,000	20	4.0	84.3	3.9	96.3	3.7	92.1
1,000	40	2.9	83.1	2.7	93.4	3.9	90.4
2,000	25	4.3	92.6	4.4	98.7	4.1	96.8
2,000	50	4.1	93.7	3.7	96.2	3.3	98.2
4,000	31	3.9	97.4	3.8	99.3	4.2	98.9
4,000	63	4.5	98.9	3.9	99.7	4.3	99.8

Nominal level is 5%; number of replication is 1,000. CUSUM = cumulative sum.

Table 3: Size (%) and power (%) of CUSUMSQ test Q_n^V

n	q	D_1		D_2		D_3	
		Size	Power	Size	Power	Size	Power
1,000	20	21.9	40.4	23.4	47.0	27.6	44.1
1,000	40	4.3	13.2	5.6	23.4	6.5	16.8
2,000	25	23.4	45.9	20.9	54.5	25.9	44.7
2,000	50	6.2	35.3	5.7	30.1	7.0	24.1
4,000	31	21.3	72.1	15.0	66.7	20.7	51.8
4,000	63	5.6	51.8	5.5	62.5	5.3	29.4

Nominal level is 5%; number of replication is 1,000. CUSUMSQ = cumulative sum of squares.

Table 2 show that Q_n^M has no severe size distortions in most cases. The empirical sizes are reasonably close to the nominal level 0.05 as n increases. Meanwhile, we can see that the powers are close to 0.9 when the sample size n is over 2000.

In Table 3, the size block shows that Q_n^V has unstable sizes. In addition, the power values substantially decrease as q increases in ARCH(10) model. Since $f(u_t)$ is strongly autocorrelated when $f(u_t) = u_t^2$, it is important to estimate long-run variance. The performance of estimator is sensitive to bandwidth q which is used to estimate σ^2 and represents another research area in selecting an optimal bandwidth.

3.3. Real data analysis

In this section, we apply our tests to three real data sets: log-returns of the KOSPI, the S&P500 index, and the KRW/USD exchange rate during the period from January 2, 2007 to December 29, consisting of 2480, 2480, 2518 observations.

In Figure 1, we observe that the log-returns rapidly fluctuate and spike to a peak around the year 2009. It shows the volatility change during global financial crisis of 2008. Through the graphs, we find that three log-returns might have some breaks: in 2008 and in 2011.

We first apply the goodness-of-fit test to examine whether the ARCH(10) model fits the data well. Since the obtained p-values are 0.9467, 0.7265, and 0.8580, respectively, we conclude that these three data sets are well fitted to ARCH(10) model. We perform the CUSUM tests and CUSUMSQ tests for these data sets.

We see significant CUSUM test for the S&P500 index with p-values 2.1%, which implies the

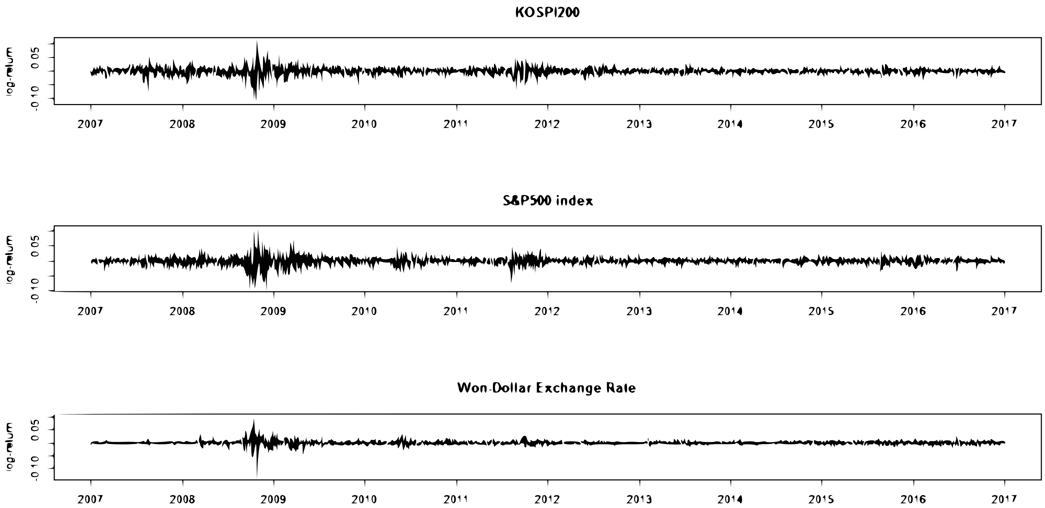

Figure 1: *Time series plots of log-returns for 2007–2016.*

Table 4: CUSUM test Q_n^M for log-returns for 2007–2016

	Q_n^M	p-value(%)
KOSPI	0.768	59.7
S&P500	1.511	2.1
KRW/USD	1.179	12.4

CUSUM = cumulative sum.

Table 5: CUSUMSQ test Q_n^V for log-returns for 2007–2016

	Q_n^V	p-value(%)
KOSPI	2.289	0.01
KRW/USD	1.610	1.10

CUSUMSQ = cumulative sum of squares.

presence of at least one mean break. However, the KOSPI and the KRW/USD exchange rate have no significant p-values for the CUSUM tests. Therefore, the CUSUM test does not provide us statistical evidence for mean break for the KOSPI and the KRW/USD exchange rate (Table 4).

We now perform the CUSUMSQ tests for the KOSPI and the KRW/USD exchange rate, in which no mean shifts exist. In these cases we see significant CUSUMSQ tests for the KOSPI and the KRW/USD exchange rate with p-values 0.01% and 1.1%, respectively. The two data sets have at least one variance break; however, the result does not involve the number of breaks and the dates for the break times (Table 5).

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