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Sampled-data Fuzzy Observer Design for an Attitude and Heading Reference System and Its Experimental Validation

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Abstract – In this paper, a linear matrix inequality-based sampled-data fuzzy observer design method is proposed based on the exact discretization approach. In the proposed design technique, a numerically relaxed observer design condition is obtained by using the discrete-time fuzzy Lyapunov function. Unlike the existing studies, the designed observer is robust to the uncertain premise variable because the fuzzy observer is designed under the imperfect premise matching condition, in which the membership functions of the system and observer are mismatched. In addition, we apply the proposed method to the state estimation problem of the attitude and heading reference system (AHRS). To do this, we derive a Takagi-Sugeno fuzzy model for the AHRS system, and validate the proposed method through the hardware experiment.

Keywords: Sampled-data fuzzy observer, Exact discretization approach, Discrete-time fuzzy Lyapunov function, Attitude and heading reference system (AHRS), State estimation

1. Introduction

The state estimation problem has been an important research topic for decades. In general, the state estimation of the Takagi–Sugeno (T–S) fuzzy systems [1] has been carried out based on either a fuzzy filter-based method [2, 3] or a fuzzy observer-based method [4, 5]. The fuzzy filter method provides a systematic approach to the state estimation method for the fuzzy model, and has gained much interest. However, it is disadvantageous in that it can be applied only when the given estimation model is asymptotically stable. As an alternative, fuzzy observer-based methods have been actively studied. The most severe problem of the fuzzy observer approach is that the approach requires an accurate system model. In addition, there are few researches on the fuzzy observer design that are robust to the system disturbance or the measurement noise that occur frequently in the practical applications [10].

Recently, there has been an increasing demand for the digital computer-based control engineering. In a digital computer-based sensor system, the measurement model operates in the continuous-time domain, but a sensor provides measurements only at each sampling time [6]. Thus, these sampled-data systems contain both the continuous- and discrete-time state variables at the same time, which complicates the system analysis. To simplify the problem, in [8], the authors proposed an exact discretization method for the sampled-data stabilization problem of the T-S fuzzy model. In this approach, the sampled-data system is discretized, and the system analysis is carried out in the discrete-time domain based on the resulting discretized system model [7-10]. Although this approach has been successfully adopted in analyzing sampled-data fuzzy systems, the studies on this approach have mainly focused on the controller design; thus, the study regarding the observer design are still lacking. Moreover, because the existing methods was derived based on the common quadratic Lyapunov function, the resulting stability conditions are numerically conservative [11].

In general, stabilization conditions for the fuzzy model has been derived based on the common quadratic Lyapunov function. However, this provides numerically conservative results since a single Lyapunov function is determined for satisfying all fuzzy rules. To obtain relaxed results, the authors of [12] performed a study using a discrete-time fuzzy Lyapunov function. The fuzzy Lyapunov function is composed of multiple quadratic Lyapunov functions, and each quadratic Lyapunov function is related to each fuzzy rule. Moreover, the resulting stability condition contains more decision variables than that form the common quadratic Lyapunov function, which numerically relaxes the resulting stability conditions. After the work, a lot of studies have been carried out on the discrete-time fuzzy Lyapunov function-based stability analysis for discrete-time fuzzy model [12-14]. However, there are still lacking studies on the stabilization of the sampled-data system based this Lyapunov function.

Apart from the above issues, it is notable that the demand for unmanned aerial vehicles (UAV) has been surprisingly increased for decades [15]. The most important technique in the actual implementation of the UAV is to obtain the attitude and heading of the aircraft.

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Generally, the attitude and heading reference system (AHRS) are constructed via the sensor fusion algorithm using the acceleration, gyro, and magnetic field sensor data [16]. Various approaches have been carried out to develop the AHRS. Among them, the Kalman filter method [17-19] and the complementary filter method [20, 21] have been most widely studied, which are the most commonly used state estimation methods. The Kalman filter method provides theoretically an optimal state estimation solution, but computational demand is high. As a result, in many practical applications, the AHRS has been constructed using the complementary filter that is computationally relaxed. However, there is lacking studies on systematically determining the design parameters for both the Kalman and complementary filter, which makes the use of these methods difficult. Therefore, it is required to develop a filtering method that is easy to design and computationally relaxed.

Motivated by the above, in this paper, a linear matrix inequality (LMI)-based sampled-data fuzzy observer design method is proposed based on the exact discretization approach. In the proposed design technique, a numerically relaxed observer design condition is obtained by using the discrete-time fuzzy Lyapunov function. Unlike the existing studies, the designed observer is robust to the uncertain premise variable because the fuzzy observer is designed under the imperfect premise matching condition, in which the membership functions of the system and observer are mismatched. In addition, we apply the proposed method to the state estimation problem of the AHRS. To this end, we derive a T–S fuzzy model for the AHRS system, and validate the proposed method through the hardware experiment.

The rest of the paper consists of the following sections: In the second section, the preliminaries for the sampled-data fuzzy observer system is briefly reviewed. In Section 3, the AHRS system model is introduced, and its fuzzy modeling procedure is provided. Section 4 provides the proposed LMI-based sampled-data observer design method. In the next section, the hardware experimental results are analyzed, and finally, the concluding remarks are given in Section 6.

2. Preliminaries

Throughout this paper, $\mathcal{I}_n$ represents the set, $\{1,2,\dots,n\}$, where n is a positive scalar. For any matrix X, we use the following shorthand notation $\mathrm{He}(X)=X+X^T$ for simplicity. Moreover, λ_X represents the maximum eigenvalue of $X^T(t)X(t)$ for all t.

Now, consider the following T–S fuzzy system:

$$\begin{aligned}\dot{x}(t)&=A\big(w(t)\big)x(t)+B\big(w(t)\big)\varpi(t),\\ y(t)&=C_yx(t)+Dv(t),\\ z(t)&=C_zx(t),\end{aligned}\tag{1}$$

where $x(t)\in\mathbb{R}^{n_x}$, $y(t)\in\mathbb{R}^{n_y}$, and $z(t)\in\mathbb{R}^{n_z}$ are the state, measurement, and output vectors, respectively; $\varpi(t)\in L_2^{n_\varpi}[0,\infty)$ and $v(t)\in L_2^{n_v}[0,\infty)$ are disturbance and noise vectors, respectively; $A(w(t))=\sum_{i=1}^{r} w_i(t)A_i\in\mathbb{R}^{n_x\times n_x}$ and $B(w(t))=\sum_{i=1}^{r} w_i(t)B_i\in\mathbb{R}^{n_x\times n_\varpi}$ are the state and disturbance input matrices, respectively; $C_x\in\mathbb{R}^{n_y\times n_x}$, $C_z\in\mathbb{R}^{n_z\times n_x}$, and $D\in\mathbb{R}^{n_y\times n_v}$ are the measurement, output, and noise matrices, respectively; $i\in\mathcal{I}_r$; and $w_i(\sigma(t))$ is the ith fuzzy weighting function, and satisfies the following conditions:

$$w_i\big(\sigma(t)\big)\in[0,1]\text{ and }\sum_{i=1}^{r} w_i(\sigma(t))=1,$$

where $\sigma(t)$ is a premise variable.

In this paper, we assume that the T–S fuzzy system (1) satisfies the following assumption:

Assumption 1: *The T–S fuzzy system (1) is observable, and the premise variable $\sigma(t)$ and the measurement vector $y(t)$ are measurable.*

Under the assumption, we use the following sampled-data fuzzy observer estimating $x(t)$:

$$\begin{aligned}\dot{\hat{x}}(t)&=A\big(m(t)\big)\hat{x}(t)+L\big(m(t)\big)\big(y(t_k)-\hat{y}(t_k)\big),\\ \hat{y}(t_k)&=C_y\hat{x}(t_k)\end{aligned}\tag{2}$$

for $t\in[t_k,t_{k+1})$, where $t_k=kh$ is the kth sampling time; $h:=t_{k+1}-t_k$ is a constant sampling period; $\hat{x}(t)\in\mathbb{R}^{n_x}$ and $\hat{y}(t_k)\in\mathbb{R}^{n_y}$ are the state and output vectors for the observer; $A(m(t))=\sum_{i=1}^{r} m_i(\sigma(t))A_i\in\mathbb{R}^{n_x\times n_x}$ and $L(m(t))=\sum_{i=1}^{r} m_i(\sigma(t))L_i\in\mathbb{R}^{n_x\times n_y}$ are the state and observer gain matrices, respectively; and $m_i(\sigma(t))$ is the ith fuzzy weighting function for the sampled-data fuzzy observer.

Now, by defining $e(t):=x(t)-\hat{x}(t)$, and from (1) and (2), we have the following estimation error dynamics:

$$\begin{aligned}\dot{e}(t)&=\dot{x}(t)-\dot{\hat{x}}(t)\\ &=A\big(w(t)\big)x(t)+B\big(w(t)\big)\varpi(t)\\ &\quad-\{A\big(m(t)\big)\hat{x}(t)+L\big(m(t)\big)\big(y(t_k)-\hat{y}(t_k)\big)\}\\ &=\{A\big(w(t)\big)-A\big(m(t)\big)+A\big(m(t)\big)\}x(t)\\ &\quad-A\big(m(t)\big)\hat{x}(t)+B\big(w(t)\big)\varpi(t)\\ &\quad-\mathrm{L}\big(m(t)\big)\{C_yx(t_k)+Dv(t_k)-C_y\hat{x}(t_k)\}\\ &=A\big(m(t)\big)e(t)-L\big(m(t)\big)C_ye(t_k)+B\big(w(t)\big)\varpi(t)\\ &\quad-L(m(t))Dv(t_k)+\{A(w(t))-A(m(t))\}x(t),\end{aligned}$$

and by employing the notation, $\bar{e}(t)=e(t)-e(t_k)$, we can rewrite it as follows:

$$\begin{aligned}\dot{e}(t)&=\Lambda\big(m(t)\big)e(t_k)+A\big(m(t)\big)\bar{e}(t)\\ &\quad+B\big(w(t)\big)\varpi(t)-\Phi\big(m(t)\big)v(t_k)\end{aligned}$$

$$+\Delta_A(t)x(t), \quad (3)$$

where $\Lambda(m(t)) = A(m(t)) - L(m(t))C_y$; $\Phi(m(t)) = L(m(t))D$; and $\Delta_A(t) = A(w(t)) - A(m(t))$.

3. The AHRS and Its Fuzzy Model

3.1 The system model

In this section, the quaternion-based kinematic model is employed for the attitude prediction. It has been well-studied from researchers that the quaternion-based attitude kinematics model can be represented by the following nonlinear differential equation [19]:

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\\ \dot{q}_3(t)\\ \dot{q}_4(t)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & -p(t) & -q(t) & -r(t)\\ p(t) & 0 & r(t) & -q(t)\\ q(t) & -r(t) & 0 & p(t)\\ r(t) & -q(t) & -p(t) & 0\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\\ q_4(t)\end{bmatrix}, \quad (4)$$

where $q_1(t)$, $q_2(t)$, $q_3(t)$, and $q_4(t)$ are quaternions; $p(t)$, $q(t)$, $r(t)$ are measurements from the gyroscope in the x, y, and z axis, respectively.

We can represent the quaternion-based attitude kinematics model (4) as the following nonlinear state-space equation:

$$\dot{x}(t) = A(t)x(t), \quad (5)$$

where

$$A(t) = \frac{1}{2}\begin{bmatrix}0 & -p(t) & -q(t) & -r(t)\\ p(t) & 0 & r(t) & -q(t)\\ q(t) & -r(t) & 0 & p(t)\\ r(t) & -q(t) & -p(t) & 0\end{bmatrix};$$

$$\text{and } x(t) = \begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} = \begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\\ q_4(t)\end{bmatrix}.$$

Moreover, we confine the operating regions of $p(t)$, $q(t)$, and $r(t)$ as follows:

$$p(t) \in [-M_p, M_p],\quad q(t) \in [-M_q, M_q], \text{ and } r(t) \in [-M_r, M_r], \quad (6)$$

where M_p, M_q, and M_r are given positive scalars.

By choosing the premise vector as $\sigma(t) =$, $[p(t)\quad q(t)\quad r(t)]^T$ we can construct the following membership functions:

$$\mu_p^1(p(t)) = \frac{p(t)+M_p}{2M_p}, \mu_p^2(p(t)) = 1 - \mu_p^1(p(t)),$$

$$\mu_q^1(q(t)) = \frac{q(t)+M_q}{2M_q}, \mu_q^2(q(t)) = 1 - \mu_q^1(q(t)),$$

$$\mu_r^1(r(t)) = \frac{r(t)+M_r}{2M_r}, \mu_r^2(r(t)) = 1 - \mu_r^1(r(t)).$$

Summarizing the above, we have the following T–S fuzzy model with eight rules that represents (5) under (6):

$$\dot{x}(t) = \sum_{i=1}^{r} w_i(\sigma(t))A_i x(t), \quad (7)$$

where

$$w_1(\sigma(t)) = \mu_p^1(p(t))\mu_q^1(q(t))\mu_r^1(r(t));$$
$$w_2(\sigma(t)) = \mu_p^2(p(t))\mu_q^1(q(t))\mu_r^1(r(t));$$
$$w_3(\sigma(t)) = \mu_p^1(p(t))\mu_q^2(q(t))\mu_r^1(r(t));$$
$$w_4(\sigma(t)) = \mu_p^2(p(t))\mu_q^2(q(t))\mu_r^1(r(t));$$
$$w_5(\sigma(t)) = \mu_p^1(p(t))\mu_q^1(q(t))\mu_r^2(r(t));$$
$$w_6(\sigma(t)) = \mu_p^2(p(t))\mu_q^1(q(t))\mu_r^2(r(t));$$
$$w_7(\sigma(t)) = \mu_p^1(p(t))\mu_q^2(q(t))\mu_r^2(r(t));$$
$$w_8(\sigma(t)) = \mu_p^2(p(t))\mu_q^2(q(t))\mu_r^2(r(t));$$
$$A_i = \frac{1}{2}\begin{bmatrix}0 & -p_i & -q_i & -r_i\\ p_i & 0 & r_i & -q_i\\ q_i & -r_i & 0 & p_i\\ r_i & q_i & -p_i & 0\end{bmatrix};$$

and p_i, q_i, and r_i are the ith elements of the following sets:

$$p = \{M_p, -M_p, M_p, -M_p, M_p, -M_p, M_p, -M_p\};$$
$$q = \{M_q, M_q, -M_q, -M_q, M_q, M_q, -M_q, -M_q\};$$
$$r = \{M_r, M_r, M_r, M_r, -M_r, -M_r, -M_r, -M_r\},$$

respectively.

The output model is chosen as

$$z(t) = C_z x(t) = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix}. \quad (8)$$

3.2 The measurement model

The accelerometer and magnetometer outputs are used in measurement model. It is well-known that the Euler angles can be computed by the following equation [20]:

$$\begin{bmatrix}\phi(t)\\ \theta(t)\\ \psi(t)\end{bmatrix} = \begin{bmatrix}\mathrm{atan}(a_y(t)/a_z(t))\\ \mathrm{atan}(-a_x(t)/\sqrt{a_y^2(t)+a_z^2(t)})\\ \mathrm{atan}(C_y/C_x)\end{bmatrix}, \quad (9)$$

where $\phi(t)$, $\theta(t)$, and $\psi(t)$ are Euler angles in x, y, and z axis, respectively; $a_x(t)$, $a_y(t)$, and $a_z(t)$ are the accelerometer outputs of each axis;

$$C_y = -m_y(t)\cos(\phi(t)) + m_z(t)\sin(\phi(t));$$
$$C_x = m_x(t)\cos(\theta(t)) + m_y\sin(\phi(t))\sin(\theta(t)) + m_z(t)\cos(\phi(t))\sin(\theta(t));$$

and $m_x(t)$, $m_y(t)$, and $m_z(t)$ are the magnetometer outputs of each axis.

Finally, the measurement output equation is as follows:

$$y(t) = C_y x(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix}, \tag{10}$$

where $y_s(t)$ with $s \in \mathcal{I}_4$ is the measurement output variable; $q_s(t)$ with $s \in \mathcal{I}_4$ is the quaternion, and has the following relation between the Euler angles:

$$\begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} = \begin{bmatrix} c_\phi c_\theta c_\psi + s_\phi s_\theta s_\psi \\ s_\phi c_\theta c_\psi - c_\phi s_\theta s_\psi \\ c_\phi s_\theta c_\psi + s_\phi c_\theta s_\psi \\ c_\phi c_\theta s_\psi - s_\phi s_\theta c_\psi \end{bmatrix}, \tag{11}$$

where $c_\phi = \cos(\phi(t)/2)$; $s_\phi = \sin(\phi(t)/2)$; $c_\theta = \cos(\theta(t)/2)$; $s_\theta = \sin(\theta(t)/2)$; $c_\psi = \cos(\psi(t)/2)$; and $s_\psi = \sin(\psi(t)/2)$.

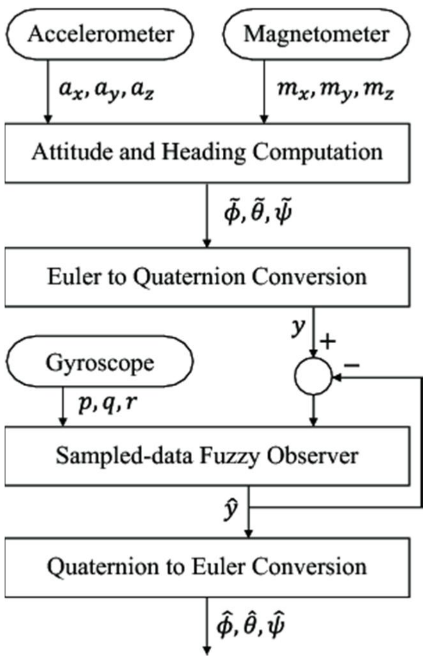

Fig. 1. The overall configuration of the proposed AHRS system

3.3 The system configuration

The overall configuration of the proposed AHRS system is shown in Fig. 1. In the figure, a_x, a_y, and a_z represent the accelerometer readings on each axis, m_x, m_y, and m_z are magnetometer readings on each axis, p, q, and r are angular velocities on each axis, $\tilde{\phi}$, $\tilde{\theta}$, and $\tilde{\psi}$ mean the computed Euler angles by the accelerometer and magnetometer readings, and $\hat{\phi}$, $\hat{\theta}$, and $\hat{\psi}$ are estimated Euler angles.

The gyroscope outputs on each axis are used to update the system model (7). The disturbance and the noise on the gyroscope readings are modeled as the disturbance vector, from which we can rewrite the system model (7) as follows:

$$\dot{x}(t) = \sum_{i=1}^{8} w_i(\sigma(t))\{A_i x(t) + B_i \varpi(t)\}, \tag{12}$$

where B_i with $i \in \mathcal{I}_8$ represents the noise specifications of the gyroscope, and is assumed to be unknown but norm bounded.

Therefore, the premise variable $\sigma(t)$ is composed of the actual noise-free angular velocities. Unfortunately, the gyroscope in real world cannot provide the actual angular velocity. The premise variable used in the observer model (2) is assumed to be disturbed, and thus, we can model the fuzzy weighting function of the fuzzy observer as follows:

$$m_i(\sigma(t)) = w_i(\sigma(t) + \Delta_\sigma(t)),$$

where $\Delta_\sigma(t)$ represents the noise on the gyroscope readings.

On the other hand, to operate the fuzzy observer (2), the measurement output on the kth sampling time, $y(t_k)$, is required. In this paper, $y(t_k)$ is updated by using (10) with (11) based on the accelerometer and magnetometer outputs.

Finally, the estimated quaternion values at $t = t_{k+1}$ are obtained by integrating (2) as follows:

$$\hat{x}(t_{k+1}) = \hat{x}(t_k) + \int_{t_k}^{t_{k+1}} \{A(m(\tau))\hat{x}(\tau) + L(m(\tau))(y(t_k) - \hat{y}(t_k))\}d\tau,$$
$$\hat{y}(t_{k+1}) = C_y \hat{x}(t_{k+1}), \tag{13}$$

under the assumption that $\hat{x}(t_k)$ is known. The integration can be done numerically by various approaches.

4. LMI-based Sampled-data Fuzzy Observer Design

In this section, we derive an LMI-based sampled-data fuzzy observer design method based on the exact discretization approach. To this end, we use the following discrete-time fuzzy Lyapunov function:

$$V(t_k) = \sum_{i=1}^{r} w_i\big(\sigma(t_k)\big)e^T(t_k)P_ie(t_k) = e^T(t_k)P(w(t_k))e(t_k), \tag{14}$$

where $0 \prec P_i = P_i^T \in \mathbb{R}^{n_x \times n_x}$ with $i \in \mathcal{I}_r$ is a positive definite matrix to be determined. Before proceeding, we introduce some lemmas and a proposition required in deriving the proposed observer design method:

Lemma 1 [22] *Given any function vector* x, *matrix* $P = P^T \succ 0$, *and* t_0, $t_f \in \mathbb{R}_{>0}$ *with* $t_0 < t_f$, *the following holds:*

$$\left\{\int_{t_0}^{t_f} x(\tau)d\tau\right\}^T P\left\{\int_{t_0}^{t_f} x(\tau)d\tau\right\} \le (t_f - t_0)\int_{t_0}^{t_f} x^T(\tau)Px(\tau)d\tau. \tag{15}$$

Lemma 2 [23] *Suppose the nonlinear system* $\dot{x} = f(t,x)$, *where* $f: [t_k, t_{k+1}) \times \mathbb{R}^n$ *is piecewise continuous in* t *and locally Lipschitz in* x, *and the matrix* $P = P^T \succ 0$; *then, the following inequality always holds:*

$$\int_{t_k}^{t_{k+1}} (x(t) - x(t_k))^T P\big(x(t) - x(t_k)\big)dt \le h^2 \int_{t_k}^{t_{k+1}} \dot{x}^T(t)P\dot{x}(t)dt, \tag{16}$$

where $h = t_{k+1} - t_k$ for any $k \in \mathbb{Z}_{\ge 0}$.

Proposition 1 *The equilibrium of the estimation error (3) is asymptotically stable under* $\varpi(t) = v(t) = 0$ *whenever the equilibrium of* $e(t_k)$ *is asymptotically stable.*

Proof. By integrating (3) and taking norm on both sides under $\varpi(t) = v(t) = 0$, we have

$$\begin{aligned}\|e(t)\| &= \|e(t_k)\| + \int_{t_k}^{t} \{\|A(m(\tau))\|\|e(\tau)\| \\ &\quad + \|L(m(\tau))C_y\|\|e(t_k)\| + \|\Delta(\tau)\|\|x(\tau)\|\}d\tau \\ &\le (1+ha)\|e(t_k)\| + \int_{t_k}^{t} \mathbf{S}_A\|e(\tau)\|d\tau + h\mathbf{S}_\Delta\mathbf{S}_x \\ &= \sqrt{(1+ha)^2 + (h\mathbf{S}_\Delta)^2}\left\|\begin{bmatrix}\mathbf{S}_x \\ e(t_k)\end{bmatrix}\right\| \\ &\quad + \int_{t_k}^{t} \mathbf{S}_A\|e(\tau)\|d\tau,\end{aligned} \tag{17}$$

where $a = \sup_\tau\|-L(m(\tau))C_y\|$; $\mathbf{S}_A = \sup_\tau\|A(m(\tau))\|$; $\mathbf{S}_\Delta = \sup_\tau\|\Delta(\tau)\|$; and $\mathbf{S}_x = \sup_\tau\|x(\tau)\|$.

Finally, an application of the Gronwall-bellman inequality to $\|e(t)\|$ constructs the following inequality:

$$\|e(t)\| \le \sqrt{(1+ha)^2 + (h\mathbf{S}_\Delta)^2}\left\|\begin{bmatrix}\mathbf{S}_x \\ e(t_k)\end{bmatrix}\right\| \exp(\mathbf{S}_A h).$$

Therefore, $\| e(t) \| \to 0$ if $\| e(t_k) \| \to 0$. This concludes the proof.

The objective of this section is to solve the following problem:

Problem 1 *For a given sampling period* h, *find a observer gain matrix such that the T–S fuzzy system (1) and the sampled-data fuzzy observer (2) satisfies the following conditions:*

1. The equilibrium of the error dynamics (3) is asymptotically stabilized under $\varpi(t) = v(t) = 0$.
2. The estimation error $e(t)$ satisfies the following H_∞ criterion under the zero initial condition:

$$\int_0^\infty e^T(\tau)Qe(\tau)d\tau \le \int_0^\infty \{\gamma_1^2 x^T(\tau)x(\tau) + \gamma_2^2\varpi^T(\tau)\varpi(\tau) + \gamma_3^2 v^T(\tau)v(\tau)\}d\tau, \tag{18}$$

where Q is a given positive definite matrix of an appropriate dimension; and γ_1, γ_2, and γ_3 are positive scalars.

The solution to Problem 1 is summarized in the following theorem:

Theorem 1 *Suppose that there exist positive definite matrices* P_i *and* R_i *with* $i \in \mathcal{I}_r$, *a symmetric matrix* N, *and any matrix* $\bar{L}_i$ *with* $i \in \mathcal{I}_r$ *such that the following optimization problem composed of LMI conditions is satisfied:*

$$\min_X(\gamma_1^2 + \gamma_2^2 + \gamma_3^2),\ X \in \{P_i, R_i, N, \bar{L}_i\} \tag{19}$$

subject to

$$\begin{bmatrix}\Psi_{ij} & * \\ \mathcal{N} & -D_\mu\end{bmatrix} \prec 0 \text{ for } (i,j) \in \mathcal{I}_r \times \mathcal{I}_r \tag{20}$$

$$\begin{bmatrix}\frac{\lambda_L^2}{\lambda_N}N & * \\ \bar{L}_i^T & I\end{bmatrix} \succ 0 \text{ for } i \in \mathcal{I}_r, \tag{21}$$

$$\begin{bmatrix}\lambda_N^2 I & * \\ N & I\end{bmatrix} \succ 0, \tag{22}$$

where λ_L and λ_N are given positive scalars; I is the identity matrix of an appropriate dimension;

$$\Psi_{ij} = \begin{bmatrix}\psi_{11}^{ij} & * & * \\ \psi_{21}^{ij} & \psi_{22}^{ij} & * \\ \psi_{31}^{ij} & \psi_{32}^{ij} & \psi_{33}^{ij}\end{bmatrix};$$

$$\psi_{11}^{ij} = P(\phi) + Q + \mathrm{He}\{NA_j - \bar{L}_jC_y\};$$

$$\psi_{21}^{ij} = P_i + P(\phi) + \beta\{NA_j - \bar{L}_jC_y\} - \alpha N;$$

$\psi_{22}^{ij} = hP_i + h^2\{P(\phi) + R_i\} - 2\beta N;$
$\psi_{31}^{ij} = \{NA_j - \bar{L}_j C_y\} + \alpha A_j^T N + Q;$
$\psi_{32}^{ij} = \beta A_j^T - N;$
$\psi_{33}^{ij} = \mathrm{He}\{NA_j\} - R_i + Q;$

$$\mathcal{N} = \begin{bmatrix} \alpha N & \beta N & N \\ \alpha N & \beta N & N \\ \alpha N & \beta N & N \end{bmatrix}; D_\mu = \begin{bmatrix} \frac{\gamma_1^2}{\lambda_\Delta} I & * & * \\ 0 & \frac{\gamma_2^2}{\lambda_B} I & * \\ 0 & 0 & \frac{\gamma_3^2}{\lambda_D \lambda_L} \end{bmatrix};$$

h is the constant sampling period; α and β are given positive scalars; and Q is a given positive definite matrix.

Then, the error dynamics (3) satisfies the conditions of Problem 1, and the maximum eigenvalues of $N^T N$ and $L_i^T L_i$ with $i \in \mathcal{I}_r$ are less than λ_N and λ_L, respectively.

Finally, the observer gain matrix is obtained as follows:

$$L_i = N^{-1} \bar{L}_i. \tag{23}$$

Proof. Based on the discrete-time fuzzy Lyapunov function (14), we have

$$\Delta V(t_k) = e^T(t_{k+1}) P(w(t_{k+1})) e(t_{k+1}) - e^T(t_k) P(w(t_k)) e(t_k), \tag{24}$$

where $P(w(t_{k+1})) = \sum_{i=1}^r w_i(\sigma(t_{k+1})) P_i$.

The exactly discretized form of the error dynamics (3) becomes

$$e(t_{k+1}) = e(t_k) + \int_{t_k}^{t_{k+1}} \dot{e}(\tau) d\tau. \tag{25}$$

On the other hand, the following is clear:

$$0 = \int_{t_k}^{t_{k+1}} \bar{e}^T(\tau) R(w(t_k)) \bar{e}(\tau) d\tau - \int_{t_k}^{t_{k+1}} \bar{e}^T(\tau) R(w(t_k)) \bar{e}(\tau) d\tau \tag{26}$$

Now, by substituting and adding (25) and (26), respectively, into (24), we obtain

$$\begin{aligned} \Delta V(t_k) &= \left\{ e(t_k) + \int_{t_k}^{t_{k+1}} \dot{e}(\tau) d\tau \right\}^T P(w(t_{k+1})) \\ &\quad \times \left\{ e(t_k) + \int_{t_k}^{t_{k+1}} \dot{e}(\tau) d\tau \right\} \\ &\quad - e^T(t_k) P(w(t_k)) e(t_k) \\ &\quad - \int_{t_k}^{t_{k+1}} \bar{e}^T(\tau) R(w(t_k)) \bar{e}(\tau) d\tau \\ &\quad + \int_{t_k}^{t_{k+1}} \bar{e}^T(\tau) R(w(t_k)) \bar{e}(\tau) d\tau \\ &= e^T(t_k) \{ P(w(t_{k+1})) - P(w(t_k)) \} e(t_k) \\ &\quad + 2 e^T(t_k) P(w(t_{k+1})) \left\{ \int_{t_k}^{t_{k+1}} \dot{e}(\tau) \mathrm{d}\tau \right\} \\ &\quad - \int_{t_k}^{t_{k+1}} \bar{e}^T(\tau) R(w(t_k)) \bar{e}(\tau) d\tau \\ &\quad + \left\{ \int_{t_k}^{t_{k+1}} \dot{e}(\tau) d\tau \right\}^T P(w(t_{k+1})) \\ &\quad \times \left\{ \int_{t_k}^{t_{k+1}} \dot{e}(\tau) d\tau \right\} \\ &\quad + \int_{t_k}^{t_{k+1}} \bar{e}^T(\tau) R(w(t_k)) \bar{e}(\tau) d\tau. \end{aligned} \tag{27}$$

From Lemma 1 and Lemma 2, (27) is majorized by the following:

$$\begin{aligned} \Delta v(t_k) &= e^T(t_k) \{ P(w(t_{k+1})) - P(w(t_k)) \} e(t_k) \\ &\quad + 2 e^T(t_k) P(w(t_{k+1})) \left\{ \int_{t_k}^{t_{k+1}} \dot{e}(\tau) d\tau \right\} \\ &\quad - \int_{t_k}^{t_{k+1}} \bar{e}^T(\tau) R(w(t_k)) \bar{e}(\tau) d\tau \\ &\quad + h \int_{t_k}^{t_{k+1}} \dot{e}^T(\tau) P(w(t_{k+1})) \dot{e}(\tau) d\tau \\ &\quad + h^2 \int_{t_k}^{t_{k+1}} \dot{e}^T(\tau) R(w(t_k)) \dot{e}(\tau) d\tau. \end{aligned} \tag{28}$$

From the error dynamics (3), the following holds for any symmetric matrix N of an appropriate dimension:

$$\begin{aligned} 0 &= 2 \int_{t_k}^{t_{k+1}} \{ N\bar{e}(\tau) + \alpha N e(t_k) + \beta N \dot{e}(\tau) \}^T \\ &\quad \times \{ -\dot{e}(\tau) + \Lambda(m(\tau)) e(t_k) + A(m(\tau)) \bar{e}(\tau) \\ &\quad + B(w(\tau)) \varpi(\tau) - \Phi(m(\tau)) v(t_k) \\ &\quad + \Delta_A(\tau) x(\tau) \} d\tau, \end{aligned} \tag{29}$$

where α and β are given positive scalars.

Before proceeding, we introduce the following term:

$$\int_{t_k}^{t_{k+1}} \Gamma(\tau) d\tau, \tag{30}$$

where $\Gamma(\tau) = e^T(\tau) Q e(\tau) - \gamma_1^2 x^T(\tau) x(\tau) - \gamma_2^2 \varpi^T(\tau) \times \varpi(\tau) - \gamma_3^2 v^T(\tau) v(\tau)$; γ_1, γ_2, and γ_3 are positive scalars to be determined; and Q is a given positive definite matrix of an appropriate dimension.

Now, by adding (29) and (30) into (28), we have

$$\Delta V(t_k) + \int_{t_k}^{t_{k+1}} \Gamma(\tau)d\tau \le \int_{t_k}^{t_{k+1}} \Theta(\tau)d\tau, \tag{31}$$

where

$$\begin{aligned}\Theta(\tau) = &\frac{1}{h}e^T(t_k)\{P(w(t_{k+1})) - P(w(t_k))\}e(t_k)\\ &+2e^T(t_k)P(w(t_{k+1}))\dot{e}(\tau) - \bar{e}^T(\tau)R(w(t_k))\bar{e}(\tau)\\ &+\dot{e}^T(\tau)\{hP(w(t_{k+1})) + h^2R(w(t_k))\}\dot{e}(\tau)\\ &+2\{N\bar{e}(\tau) + \alpha Ne(t_k) + \beta N\dot{e}(\tau)\}^T\\ &\times\{-\dot{e}(\tau) + \Lambda(m(\tau))e(t_k) + A(m(\tau))\bar{e}(\tau)\}\\ &+2\{N\bar{e}(\tau) + \alpha Ne(t_k) + \beta N\dot{e}(\tau)\}^T\\ &\times\{B(w(\tau))\varpi(\tau) - \Phi(m(\tau))v(t_k) + \Delta_A(\tau)x(\tau)\}\\ &+e^T(\tau)Qe(\tau) - \gamma_1^2x^T(\tau)x(\tau)\\ &-\gamma_2^2\varpi^T(\tau)\varpi(\tau) - \gamma_3^2v^T(\tau)v(\tau),\end{aligned}$$

from which we know that

$$\Delta V(t_k) + \int_{t_k}^{t_{k+1}} \Gamma(\tau)d\tau \le 0$$

holds if

$$\Theta(\tau) \le 0 \tag{32}$$

for $\tau \in [t_k, t_{k+1})$ is satisfied.

On the other hand, considering that

$$\begin{aligned}w_i(t_{k+1}) &= w_i(t_k) + \int_{t_k}^{t_{k+1}} \dot{w}_i(\tau)d\tau\\ &\le w_i(t_k) + \int_{t_k}^{t_{k+1}} \phi_i d\tau\\ &= w_i(t_k) + h\phi_i,\end{aligned} \tag{33}$$

where ϕ_i with $i \in \mathcal{I}_r$ is a given positive scalar that satisfies $|\dot{w}_i(\tau)| \le \phi_i$ for all τ, we obtain

$$\begin{aligned}P(w(t_{k+1})) &= \sum_{i=1}^{r} w_i(t_{k+1})P_i\\ &\le \sum_{i=1}^{r} \{w_i(t_k) + h\phi_i\}P_i\\ &= P(w(t_k)) + hP(\phi),\end{aligned} \tag{34}$$

where $P(\phi) = \sum_{i=1}^{r} \phi_i P_i$.

Moreover, we introduce the following matrix inequality:

$$2X^TY \le \mu X^TX + \mu^{-1}Y^TY, \tag{35}$$

where X and Y are any matrices, and μ is a given positive scalar.

Then, from (35), we know that

$$\begin{aligned}&2\{N\bar{e}(\tau) + \alpha Ne(t_k) + \beta N\dot{e}(\tau)\}^T\\ &\times\{B(w(\tau))\varpi(\tau) - \Phi(m(\tau))v(t_k) + \Delta_A(\tau)x(\tau)\}\end{aligned} \tag{36}$$

$$\begin{aligned}&\le \{N\bar{e}(\tau) + \alpha Ne(t_k) + \beta N\dot{e}(\tau)\}^T(\mu_1I + \mu_2I + \mu_3I)\\ &\quad\times\{N\bar{e}(\tau) + \alpha Ne(t_k) + \beta N\dot{e}(\tau)\}\\ &\quad+\mu_1^{-1}x^T(\tau)\Delta_A^T(\tau)\Delta_A(\tau)x(\tau)\\ &\quad+\mu_2^{-1}\varpi^T(\tau)B^T(w(\tau))B(w(\tau))\varpi(\tau)\\ &\quad+\mu_3^{-1}v^T(\tau)\Phi^T(m(\tau))\Phi(m(\tau))v(\tau)\end{aligned} \tag{37}$$

holds with given positive scalars μ_1, μ_2, and μ_3.

From (34) and (37), (32) is further majorized by

$$\begin{aligned}\Theta(\tau) \le &\frac{1}{h}e^T(t_k)\{P(w(t_k)) + hP(\phi) - P(w(t_k))\}e(t_k)\\ &+2e^T(t_k)\{P(w(t_k)) + hP(\phi)\}\dot{e}(\tau)\\ &+\dot{e}^T(\tau)\{hP(w(t_k)) + h^2P(\phi)\\ &\qquad + h^2R(w(t_k))\}\dot{e}(\tau)\\ &-\bar{e}^T(\tau)R(w(t_k))\bar{e}(\tau)\\ &+2\{N\bar{e}(\tau) + \alpha Ne(t_k) + \beta N\dot{e}(\tau)\}^T\\ &\times\{-\dot{e}(\tau) + \Lambda(m(\tau))e(t_k) + A(m(\tau))\bar{e}(\tau)\}\\ &+\{\bar{e}(\tau) + e(t_k)\}^TQ\{\bar{e}(\tau) + e(t_k)\}\\ &+\{N\bar{e}(\tau) + \alpha Ne(t_k) + \beta N\dot{e}(\tau)\}^T\\ &\times(\mu_1I + \mu_2I + \mu_3I)\\ &\times\{N\bar{e}(\tau) + \alpha Ne(t_k) + \beta N\dot{e}(\tau)\}\\ &+x^T(\tau)\{\mu_1^{-1}\Delta_A^T(\tau)\Delta_A(\tau) - \gamma_1^2I\}x(\tau)\\ &+\varpi^T(\tau)\{\mu_2^{-1}B^T(w(\tau))B(w(\tau)) - \gamma_2^2I\}\varpi(\tau)\\ &+v^T(\tau)\{\mu_3^{-1}\Phi^T(m(\tau))\Phi(m(\tau)) - \gamma_3^2I\}v(t)\\ &\le 0,\end{aligned} \tag{38}$$

where $e(\tau) = \bar{e}(\tau) + e(t_k)$.

Thus, the sufficient condition for guaranteeing (38) is as follows:

$$\begin{bmatrix} e(t_k)\\ \dot{e}(\tau)\\ \bar{e}(\tau)\end{bmatrix}^T \{\Psi(\tau) + \mathcal{N}^TD_\mu\mathcal{N}\}\begin{bmatrix} e(t_k)\\ \dot{e}(\tau)\\ \bar{e}(\tau)\end{bmatrix} \le 0, \tag{39}$$

$$\mu_1^{-1}\lambda_\Delta - \gamma_1^2 = 0, \tag{40}$$

$$\mu_2^{-1}\lambda_B - \gamma_2^2 = 0, \tag{41}$$

$$\mu_3^{-1}\lambda_\Phi - \gamma_3^2 = 0. \tag{42}$$

where

$$\Psi(\tau) = \begin{bmatrix}\psi_{11} & * & *\\ \psi_{21} & \psi_{22} & *\\ \psi_{31} & \psi_{32} & \psi_{33}\end{bmatrix};$$

$\psi_{11} = P(\phi) + Q + \mathrm{He}\{N^T\Lambda(m(\tau))\}$;

$\psi_{21} = P(w(t_k)) + hP(\phi) + \beta N^T\Lambda(m(\tau)) - \alpha N$;

$\psi_{22} = hP(w(t_k)) + h^2\{P(\phi) + R(w(t_k))\}$;

$$\psi_{31} = N^T \Lambda(m(\tau)) + \alpha A^T(m(\tau)) + Q;$$
$$\psi_{32} = \beta A^T(m(\tau)) - N;$$
$$\psi_{33} = \mathrm{He}\{N^T A(m(\tau))\} - R(w(t_k)) + Q;$$
$$\mathcal{N} = \begin{bmatrix} \alpha N & \beta N & N \\ \alpha N & \beta N & N \\ \alpha N & \beta N & N \end{bmatrix}; \text{ and } D_\mu = \begin{bmatrix} \mu_1 I & * & * \\ 0 & \mu_2 I & * \\ 0 & 0 & \mu_3 I \end{bmatrix}.$$

Now, by applying the Schur complements on (39), we have

$$\begin{bmatrix} \Psi(\tau) & * \\ \mathcal{N} & -D_\mu^{-1} \end{bmatrix} \prec 0; \quad (43)$$

thus, we know that $\Delta V(t_k) + \int_{t_k}^{t_{k+1}} \Gamma(\tau) d\tau \leq 0$ is satisfied if (43) holds.

From (40), (41), and (42), we have

$$\mu_1^{-1} = \frac{\gamma_1^2}{\lambda_\Delta},\ \mu_2^{-1} = \frac{\gamma_2^2}{\lambda_B},\ \text{and}\ \mu_3^{-1} = \frac{\gamma_3^2}{\lambda_L \lambda_D}, \quad (44)$$

because of $\lambda_\Phi = \lambda_L \lambda_d$, from which we obtain

$$D_\mu^{-1} = \begin{bmatrix} \mu_1^{-1} I & * & * \\ 0 & \mu_2^{-1} I & * \\ 0 & 0 & \mu_3^{-1} I \end{bmatrix} = \begin{bmatrix} \frac{\gamma_1^2}{\lambda_\Delta} I & * & * \\ 0 & \frac{\gamma_2^2}{\lambda_B} I & * \\ 0 & 0 & \frac{\gamma_3^2}{\lambda_\Phi} \end{bmatrix} = D_\gamma. \quad (45)$$

Finally, without the shorthand notation and using (45), we can rewrite (43) as follows:

$$\sum_{i=1}^{r} \sum_{j=1}^{r} w_i(t_k) m_j(\tau) \begin{bmatrix} \Psi_{ij} & * \\ \mathcal{N} & -D_\gamma \end{bmatrix} \prec 0, \quad (46)$$

from which we obtain the LMI given in (20).

Moreover, supposed that the following holds:

$$N^{\frac{1}{2}} L^T(\tau) L(\tau) N^{\frac{1}{2}} \prec \frac{\lambda_L^2}{\lambda_N} I \quad (47)$$

then, we have the following form the Schur complements:

$$\begin{bmatrix} \frac{\lambda_L^2}{\lambda_N} I & * \\ L^T(\tau) N^{\frac{1}{2}} & I \end{bmatrix} \succ 0. \quad (48)$$

Now, by applying the congruence transformation with $diag\{N^{\frac{1}{2}}, I\}$, we have

$$\begin{bmatrix} \frac{\lambda_L^2}{\lambda_N} N & * \\ L^T(\tau) & I \end{bmatrix} \succ 0, \quad (49)$$

which implies the LMI condition (21).

In addition, by applying the Schur complements on

$$N^T N \prec \lambda_N^2 I, \quad (50)$$

we have the LMI condition (22).

Thus, if (21) and (22) are satisfied, the maximum eigenvalues of $L(\tau)$ and N are less than λ_L and λ_N, respectively.

Furthermore, we know that the following holds if the LMI (20) are satisfied:

$$\Delta V(t_k) + \int_{t_k}^{t_{k+1}} \Gamma(\tau) d\tau \leq 0, \quad (51)$$

which implies that the equilibrium of (3) is asymptotically stabilized from Proposition 1.

Moreover, summing (51) from $k = 0$ to $k = \infty$ yields

$$V(\infty) - V(0) + \int_0^\infty \Gamma(\tau) d\tau \leq 0.$$

Because $V(\infty) \to 0$, we conclude that if there exists a solution to the LMI-based optimization problem (19), the sampled-data fuzzy observer (2) achieves the conditions of Problem 1.

Remark 1 *The major distinguishable features of the proposed sampled-data fuzzy observer design method compared with the existing approaches are as follows:*

1. An LMI-based optimization problem is proposed based on the discrete-time fuzzy Lyapunov function, by which a numerically relaxed observer design conditions are obtained.
2. The observer gain matrix is determined using the norm upper bounds of the system disturbance and measurement noise.
3. The designed observer is robust to the uncertain premise variable by allowing the membership function of the observer to be different from that of the system.

5. Experimental Results

Throughout the section, we design the sampled-data fuzzy observer (2) using Theorem 1, and validate its performance using the commercial AHRS unit. This section consists of two subsections; in the first subsection, the fuzzy observer gain matrix is determined by numerically solving the optimization problem with the LMIs. The LMI conditions was solved via YALMIP and SeDuMi [24, 25] running on MATLAB 2016a. After constructing the fuzzy observer, in the next subsection, we perform real-time experiments, by which the performance of our approach is validated.

5.1 Observer design

The first step is to determine parameters $(\alpha,\beta,Q,h,\lambda_\Delta,\lambda_B,\lambda_D,\lambda_L,\lambda_N)$. The free parameters are chosen as $(\alpha,\beta,Q,\lambda_L,\lambda_N)=(0.1,0.1,I,10,100)$. Moreover, the sampling period is set to $h=2.5ms=400Hz$, and the operating regions (6) is confined by $M_p=4\text{rad/sec}$, $M_q=4\text{rad/sec}$, and $M_r=2\text{rad/sec}$. The remained parameters λ_B, λ_D, and λ_Δ are tuned according to the sensor noise and the disturbance on the experimental setup.

By numerically solving the optimization problem (19) with LMIs (20)-(22), we obtain the following observer gain matrices:

$$L_1=\begin{bmatrix}2.4099 & 0.0000 & -0.0000 & -0.0000\\ 0.0000 & 2.4099 & 0.0000 & -0.0000\\ -0.0000 & 0.0000 & 2.4099 & 0.0000\\ -0.0000 & -0.0000 & 0.0000 & 2.4099\end{bmatrix},$$

$$L_2=\begin{bmatrix}2.4181 & 0.0000 & -0.0000 & -0.0000\\ 0.0000 & 2.4181 & -0.0000 & 0.0000\\ -0.0000 & -0.0000 & 2.4181 & 0.0000\\ -0.0000 & 0.0000 & 0.0000 & 2.4181\end{bmatrix},$$

$$L_3=\begin{bmatrix}2.4126 & 0.0000 & -0.0000 & -0.0000\\ 0.0000 & 2.4126 & 0.0000 & -0.0000\\ -0.0000 & 0.0000 & 2.4126 & 0.0000\\ -0.0000 & -0.0000 & 0.0000 & 2.4126\end{bmatrix},$$

$$L_4=\begin{bmatrix}2.4205 & 0.0000 & -0.0000 & -0.0000\\ 0.0000 & 2.4205 & 0.0000 & -0.0000\\ -0.0000 & 0.0000 & 2.4205 & 0.0000\\ -0.0000 & -0.0000 & 0.0000 & 2.4205\end{bmatrix},$$

$$L_5=\begin{bmatrix}2.4099 & 0.0000 & -0.0000 & -0.0000\\ 0.0000 & 2.4099 & 0.0000 & -0.0000\\ -0.0000 & 0.0000 & 2.4099 & 0.0000\\ -0.0000 & -0.0000 & 0.0000 & 2.4099\end{bmatrix},$$

$$L_6=\begin{bmatrix}2.4181 & 0.0000 & -0.0000 & -0.0000\\ 0.0000 & 2.4181 & -0.0000 & 0.0000\\ -0.0000 & -0.0000 & 2.4181 & 0.0000\\ -0.0000 & 0.0000 & 0.0000 & 2.4181\end{bmatrix},$$

$$L_7=\begin{bmatrix}2.4126 & 0.0000 & -0.0000 & -0.0000\\ 0.0000 & 2.4126 & 0.0000 & -0.0000\\ -0.0000 & 0.0000 & 2.4126 & 0.0000\\ -0.0000 & -0.0000 & 0.0000 & 2.4126\end{bmatrix},$$

$$L_8=\begin{bmatrix}2.4205 & 0.0000 & -0.0000 & -0.0000\\ 0.0000 & 2.4205 & 0.0000 & 0.0000\\ -0.0000 & 0.0000 & 2.4205 & 0.0000\\ -0.0000 & 0.0000 & 0.0000 & 2.4205\end{bmatrix}.$$

As can be seen from the resulting gain matrices, the maximum eigen values for the resulting observer gains are less than λ_L.

5.2 Experimental results

The experiments were carried out using MTi-30 AHRS from Xsens, from which we can obtain both calibrated inertial sensor data and Euler angles at the same time up to $400Hz$. In this experiment, it is assumed that the Euler angle from MTi-30 AHRS is true angle of the sensor. The proposed sampled-data fuzzy observer computes the estimated Euler angles using the raw sensor data obtained by MTi-30 AHRS. The performance of the proposed method is verified by analyzing the error between the Euler angles acquired by the proposed method and MTi-30 AHRS.

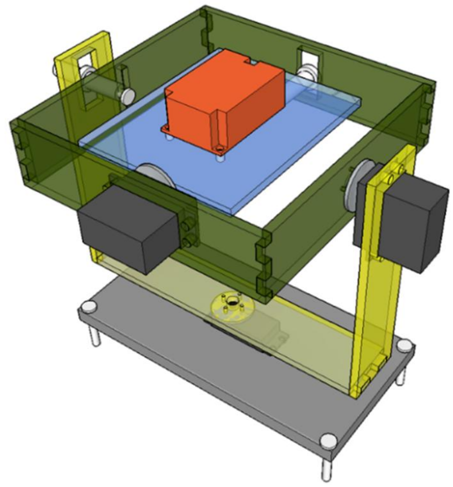

Fig. 2. The 3-axis motion table

Table 1. Comparison of the estimation performance

Methods	RMS(ROLL) STD(ROLL)	RMS(Pitch) STD(Pitch)	RMS(YAW) STD(Yaw)
Proposed	0.3218 0.2839	0.4204 0.3643	4.8338 4.5766
Madgwick et al. [16]	0.4148 0.4026	0.4845 0.4673	2.0804 2.062

To this end, we built the 3-axis motion table as shown in Fig. 2. In this experiment, each axis turns from −40deg to 40deg for five times using the 3-axis motion table. Fig. 3 shows the raw sensor data acquired by MTi-30 AHRS. Due to the weak coupling between the axes of the experimental equipment, severe disturbances affect on each sensor. Fig. 4 shows the Euler angles acquired by the proposed method, the conventional approach [16], and MTi-30 AHRS. Moreover, Table 1 summarizes the error statistics. As can be seen from the results, although there is a strong disturbance, sufficient Roll and Pitch estimation performance can be obtained through the proposed method. Moreover, the proposed method provides better performance than the conventional approach for Roll and Pitch estimation. However, the Yaw estimation performance of the proposed method is not appropriate. The reason is as follows: In the measurement Eq. (9), Roll and Pitch angles measured by the acceleration are used to compute Yaw measurement. Since there is a severe disturbance in the experimental environment, the Roll and Pitch measurements are inaccurate; thus, the Yaw measurement is unreliable. However, this problem can be easily solved by the following methods: The first method is to divide the estimation procedure into the two stages. In

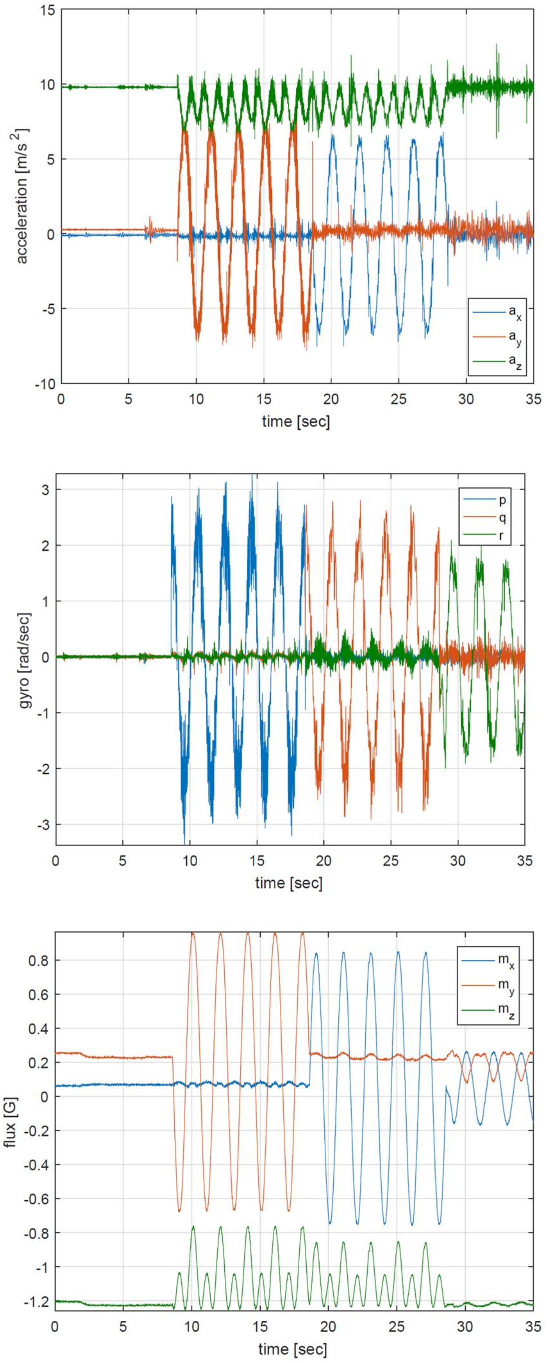

Fig. 3. The raw sensor readings acquired by MTi-30 AHRS

the first stage, the state estimation results for the Roll and Pitch are obtained, and then the results are used as measurements at the Yaw estimation stage. The block diagram for this proposed two-step procedure is shown in Fig. 5. The second method is to apply a low-pass filter to the acceleration sensor data to attenuate the effect on the disturbance. When we applied the two-step procedure, the Yaw estimation performance can be enhanced. The RMS and standard variance of the improved Yaw estimation are 2.3497 and 2.3362, respectively, which means that the improvement rate is 51.39%.

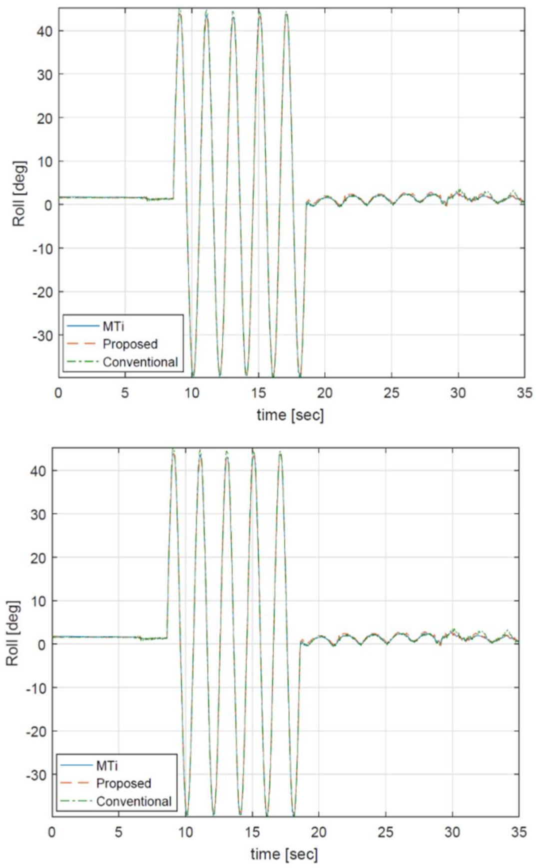

Fig. 4. The Euler angles obtained by MTi-30 AHRS (solid line), the proposed method (dashed-line), and the conventional method (dash-dotted line)

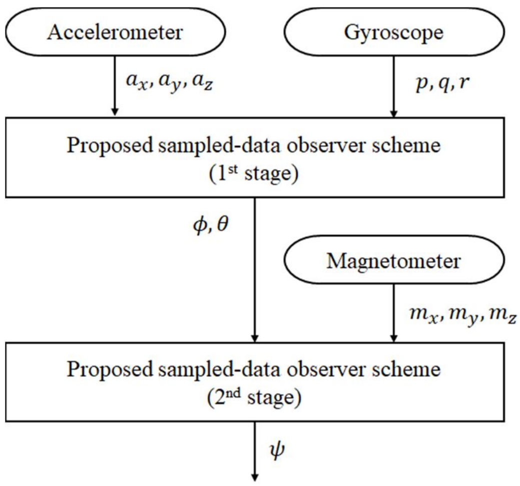

Fig. 5. The block diagram for the proposed two-step procedure

5. Conclusion

Throughout the paper, the exact discretization approach to the LMI-based sampled-data fuzzy observer design method has been dealt with. The resulting observer design condition is numerically relaxed because it was derived based on the discrete-time fuzzy Lyapunov function that contains more decision variables than the conventional common quadratic Lyapunov function. Moreover, the observer used in this paper is robust to the uncertain premise variable because the membership function of the observer can be freely designed independently from the system membership function. In addition, we derived the fuzzy model for the AHRS, and applied the proposed design technique to the state estimation problem for the AHRS. Finally, throughout the hardware experiment, we analyzed the state estimate performance of the proposed method and compared it with the conventional approach, by which we concluded that the resulting observer was well constructed.

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